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QUINTIC DEFICIENT SPLINE WAVELETS

F. BASTIN and P. LAUBIN*

Abstract

We show explicitely how to construct scaling functions and wavelets which are quintic deficient splines with compact support and symmetry properties

1 Introduction

For $m \in \mathbb{N}$, it is well known that the functions $N_{m+1}(\cdot - k)$ $(k \in \mathbb{Z})$, where $N_{m+1} = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($m+1$ factors), constitute a Riesz basis of the set of smoothest splines of degree m,

$$\mathcal{V}_0 = \{f \in L_2(\mathbb{R}) : f|_{[k,k+1]} = P_k^{(m)}, k \in \mathbb{Z} \text{ and } f \in C_{m-1}(\mathbb{R})\}$$

where $P_k^{(m)}$ is a polynomial of degree at most m; for $m = 0$, it is simply the set of functions in $L^2(\mathbb{R})$ which are constant on every interval $[k, k+1]$, $k \in \mathbb{Z}$

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*P Laubin died on February 21, 2001 Bastin and Laubin started working on this paper together That's why both are quoted as authors

Moreover, if we define

$$\mathcal{V}_j = \{f \in L_2(\mathbb{R}) : f(2^{-j}.) \in \mathcal{V}_0\}, \quad j \in \mathbb{Z}$$

then the sets $\mathcal{V}_j$ ($j \in \mathbb{Z}$) constitute a multiresolution analysis of $L^2(\mathbb{R})$ and the function N_{m+1} is a scaling function for it; hence one gets bases of wavelets from standard constructions ([6]; Chui-Wang biorthogonal wavelets,[2]; Battle-Lemarié orthonormal wavelets, [7]).

For numerical analysis purposes, splines of odd degree are of special interest; moreover, it is also useful to consider the set of deficient splines of degree $2m+1$ ($m \in \mathbb{N}$), that is to say

$$\boxed{V_0 := \{f \in L_2(\mathbb{R}) : \ f|_{[k,k+1]} = P_k^{(2m+1)}, k \in \mathbb{Z} \text{ and } f \in C_{m+1}(\mathbb{R})\}}$$

(see [3], [8]). As for the space $\mathcal{V}_0$, a standard argument shows that the space V_0 is a closed subspace of $L^2(\mathbb{R})$. For $m = 1$, this is the set of smoothest cubic splines; for $\boxed{m = 2}$, we denote this set as the set of

deficient quintic splines.

In what follows, we want to show explicitely how to construct scaling functions and wavelets which are quintic deficient splines with compact support and symmetry properties.

We go straithforward to the heart of the problem of the construction of the multiresolution analysis, with all direct computations and without referring or using other results. The construction of the wavelets is also a direct computation adapted to the problem. The idea of the proof that they are a Riesz basis comes from [4], [5]. For the sake of completeness, we give here all the justifications.

2 Definitions and notations

We say that a sequence of functions f_k ($k \in \mathbb{Z}$) in a Hilbert space $(H, \|.\|)$ satisfies the Riesz condition if they are $A, B > 0$, $A \leq B$ such that

$$A \sum_{(k)} |c_k|^2 \leq \| \sum_{(k)} c_k f_k \|^2 \leq B \sum_{(k)} |c_k|^2 \qquad \text{(RC)}$$

for every finite sequence (c_k) of complex numbers. If we denote by L the closed linear hull of the f_k ($k \in \mathbb{Z}$) then the map

$$T : l^2 \to L \qquad (c_k)_{k \in \mathbb{Z}} \mapsto \sum_{k=-\infty}^{+\infty} c_k f_k$$

is then a topological isomorphism. We say that the functions f_k ($k \in \mathbb{Z}$) constitute a Riesz basis for L.

We use the notation $\widehat{f}(\xi)$ for the Fourier transform $\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx$ of f.

In case $H = L^2(\mathbb{R})$ and $f_k(x) = f(x-k)$ ($k \in \mathbb{Z}$), taking Fourier transforms, the inequality (RC) of the Riesz condition can be written as follows

$$A \sum_{(k)} |c_k|^2 \leq \|p\sqrt{w}\|^2_{L^2([0,2\pi])} \leq B \sum_{(k)} |c_k|^2 \qquad \text{(RCF)}$$

with

$$w(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{f}(\xi + 2l\pi)|^2 \in L^1_{loc}, \quad p(\xi) = \sum_{(k)} c_k e^{-ik\xi}.$$

Finally, using a classical argument (based on Fejer kernel for example), one shows that (RFC) is satisfied for every finite sequence (c_k) if and only if

$$A \leq w(\xi) \leq B \quad a.e.$$

(see for example [1],[7]).

For the sake of completeness, we also recall the standard definition of multiresolution analysis. We say that a sequence of closed linear subspaces V_j ($j \in \mathbb{Z}$) of $L^2(\mathbb{R})$ constitutes a *multiresolution analysis* of $L^2(\mathbb{R})$ if the following properties hold:

(i) $V_j \subset V_{j+1}\ \forall j \in \mathbb{Z}$, $\overline{\cup_{j\in\mathbb{Z}} V_j}_{L^2} = L^2(\mathbb{R})$, $\cap_{j\in\mathbb{Z}} V_j = \{0\}$

(ii) $f \in V_0 \Leftrightarrow f(. - k) \in V_0\ \forall k \in \mathbb{Z}$

(iii) $\forall j \in Z$, $f \in V_j \Leftrightarrow f(2^{-j}.) \in V_0$

(iv) there is $\varphi \in L^2(\mathbb{R})$ such that the functions $\varphi(. - k)$, $k \in \mathbb{Z}$, are a Riesz basis for V_0.

From a mutiresolution analysis, one constructs a Riesz basis of $L^2(\mathbb{R})$ from a standard procedure (see for example [6], [7]), using the spaces W_j, orthogonal complement of V_j in V_{j+1} ($j \in \mathbb{Z}$).

Here we use this procedure but with two functions instead of one for property (iv).

3 Construction of a multiresolution analysis

Let us denote by V_0 the following set of quintic splines

$$V_0 := \{f \in L_2(\mathbb{R}) :\ f|_{[k,k+1]} = P_k^{(5)}, k \in \mathbb{Z} \text{ and } f \in C_3(\mathbb{R})\}.$$

Looking for $f \in V_0$ with support $[0,3]$ (smaller interval does not give anything), we are lead to a homogenous linear system of 18 unknowns and 16 equations; this let us think that two scaling functions will be needed to generate V_0.

Proposition 3.1 *A function f with support $[0,3]$ belongs to V_0 if and only if*

$$f(x) = \begin{cases} nx^4 + ax^5 & \text{if } x \in [0,1] \\ b\left(x-\frac{3}{2}\right)^5 + c\left(x-\frac{3}{2}\right)^4 + d\left(x-\frac{3}{2}\right)^3 & \\ \qquad +e\left(x-\frac{3}{2}\right)^2 + f\left(x-\frac{3}{2}\right) + g & \text{if } x \in [1,2] \\ h(3-x)^4 + j(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x \notin [0,3] \end{cases}$$

with

$$b = 19a + \tfrac{57}{5}n \qquad c = \tfrac{15}{2}a + \tfrac{11}{2}n \qquad d = -\tfrac{45}{2}a - \tfrac{27}{2}n \qquad e = -\tfrac{45}{4}a - \tfrac{33}{4}n$$

$$f = \tfrac{135}{16}a + \tfrac{81}{16}n \qquad g = \tfrac{117}{32}a + \tfrac{627}{160}n \qquad h = 15a + 10n \qquad j = -10a - \tfrac{33}{5}n$$

Proof. The particular form in which we write the polynomials are due to the fact that we have in mind to construct functions with symmetry. Moreover, the polynomial on $[0,1]$ (resp. $[2,3]$) can immediately be written in this form because we want C_3 regularity at the point 0 (resp. 3) and support in $[0,3]$.

The coefficients are obtained using the definition of the quintic splines; we get an homogenous system of 8 linear equations with 10 unknowns.□

Among the functions described above, there exists symmetric and antisymmetric ones (the symmetry is naturally considered relatively to $\frac{3}{2}$). We are also going to show that they generate V_0.

Theorem 3.2 *The following functions φ_a and φ_s*

$$\varphi_a(x) = \begin{cases} x^4 - \frac{11}{15}x^5 & \text{if } x \in [0,1] \\ -\frac{9}{8}(x-\frac{3}{2}) + 3(x-\frac{3}{2})^3 - \frac{38}{15}(x-\frac{3}{2})^5 & \text{if } x \in [1,2] \\ -(3-x)^4 + \frac{11}{15}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

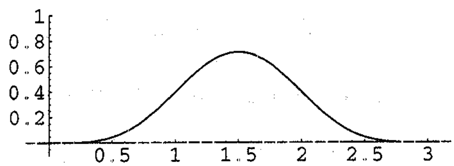
$$\varphi_s(x) = \begin{cases} x^4 - \frac{3}{5}x^5 & \text{if } x \in [0,1] \\ \frac{57}{80} - \frac{3}{2}(x-\frac{3}{2})^2 + (x-\frac{3}{2})^4 & \text{if } x \in [1,2] \\ (3-x)^4 - \frac{3}{5}(3-x)^5 & \text{if } x \in [2,3] \\ 0 & \text{if } x < 0 \text{ or } x > 3 \end{cases}$$

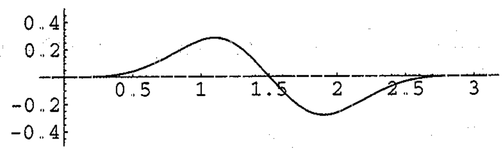
are respectively antisymmetric and symmetric with respect to $\frac{3}{2}$ and the family

$$\{\varphi_a(\cdot - k),\ k \in \mathbb{Z}\} \cup \{\varphi_s(\cdot - k),\ k \in \mathbb{Z}\}$$

constitutes a Riesz basis of V_0.

Here is a picture of φ_s, φ_a.





Proof. *Construction of* φ_a, φ_s.

We use the notations and the result of Proposition (3.1). We look for a, n such that

$$\begin{cases} n = h \\ a = j \\ b = 0 \\ d = 0 \\ f = 0 \end{cases} \qquad \text{(resp.} \quad \begin{cases} n = -h \\ a = -j \\ c = 0 \\ d = 0 \\ g = 0. \end{cases} \text{)}$$

This system is equivalent to the single equation

$$5a + 3n = 0 \qquad \text{(resp. } 15a + 11n = 0\text{).}$$

With $n = 1, a = -\frac{3}{5}$ (resp. $n = 1, a = -\frac{11}{15}$), we get φ_s (resp. φ_a).

Riesz condition.

For every $k \in \mathbb{Z}$, we define

$$\varphi_{a,k}(x) = \varphi_a(x - k) \quad \text{and} \quad \varphi_{s,k}(x) = \varphi_s(x - k).$$

We first prove that the functions $\varphi_{a,k}$ ($k \in \mathbb{Z}$) (resp. $\varphi_{s,k}$ ($k \in \mathbb{Z}$)) form a Riesz family. Indeed, since we have

$$\|\sum_{(k)} c_k \varphi_{a,k}\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\sum_{(k)} c_k e^{-ik\xi}\widehat{\varphi_a}(\xi)\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\sum_{(k)} c_k e^{-ik\xi}\sqrt{\omega_a(\xi)}\|^2_{L^2([0,2\pi])}$$

for every finite sequence (c_k) of complex numbers and where

$$\omega_a(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_a}(\xi + 2l\pi)|^2,$$

it suffices to show that there are constants $c, C > 0$ such that

$$c \le \omega_a(\xi) \le C, \; \xi \in [0, 2\pi].$$

Using the definition of φ_a, we get

$$\begin{aligned} \widehat{\varphi_a}(\xi) &= \frac{-16i}{\xi^6} e^{-3i\xi/2}\left(3\xi(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) - 11\sin(\frac{3\xi}{2}) - 27\sin(\frac{\xi}{2})\right) \\ &= \frac{-16i}{\xi^6} e^{-3i\xi/2}\left(6\xi\cos(\frac{\xi}{2})(4+\cos\xi) - 2\sin(\frac{\xi}{2})(19 + 11\cos\xi)\right). \end{aligned}$$

Using

$$\sum_{l=-\infty}^{+\infty} \frac{1}{(\xi + k)^{r+2}} = \frac{(-1)^r}{(r+1)!} D^r_\xi \frac{\pi^2}{\sin^2(\pi\xi)}, \; r \in \mathbb{N}, \; \xi \in \mathbb{R} \setminus \mathbb{Z},$$

some computations lead to

$$\omega_a(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_a}(\xi + 2l\pi)|^2 = \frac{23247 - 21362\cos\xi - 385\cos(2\xi)}{311850}$$

hence to the conclusion. The same can be done for φ_s. We get

$$\widehat{\varphi_s}(\xi) = \frac{96}{\xi^6} e^{-3i\xi/2}\sin(\frac{\xi}{2})\left(\xi(2+\cos\xi) - 3\sin\xi\right)$$

and

$$\omega_s(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_s}(\xi + 2l\pi)|^2 = \frac{14445 + 7678\cos\xi + 53\cos(2\xi)}{34650}.$$

Now, let us consider both families $\varphi_{a,k}$ ($k \in \mathbb{Z}$) and $\varphi_{s,k}$ ($k \in \mathbb{Z}$) together. For every finite sequence (c_k) and (d_k) of complex numbers, we have

$$\|\sum_{(k)}(c_k\varphi_{a,k} + d_k\varphi_{s,k})\|^2_{L^2(\mathbb{R})} = \sum_{j=-\infty}^{+\infty} \|\sum_{(k)}(c_k\varphi_{a,k-j} + d_k\varphi_{s,k-j})\|^2_{L^2([0,1])}.$$

On $[0,1]$, only $\varphi_{a,l}, \varphi_{s,l}$ with $l=-2,-1,0$ are not identically 0; moreover, these functions are linearly independant (see appendix for a proof). As on a finite dimensional space, all norms are equivalent, we get that there are $r, R > 0$ such that

$$\begin{aligned} r & \left(\| \sum_{(k)} c_k \varphi_{a,k-j} \|^2_{L^2([0,1])} + \| \sum_{(k)} d_k \varphi_{s,k-j} \|^2_{L^2([0,1])} \right) \\ & \leq \| \sum_{(k)} (c_k \varphi_{a,k-j} + d_k \varphi_{s,k-j}) \|^2_{L^2([0,1])} \\ & \leq R \left(\| \sum_{(k)} c_k \varphi_{a,k-j} \|^2_{L^2([0,1])} + \| \sum_{(k)} d_k \varphi_{s,k-j} \|^2_{L^2([0,1])} \right) \end{aligned}$$

Now, writing again

$$\sum_{j=-\infty}^{+\infty} \| \sum_{(k)} c_k \varphi_{a,k-j} \|^2_{L^2([0,1])} = \| \sum_{(k)} c_k \varphi_{a,k} \|^2_{L^2(\mathbb{R})},$$

$$\sum_{j=-\infty}^{+\infty} \| \sum_{(k)} d_k \varphi_{s,k-j} \|^2_{L^2([0,1])} = \| \sum_{(k)} d_k \varphi_{s,k} \|^2_{L^2(\mathbb{R})}$$

and using what has been done on each family separately, we conclude.

Riesz basis for V_0.

Let us show that V_0 is the closed linear hull of the $\varphi_{a,k}, \varphi_{s,k}$ $(k \in \mathbb{Z})$.

On one hand, as the set V_0 is a closed subspace of $L^2(\mathbb{R})$ containing each $\varphi_{a,k}$ and $\varphi_{s,k}$, it contains the closed linear hull of these functions.

On the other hand, using Fourier transforms, we see that it suffices to show that for every $f \in V_0$, there are $p, q \in L^2_{loc}$ and $2\pi-$ periodic such that

$$\widehat{f}(\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi) \quad a.e.$$

Let $f \in V_0$. Because of the definition of V_0, there are $(c_k)_{k\in\mathbb{Z}}, (d_k)_{k\in\mathbb{Z}} \in l^2$ such that

$$D^6 f = \lim_{m\to+\infty} \sum_{k=-m}^{m} (c_k \delta_k + d_k \delta'_k)$$

in the distribution sense, where δ_k and δ'_k are respectively the Dirac and the derivative of the Dirac distribution at k (see appendix for proof). Taking Fourier transforms, we get also

$$(i\xi)^6 \widehat{f}(\xi) = \lim_{m\to+\infty} \sum_{k=-m}^{m} (c_k e^{-ik\xi} + i d_k \xi e^{-ik\xi});$$

it follows that there are $m(\xi),\ n(\xi) \in L^2_{loc}$ and $2\pi-$ periodic such that

$$(i\xi)^6 \widehat{f}(\xi) = m(\xi) + \xi n(\xi) \quad a.e.$$

Hence the problem is to find $p, q \in L^2_{loc}$ and $2\pi-$ periodic such that

$$\frac{m(\xi) + \xi n(\xi)}{(i\xi)^6} = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi).$$

Using the explicit expression of the Fourier transform of φ_s and φ_a, we are lead to look for p, q such that

$$\left\{ \begin{array}{rcl} -m(\xi) &=& e^{-3i\xi/2}\left(-3\ 96 \sin\xi \sin(\frac{\xi}{2})p(\xi) + 16i(11\sin(\frac{3\xi}{2}) + 27\sin(\frac{\xi}{2}))q(\xi)\right) \\ -n(\xi) &=& e^{-3i\xi/2}\left(96(2+\cos\xi)\sin(\frac{\xi}{2})p(\xi) - 48i(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2}))q(\xi)\right). \end{array} \right. \qquad (*)$$

For fixed ξ, this is a linear system of two equations and two unknowns; as

$$det \begin{pmatrix} -3.\ 96\sin\xi\sin(\frac{\xi}{2}) & 16i(11\sin(\frac{3\xi}{2}) + 27\sin(\frac{\xi}{2})) \\ 96(2+\cos\xi)\sin(\frac{\xi}{2}) & -48i(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) \end{pmatrix} = C\sin^6\frac{\xi}{2}$$

with $C = -3\ 2^{12}i$, we get

$$p(\xi) = \frac{16ie^{3i\xi/2}}{C\sin^6(\frac{\xi}{2})}\left(3m(\xi)(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) + n(\xi)(11\sin(\frac{3\xi}{2}) + 27\sin(\frac{\xi}{2}))\right)$$

and

$$q(\xi) = \frac{96e^{3i\xi/2}}{C\sin^6(\frac{\xi}{2})}\left(3n(\xi)\sin\xi\sin(\frac{\xi}{2}) + m(\xi)(2+\cos\xi)\sin(\frac{\xi}{2})\right).$$

These functions are $2\pi-$ periodic; it remains to prove that they are L^2_{loc}. Indeed, using $m(\xi) = -\xi n(\xi) - \xi^6 \widehat{f}(\xi)$ we get

$$\begin{aligned} & 3m(\xi)(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) + n(\xi)(11\sin(\frac{3\xi}{2}) + 27\sin(\frac{\xi}{2})) \\ = & -3\xi^6\widehat{f}(\xi)\left(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})\right) + n(\xi)\left[\frac{3}{280}\xi^7 + O(\xi^9)\right] \end{aligned}$$

and

$$\begin{aligned} & 3n(\xi)\sin\xi\sin(\frac{\xi}{2}) + m(\xi)(2+\cos\xi)\sin(\frac{\xi}{2}) \\ = & \sin(\frac{\xi}{2})\left(-\xi^6(2+\cos\xi)\widehat{f}(\xi) + n(\xi)\left[-\frac{1}{60}\xi^5 + O(\xi^7)\right]\right) \end{aligned}$$

and we conclude.□

Remark 3.3 *1) The previous proof also shows that a function f of $L^2(\mathbb{R})$ belongs to V_0 if and only if there exist $m, n \in L^2_{loc}$, $2\pi-$ periodic such that*

$$(i\xi)^6 \widehat{f}(\xi) = m(\xi) + \xi n(\xi) \quad a.e.$$

2) Since

$$3\xi\left(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})\right) - 11\sin(\frac{3\xi}{2}) - 27\sin(\frac{\xi}{2}) = \frac{-3}{280}\xi^7 + O(\xi^9)$$

and

$$\xi(2+\cos\xi) - 3\sin\xi = \frac{1}{60}\xi^5 + O(\xi^7)$$

we get

$$\widehat{\varphi_a}(0) = 0, \qquad \widehat{\varphi_s}(0) = \frac{4}{5}$$

For every $j \in \mathbb{Z}$ we define

$$V_j = \{f \in L^2(\mathbb{R}) \ : \ f(2^{-j}\cdot) \in V_0\}$$

Proposition 3.4 *The sequence V_j ($j \in \mathbb{Z}$) is an increasing sequence of closed sets of $L^2(\mathbb{R})$ and*

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}, \qquad \overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$$

Moreover, the functions φ_a, φ_s satisfy the following scaling relation

$$\boxed{\begin{pmatrix} \widehat{\varphi_s}(2\xi) \\ \widehat{\varphi_a}(2\xi) \end{pmatrix} = M_0(\xi) \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_a}(\xi) \end{pmatrix}}$$

where $M_0(\xi)$ is the matrix (called filter matrix)

$$\begin{aligned} M_0(\xi) &= e^{-3i\xi/2} \begin{pmatrix} \frac{1}{32}\cos(\frac{\xi}{2})(19+13\cos\xi) & \frac{-9i}{16}\cos^2(\frac{\xi}{2})\sin(\frac{\xi}{2}) \\ \frac{i}{32}\sin(\frac{\xi}{2})(16+11\cos\xi) & \frac{1}{32}\cos(\frac{\xi}{2})(8-7\cos\xi)) \end{pmatrix} \\ &= \frac{e^{-3i\xi/2}}{64} \begin{pmatrix} 51\cos(\frac{\xi}{2}) + 13\cos(\frac{3\xi}{2}) & -9i(\sin(\frac{\xi}{2}) + \sin(\frac{3\xi}{2})) \\ i(11\sin(\frac{3\xi}{2}) + 21\sin(\frac{\xi}{2})) & -7\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) \end{pmatrix} \end{aligned}$$

Expressed in terms of the functions instead of the Fourier transform, the scaling relation can be written as follows

$$\begin{aligned}
\varphi_s(\frac{\xi}{2}) &= \frac{1}{64}(13\varphi_s(x) + 51\varphi_s(x-1) + 51\varphi_s(x-2) + 13\varphi_s(x-3) \\
&\quad -9\varphi_a(x) - 9\varphi_a(x-1) + 9\varphi_a(x-2) + 9\varphi_a(x-3)) \\
\varphi_a(\frac{x}{2}) &= \frac{1}{64}(11\varphi_s(x) + 21\varphi_s(x-1) - 21\varphi_s(x-2) - 11\varphi_s(x-3) \\
&\quad -7\varphi_a(x) + 9\varphi_a(x-1) + 9\varphi_a(x-2) - 7\varphi_a(x-3))
\end{aligned}$$

Proof. Using the definition of V_0 and of the V_j ($j \in \mathbb{Z}$), it is clear that $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$. The density of the union is due to the facts that a smoothest spline is also a deficient spline ($\mathcal{V}_j \subset V_j$) and that the union $\cup_{j\in\mathbb{Z}} \mathcal{V}_j$ is dense in $L^2(\mathbb{R})$.

Now, let $f \in \cap_{j\in\mathbb{Z}} V_j$. For every $j \le 0$, there is then a polynomial P_j (resp. Q_j) such that $f = P_j$ on $[0, 2^{-j}]$ (resp. $f = Q_j$ on $[-2^{-j}, 0]$). It follows that $P_j = P_{j'}$ (resp. $Q_j = Q_{j'}$) for every $j, j' \le 0$ hence f is a polynomial on $[0, +\infty[$ (resp. $]-\infty, 0]$). Since $f \in L^2(\mathbb{R})$, this implies $f = 0$ on $[0, +\infty[$ (resp. $]-\infty, 0]$).

Let us show how to obtain the scaling relation. We have

$$\begin{aligned}
\widehat{\varphi_s}(2\xi) &= \frac{3e^{-3i\xi}\sin\xi\ (2+\cos(2\xi))}{\xi^5} - \frac{9e^{-3i\xi}\sin\xi\sin(2\xi)}{2\xi^6} \\
\widehat{\varphi_a}(2\xi) &= \frac{-3ie^{-3i\xi}(\cos(3\xi)+9\cos\xi)}{2\xi^5} + \frac{ie^{-3i\xi}(11\sin(3\xi)+27\sin\xi)}{4\xi^6}
\end{aligned}$$

We define

$$m_s(\xi) = \frac{9}{2}e^{-3i\xi}\sin\xi\sin(2\xi), \qquad n_s(\xi) = -3e^{-3i\xi}\sin\xi\ (2+\cos(2\xi))$$

$$m_a(\xi) = \frac{-i}{4}e^{-3i\xi}(11\sin(3\xi)+27\sin\xi), \qquad n_a(\xi) = \frac{3i}{2}e^{-3i\xi}(\cos(3\xi)+9\cos\xi)$$

and use the resolution of the linear system (*) occuring in the proof of Theorem 3.2 to get

$$\begin{aligned}
p_s(\xi) &= \frac{-2^{-8}e^{3i\xi/2}}{3\sin^6(\frac{\xi}{2})}\left(3m_s(\xi)(\cos(\frac{3\xi}{2}) + 9\cos(\frac{\xi}{2})) + n_s(\xi)(11\sin(\frac{3\xi}{2}) + 27\sin(\frac{\xi}{2}))\right) \\
&= \frac{e^{-3i\xi/2}}{32}\cos(\frac{\xi}{2})\ (19+13\cos\xi) \\
q_s(\xi) &= \frac{2^{-7}ie^{3i\xi/2}}{\sin^6(\frac{\xi}{2})}\left(3n_s(\xi)\sin\xi\sin(\frac{\xi}{2}) + m_s(\xi)(2+\cos\xi)\sin(\frac{\xi}{2})\right).
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-9e^{-3i\xi/2}}{16}\cos^2(\frac{\xi}{2})\sin(\frac{\xi}{2}) \\
p_a(\xi) &= \frac{-2^{-8}e^{3i\xi/2}}{3\sin^6(\frac{\xi}{2})}\left(3m_a(\xi)(\cos(\frac{3\xi}{2})+9\cos(\frac{\xi}{2}))+n_a(\xi)(11\sin(\frac{3\xi}{2})+27\sin(\frac{\xi}{2}))\right) \\
&= \frac{ie^{-3i\xi/2}}{32}\sin(\frac{\xi}{2})\ (16+11\cos\xi) \\
q_a(\xi) &= \frac{2^{-7}ie^{3i\xi/2}}{\sin^6(\frac{\xi}{2})}\left(3n_a(\xi)\sin\xi\sin(\frac{\xi}{2})+m_a(\xi)(2+\cos\xi)\sin(\frac{\xi}{2})\right). \\
&= \frac{e^{-3i\xi/2}}{64}(9\cos(\frac{\xi}{2})-7\cos(\frac{3\xi}{2}))
\end{aligned}
$$

such that

$$
\begin{aligned}
\widehat{\varphi_s}(2\xi) &= p_s(\xi)\widehat{\varphi_s}(\xi)+q_s(\xi)\widehat{\varphi_a}(\xi) \\
\widehat{\varphi_a}(2\xi) &= p_a(\xi)\widehat{\varphi_s}(\xi)+q_a(\xi)\widehat{\varphi_a}(\xi)
\end{aligned}
$$

□

The scaling relation leads to the following formula[1]

Property 3.5 *We have*

$$W(2\xi) = M_0(\xi)W(\xi)M_0^*(\xi) \ + \ M_0(\xi+\pi)W(\xi+\pi)M_0^*(\xi+\pi) \qquad (R1)$$

where

$$\boxed{W(\xi) = \begin{pmatrix} \omega_s(\xi) & \omega_m(\xi) \\ \overline{\omega_m(\xi)} & \omega_a(\xi) \end{pmatrix}}$$

with

$$\omega_a(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_a}(\xi+2l\pi)|^2 = \frac{23247-21362\cos\xi-385\cos(2\xi)}{311850}$$

[1] In case V_0 is generated by one single function φ, we recall that we have

$$|m_0(\xi)|^2\omega(\xi)+|m_0(\xi+\pi)|^2\omega(\xi+\pi)=\omega(2\xi)$$

where m_0 is the filter and where

$$\omega(\xi) = \sum_{k=-\infty}^{+\infty} |\widehat{\varphi}(\xi+2k\pi)|^2$$

$$\omega_s(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\varphi_s}(\xi + 2l\pi)|^2 = \frac{14445 + 7678\cos\xi + 53\cos(2\xi)}{34650}$$

$$\omega_m(\xi) = \sum_{l=-\infty}^{+\infty} \widehat{\varphi_s}(\xi + 2l\pi)\overline{\widehat{\varphi_a}(\xi + 2l\pi)} = -\frac{i}{51975}\sin\xi\ (6910 + 193\cos\xi).$$

Proof. Define

$$\phi(\xi) = \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_a}(\xi) \end{pmatrix}.$$

Using the scaling relation, we have

$$\phi(2\xi)\ \phi^*(2\xi) = M_0(\xi)\phi(\xi)\ \phi^*(\xi)M_0^*(\xi). \qquad (**)$$

As we also have

$$\phi(\xi)\ \phi^*(\xi) = \begin{pmatrix} |\widehat{\varphi_s}(\xi)|^2 & \widehat{\varphi_s}(\xi)\ \overline{\widehat{\varphi_a}}(\xi) \\ \widehat{\varphi_a}(\xi)\ \overline{\widehat{\varphi_s}}(\xi) & |\widehat{\varphi_a}(\xi)|^2 \end{pmatrix}$$

hence

$$\sum_{l=-\infty}^{+\infty} \phi(\xi + 2l\pi)\ \phi^*(\xi + 2l\pi) = W(\xi)$$

we finally get from (**)

$$W(2\xi) = M_0(\xi)W(\xi)M_0^*(\xi) + M_0(\xi + \pi)W(\xi + \pi)M_0^*(\xi + \pi).$$

□

From the previous results, we obtain that the closed subspaces V_j ($j \in \mathbb{Z}$) form a multiresolution analysis of $L^2(\mathbb{R})$ with the difference that V_0 is generated using two functions.

A next step is then to define W_0 as the orthogonal complement of V_0 in V_1 and to construct mother wavelets in that context, that is to say functions which will genererate W_0 and which will be compactly supported deficient splines with symmetry properties.

4 Construction of wavelets

Proposition 4.1 *A function f belongs to W_0 if and only if there exists $p, q \in L^2_{loc}$, $2\pi-$ periodic such that*

$$\hat{f}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi)$$

and

$$\overline{M_0(\xi)}\ \overline{W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}\ \overline{W(\xi+\pi)} \begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \ a.e. \qquad (***)$$

where M_0 is the filter matrix obtained in Proposition 3.4 and $W(\xi)$ is the matrix defined in Property 3.5.

Proof. We have

$$\begin{aligned} f \in W_0 &\Leftrightarrow f \in V_1 \text{and } f \perp V_0 \\ &\Leftrightarrow \exists p, q \in L^2_{loc}, 2\pi - \text{per.} : \widehat{f}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi) \text{and } f \perp V_0. \end{aligned}$$

Let us develop the orthogonality condition, assuming the decomposition of f in terms of p, q. We have

$$\begin{aligned} f \perp V_0 &\Leftrightarrow \langle f, \varphi_{s,k} \rangle = 0 \text{ and } \langle f, \varphi_{a,k} \rangle = 0 \quad \forall k \in \mathbb{Z} \\ &\Leftrightarrow \int_{\mathbb{R}} d\xi\ e^{2ik\xi}(p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi))\overline{\widehat{\phi}(2\xi)} = 0 \quad \forall k \in \mathbb{Z} \end{aligned}$$

where

$$\phi(\xi) = \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_s}(\xi) \end{pmatrix}.$$

using the scaling relation $\phi(2\xi) = M_0(\xi)\phi(\xi)$ we get

$$\begin{aligned} f \perp V_0 &\Leftrightarrow \int_{\mathbb{R}} d\xi\ e^{2ik\xi}\overline{M_0(\xi)}(p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi))\overline{\widehat{\phi}(\xi)} = 0 \quad \forall k \in \mathbb{Z} \\ &\Leftrightarrow \int_0^{2\pi} d\xi\ e^{2ik\xi}\overline{M_0(\xi)} \begin{pmatrix} p(\xi)\omega_s(\xi) + q(\xi)\overline{\omega_m(\xi)} \\ p(\xi)\omega_m(\xi) + q(\xi)\omega_a(\xi) \end{pmatrix} = 0 \quad \forall k \in \mathbb{Z} \end{aligned}$$

We finally obtain

$$\begin{aligned} f \perp V_0 &\Leftrightarrow \int_0^{2\pi} d\xi\ e^{2ik\xi}\overline{M_0(\xi)W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} = 0 \quad \forall k \in \mathbb{Z} \\ &\Leftrightarrow \overline{M_0(\xi)}\ \overline{W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}\ \overline{W(\xi+\pi)} \begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \text{ a.e.} \end{aligned}$$

□

Property 4.2 *Define*

$$p(\xi) = \sum_{k=0}^{8} p_k e^{-ik\xi}, \quad q(\xi) = \sum_{k=0}^{8} q_k e^{-ik\xi}.$$

Then

$$\overline{M_0(\xi)}\;\overline{W(\xi)}\begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}\;\overline{W(\xi+\pi)}\begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0$$

if and only if

$$\begin{aligned}
p_0 &= -\frac{3889626976749167\ q_6}{5994139826128818} + \frac{131897103348532083\ q_7}{1998046608709606}\\
p_1 &= -\frac{309465997116423653\ q_6}{5994139826128818} + \frac{3147541171812450527 5\ q_7}{5994139826128818}\\
p_2 &= -\frac{2910616639302037153\ q_6}{11988279652257636} + \frac{9846020303993086815 1\ q_7}{3996093217419212}\\
p_3 &= -\frac{63116209243492295\ q_6}{11988279652257636} + \frac{27528771579833503 39\ q_7}{11988279652257636}\\
p_4 &= \frac{1001080766452619117\ q_6}{3996093217419212} - \frac{30544260607474969369 1\ q_7}{11988279652257636}\\
p_5 &= \frac{586477042773225505\ q_6}{11988279652257636} - \frac{187024916497747840 79\ q_7}{3996093217419212}\\
p_6 &= \frac{8697\ q_6}{9722} - \frac{815185\ q_7}{29166}\\
p_7 &= \frac{q_6}{29166} + \frac{7671\ q_7}{9722}\\
p_8 &= 0\\
q_0 &= \frac{817983890541088\ q_6}{999023304354803} - \frac{83213460955642643\ q_7}{999023304354803}\\
q_1 &= \frac{83213460955642643\ q_6}{999023304354803} - \frac{846408115943922303 0\ q_7}{999023304354803}\\
q_2 &= \frac{3076490626693617437\ q_6}{3996093217419212} - \frac{31258164744637865992 9\ q_7}{3996093217419212}\\
q_3 &= \frac{6206512064613183305\ q_6}{3996093217419212} - \frac{62760971622352183898 1\ q_7}{3996093217419212}\\
q_4 &= \frac{3093733577622211529\ q_6}{3996093217419212} - \frac{30714520295885794338 9\ q_7}{3996093217419212}\\
q_5 &= \frac{318992113046003613\ q_6}{3996093217419212} - \frac{2869366033222211032 1\ q_7}{3996093217419212}\\
q_8 &= 0
\end{aligned}$$

It follows that there exists deficient spline wavelets with support in $[0,5]$, *i.e. functions* ψ *such that*

$$\widehat{\psi}(2\xi) = \sum_{k=0}^{7} p_k e^{-ik\xi}\widehat{\varphi_s}(\xi) \ + \ \sum_{k=0}^{7} q_k e^{-ik\xi}\widehat{\varphi_a}(\xi)$$

or

$$\frac{1}{2}\psi(x) = \sum_{k=0}^{7} p_k\varphi_s(2x-k) \;+\; \sum_{k=0}^{7} q_k\varphi_a(2x-k).$$

Proof. The degree of the polynomials p, q are due to a look to the system that has to be solved. The resolution of the linear system is a *Mathematica* computation.□

Property 4.3 *For every q_6, q_7, the function ψ has (at least) one vanishing moment.*

Proof. We have

$$\widehat{\psi}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) \;+\; q(\xi)\widehat{\varphi_a}(\xi)$$

with

$$p(\xi) = \sum_{k=0}^{7} p_k e^{-ik\xi}, \quad q(\xi) = \sum_{k=0}^{7} q_k e^{-ik\xi}.$$

As

$$\widehat{\varphi_a}(0) = 0, \; \widehat{\varphi_s}(0) \neq 0$$

it suffices to check that $p(0) = 0$.

To obtain this property, we just use the relation (***) with $\xi = 0$ (the relation is in fact an equality everywhere since p, q are polynomials in that case. Indeed, since

$$M_0(0) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{32} \end{pmatrix}, \quad M_0(\pi) = \begin{pmatrix} 0 & 0 \\ \frac{5}{32} & 0 \end{pmatrix},$$

and

$$W(0) = \begin{pmatrix} \omega_s(0) & 0 \\ 0 & \omega_a(0) \end{pmatrix}, \quad W(\pi) = \begin{pmatrix} \omega_s(\pi) & 0 \\ 0 & \omega_a(\pi) \end{pmatrix},$$

from (***) we obtain $\omega_0(0)p(0) = 0$ hence the conclusion. □

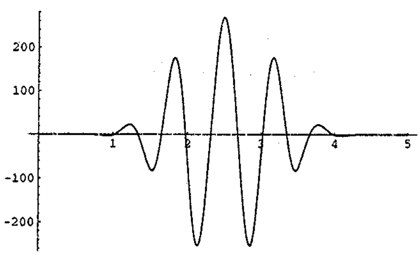
Moreover, symmetric compactly supported wavelets can be constructed: take q_6, q_7 such that $p_0 = p_7$; then $p_1 = p_6, p_2 = p_5, p_3 = p_4, q_0 = -q_7, q_1 = -q_6, q_2 = -q_5; q_3 = -q_4$ (we denote these coefficients with an "s") and we get (after some normalisation)

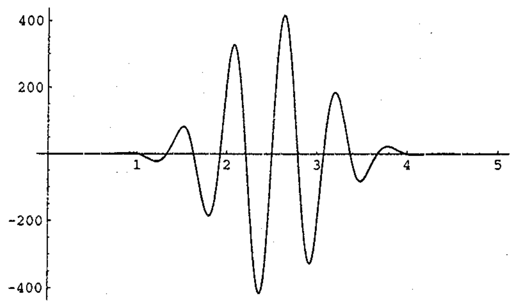
$$\begin{aligned}
&\frac{1}{2}\psi_s\left(\frac{x}{2}\right) \\
= \;& -17951959(\varphi_s(x)+\varphi_s(x-7)) - \frac{12632556065}{9}(\varphi_s(x-1)+\varphi_s(x-6)) \\
& -\frac{16090899067}{3}(\varphi_s(x-2)+\varphi_s(x-5)) + \frac{61066820897}{9}(\varphi_s(x-3)+\varphi_s(x-4)) \\
& +\frac{67958549}{3}(\varphi_a(x)-\varphi_a(x-7)) + 2276806815(\varphi_a(x-1)-\varphi_a(x-6)) \\
& +\frac{57273621163}{3}(\varphi_a(x-2)-\varphi_a(x-5)) + 21550944929(\varphi_a(x-3)-\varphi_a(x-4))
\end{aligned}$$

In the same way, antisymmetric compactly supported wavelets can be constructed: take q_6, q_7 such that $p_0 = -p_7$; then $p_1 = -p_6, p_2 = -p_5, p_3 = -p_4, q_0 = q_7, q_1 = q_6, q_2 = q_5; q_3 = q_4$ (we denote these coefficients with an "a") and we get (after some normalisation)

$$
\begin{aligned}
&\frac{1}{2}\psi_a(\frac{x}{2}) \\
&= -28619155(\varphi_s(x) - \varphi_s(x-7)) - 2316324977(\varphi_s(x-1) - \varphi_s(x-6)) \\
&\quad -\frac{25729608221}{2}(\varphi_s(x-2) - \varphi_s(x-5)) - \frac{22560506027}{2}(\varphi_s(x-3) - \varphi_s(x-4)) \\
&\quad +36109536(\varphi_a(x) + \varphi_a(x-7)) + 3717522762(\varphi_a(x-1) + \varphi_a(x-6)) \\
&\quad +\frac{74946039675}{2}(\varphi_a(x-2) + \varphi_a(x-5)) + \frac{205277609199}{2}(\varphi_a(x-3) + \varphi_a(x-4))
\end{aligned}
$$

Here are ψ_s, ψ_a (up to a multiplicative constant)





The preceeding definitions can also be written using Fourier transforms. We define

$$
p_s(\xi) = \sum_{k=0}^{7} p_k^s e^{-ik\xi}, \quad q_s(\xi) = \sum_{k=0}^{7} q_k^s e^{-ik\xi}
$$

$$
p_a(\xi) = \sum_{k=0}^{7} p_k^a e^{-ik\xi}, \quad q_a(\xi) = \sum_{k=0}^{7} q_k^a e^{-ik\xi}
$$

With

$$
M_1(\xi) = \begin{pmatrix} p_s(\xi) & q_s(\xi) \\ p_a(\xi) & q_a(\xi) \end{pmatrix}
$$

we get (from (***))

$$
M_1(\xi)W(\xi)M_0^*(\xi) + M_1(\xi+\pi)W(\xi+\pi)M_0^*(\xi+\pi) = 0 \qquad \text{(R2)}
$$

and

$$\boxed{\begin{pmatrix} \widehat{\psi_s}(2\xi) \\ \widehat{\psi_a}(2\xi) \end{pmatrix} = M_1(\xi) \begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_a}(\xi) \end{pmatrix}.}$$

Now, we want to show that the family $\{\psi_{s,k} : k \in \mathbb{Z}\} \cup \{\psi_{a,k} : k \in \mathbb{Z}\}$ is a Riesz basis for W_0. First, we give a lemma which will be of great help to get the Riesz condition. We note here that this way of proving the Riesz condition is different from the one used for the scaling functions. We could have used the same method but computations became much more complicated; that's why we tried to get the result through another way.

Lemma 4.4 *([5]) Let $f, g \in L^2(\mathbb{R})$. We define $f_k(x) = f(x-k)$, $g_k(x) = g(x-k)$, $k \in \mathbb{Z}$ and*

$$H(\xi) = \begin{pmatrix} \omega_{f,f}(\xi) & \omega_{f,g}(\xi) \\ \overline{\omega_{f,g}(\xi)} & \omega_{g,g}(\xi) \end{pmatrix}$$

where

$$\begin{aligned} \omega_{f,f}(\xi) &= \sum_{k=-\infty}^{+\infty} |\widehat{f}(\xi + 2k\pi)|^2 \\ \omega_{g,g}(\xi) &= \sum_{k=-\infty}^{+\infty} |\widehat{g}(\xi + 2k\pi)|^2 \\ \omega_{f,g}(\xi) &= \sum_{k=-\infty}^{+\infty} \widehat{f}(\xi + 2k\pi)\, \overline{\widehat{g}(\xi + 2k\pi)}. \end{aligned}$$

The following properties are equivalent:
(i) the family $\{f_k : k \in \mathbb{Z}\} \cup \{g_k : k \in \mathbb{Z}\}$ satisfies the Riesz condition
(ii) there exists $A, B > 0$ such that

$$A(\|(c_k)\|_{l^2}^2 + \|(d_k)\|_{l^2}^2) \le \int_0^{2\pi} \left\langle H(\xi) \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix}, \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} \right\rangle d\xi \le B(\|(c_k)\|_{l^2}^2 + \|(d_k)\|_{l^2}^2)$$

for every finite sequences (c_k), (d_k) and where

$$p(\xi) = \sum_{(k)} c_k e^{-ik\xi}, \; q(\xi) = \sum_{(k)} d_k e^{-ik\xi}$$

(iii) there exists $A, B > 0$ such that

$$A \le \lambda_i(\xi) \le B \qquad (i = 1, 2)$$

where $\lambda_1(\xi), \lambda_2(\xi)$ are the eigenvalues of $H(\xi)$.

Proof. We have

$$
\begin{aligned}
\| \sum_{(k)} c_k f_k \;+\; \sum_{(k)} d_k g_k \|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{2\pi} \| \sum_{(k)} c_k e^{-ik\xi} \widehat{f}(\xi) \;+\; \sum_{(k)} d_k e^{-ik\xi} \widehat{g}(\xi) \|^2_{L^2(\mathbb{R})} \\
&= \frac{1}{2\pi} \int_0^{2\pi} |p(\xi)|^2 \omega_{ff}(\xi) + |q(\xi)|^2 \omega_{aa}(\xi) + p(\xi)\overline{q(\xi)}\omega_{fg}(\xi) + \overline{p(\xi)} q(\xi) \overline{\omega_{fg}(\xi)} \, d\xi \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left\langle H(\xi) \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix}, \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} \right\rangle d\xi,
\end{aligned}
$$

which shows that (i) and (ii) are equivalent.

Now, for every ξ, the matrix $H(\xi)$ is hermitian. Therefore, for every ξ, there is a unitary matrix $U(\xi)$ such that $U^*(\xi)H(\xi)U(\xi) = \mathrm{diag}(\lambda_1(\xi), \lambda_2(\xi))$. As we have

$$
\begin{aligned}
\left\| U \begin{pmatrix} p \\ q \end{pmatrix} \right\|^2_{L^2([0,2\pi])} &= \int_0^{2\pi} \left\langle U(\xi) \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix}, U(\xi) \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} \right\rangle d\xi \\
&= \int_0^{2\pi} \left\langle \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix}, \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} \right\rangle d\xi \\
&= \left\| \begin{pmatrix} p \\ q \end{pmatrix} \right\|^2_{L^2([0,2\pi])} \\
&= \|(c_k)\|^2_{l^2} + \|(d_k)\|^2_{l^2}
\end{aligned}
$$

we obtain that (ii) is equivalent to

$$
A(\|(c_k)\|^2_{l^2} + \|(d_k)\|^2_{l^2}) \leq \int_0^{2\pi} (\lambda_1(\xi)p(\xi) + \lambda_2(\xi)q(\xi)) \, d\xi \leq B(\|(c_k)\|^2_{l^2} + \|(d_k)\|^2_{l^2})
$$

for every finite sequences $(c_k), (d_k)$. Now, it is clear that (iii) implies (ii). To get that (ii) implies (iii), it suffices for example to use the Fejer kernel as p, q (same proof as for the Riesz condition). □

Now we want to use this lemma to obtain the desired result about the wavelets. Let us give some notations: define the matrix

$$
\boxed{W_\psi(\xi) = \begin{pmatrix} \omega_{\psi_s}(\xi) & \omega_{\psi_s,\psi_a}(\xi) \\ \overline{\omega_{\psi_s,\psi_a}(\xi)} & \omega_{\psi_a}(\xi) \end{pmatrix}}
$$

where

$$\omega_{\psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\psi_a}(\xi + 2l\pi)|^2$$

$$\omega_{\psi_s}(\xi) = \sum_{l=-\infty}^{+\infty} |\widehat{\psi_s}(\xi + 2l\pi)|^2$$

$$\omega_{\psi_s,\psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} \widehat{\psi_s}(\xi + 2l\pi)\overline{\widehat{\psi_a}(\xi + 2l\pi)}.$$

Theorem 4.5 *The family* $\{\psi_{s,k} : k \in \mathbb{Z}\} \cup \{\psi_{a,k} : k \in \mathbb{Z}\}$ *constitutes a Riesz basis for* W_0. *The functions with index s (resp. a) are symmetric (resp. antisymmetric). The support of* $\psi_{s,0}$ *and* $\psi_{a,0}$ *is included in* $[0, 5]$.

It follows that the functions

$$2^{j/2}\psi_s(2^j x - k),\ 2^{j/2}\psi_a(2^j x - k) \quad (j, k \in \mathbb{Z})$$

form a Riesz basis of compactly supported deficient splines of $L^2(\mathbb{R})$ *with symmetry properties.*

Proof. Using the expression of ψ_a, ψ_s in terms of φ_a, φ_s, i.e.

$$\begin{pmatrix} \widehat{\psi_s}(2\xi) \\ \widehat{\psi_a}(2\xi) \end{pmatrix} = M_1(\xi)\begin{pmatrix} \widehat{\varphi_s}(\xi) \\ \widehat{\varphi_a}(\xi) \end{pmatrix}$$

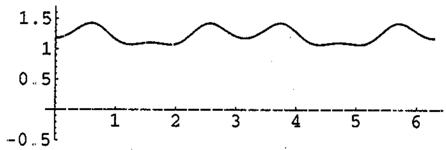
and by a computation similar to the one leading to (R1), we get

$$W_\psi(2\xi) = M_1(\xi)W(\xi)M_1{}^*(\xi) \ + \ M_1(\xi+\pi)W(\xi+\pi)M_1{}^*(\xi+\pi). \qquad \text{(R3)}$$

Then, since $W(\xi)$ is hermitian positive definite for every ξ, the matrix W_ψ has the same property if and only if the matrices $M_1(\xi)$ and $M_1(\xi+\pi)$ are not simultaneously singular. This is the case since we have (up to an exponential function and a multiplicative constant)

$$\begin{aligned}
&\det M_1(\xi)\\
&= \sin^2(\xi/2)\ (-64944404321059950\\
&\quad +1483142106949117120\cos\xi + 1192353539007974745\cos(2\xi)\\
&\quad +605163081148101400\cos(3\xi) + 249900649739435294\cos(4\xi)\\
&\quad +25542907675492680\cos(5\xi) + 250030917177111\cos(6\xi))
\end{aligned}$$

which gives the graph for $10^{-37}(\det M_1(\xi))^2 + (\det M_1(\xi+\pi))^2$



Finally, since the elements of W_ψ are trigonometric 2π-periodic polynomials, the eigenvalues are also periodic and continuous. Since they are strictly positive, condition (iii) of Lemma 4.4 follows. Hence the family of wavelets satisfies the Riesz condition.

To prove that the closure of the linear hull of the functions $\psi_{s,k}, \psi_{a,k}$ ($k \in \mathbb{Z}$) is W_0, it remains to show that

$$f \in W_0, \begin{cases} \langle f, \psi_{s,k} \rangle = 0 \\ \langle f, \psi_{a,k} \rangle = 0 \end{cases} \quad \Rightarrow \quad f = 0.$$

For $f \in W_0$, we have (see Proposition 4.1) $p, q \in L^2_{loc}$, $2\pi-$ periodic such that

$$\widehat{f}(2\xi) = p(\xi)\widehat{\varphi_s}(\xi) + q(\xi)\widehat{\varphi_a}(\xi)$$

and

$$\overline{M_0(\xi)}\;\overline{W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_0(\xi+\pi)}\;\overline{W(\xi+\pi)} \begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \text{ a.e.} \qquad (1)$$

The same computation as the one leading to the equality above in Proposition 4.1, but using orthogonality to $\psi_{s,k}, \psi_{a,k}$ instead of to $\varphi_{s,k}, \varphi_{a,k}$, leads to

$$\overline{M_1(\xi)}\;\overline{W(\xi)} \begin{pmatrix} p(\xi) \\ q(\xi) \end{pmatrix} + \overline{M_1(\xi+\pi)}\;\overline{W(\xi+\pi)} \begin{pmatrix} p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \text{ a.e.} \qquad (2)$$

Then (1) and (2) are equivalent to

$$\begin{pmatrix} \overline{M_0(\xi)}\;\overline{W(\xi)} & \overline{M_0(\xi+\pi)}\;\overline{W(\xi+\pi)} \\ \overline{M_1(\xi)}\;\overline{W(\xi)} & \overline{M_1(\xi+\pi)}\;\overline{W(\xi+\pi)} \end{pmatrix} \begin{pmatrix} p(\xi) \\ q(\xi) \\ p(\xi+\pi) \\ q(\xi+\pi) \end{pmatrix} = 0 \text{ a.e.} \qquad (3)$$

We have

$$\begin{pmatrix} \overline{M_0(\xi)}\;\overline{W(\xi)} & \overline{M_0(\xi+\pi)}\;\overline{W(\xi+\pi)} \\ \overline{M_1(\xi)}\;\overline{W(\xi)} & \overline{M_1(\xi+\pi)}\;\overline{W(\xi+\pi)} \end{pmatrix}$$
$$= \begin{pmatrix} \overline{M_0(\xi)} & \overline{M_0(\xi+\pi)} \\ \overline{M_1(\xi)} & \overline{M_1(\xi+\pi)} \end{pmatrix} \begin{pmatrix} \overline{W(\xi)} & 0 \\ 0 & \overline{W(\xi+\pi)} \end{pmatrix}.$$

Using the relations (R1), (R2), (R3), we get

$$\begin{pmatrix} \overline{M_0(\xi)} & \overline{M_0(\xi+\pi)} \\ \overline{M_1(\xi)} & \overline{M_1(\xi+\pi)} \end{pmatrix} \begin{pmatrix} W(\xi) & 0 \\ 0 & W(\xi+\pi) \end{pmatrix} \begin{pmatrix} \overline{M_0(\xi)} & \overline{M_0(\xi+\pi)} \\ \overline{M_1(\xi)} & \overline{M_1(\xi+\pi)} \end{pmatrix}^*$$

$$= \begin{pmatrix} W(2\xi) & 0 \\ 0 & W_\psi(2\xi) \end{pmatrix}.$$

For every ξ, the matrices $W(\xi), W_\psi(\xi)$ are not singular. Hence, for every ξ the matrix

$$\begin{pmatrix} \overline{M_0(\xi)} & \overline{M_0(\xi+\pi)} \\ \overline{M_1(\xi)} & \overline{M_1(\xi+\pi)} \end{pmatrix}$$

is not singular. The conclusion follows: from (3) we obtain $p(\xi) = q(\xi) = 0$ a.e.□

5 Appendix

Property 5.1 *The functions $\varphi_{a,l}|_{[0,1]}, \varphi_{s,l}|_{[0,1]}$ with $l = -2, -1, 0$ are linearly independant.*

Proof. For $x \in [0, 1]$, we have

$$\begin{aligned}
P_{a,0}(x) := \varphi_{a,0}(x) = \varphi_a(x) &= x^4 - \frac{11}{15}x^5 \\
P_{a,-1}(x) := \varphi_{a,-1}(x) = \varphi_a(x+1) &= -\frac{9}{8}(x - \frac{1}{2}) + 3(x - \frac{1}{2})^3 - \frac{38}{15}(x - \frac{1}{2})^5 \\
P_{a,-2}(x) := \varphi_{a,-2}(x) = \varphi_a(x+2) &= -(1-x)^4 + \frac{11}{15}(1-x)^5 \\
P_{s,0} := \varphi_{s,0}(x) = \varphi_s(x) &= x^4 - \frac{3}{5}x^5 \\
P_{s,-1} := \varphi_{s,-1}(x) = \varphi_s(x+1) &= \frac{57}{80} - \frac{3}{2}(x - \frac{1}{2})^2 + (x - \frac{1}{2})^4 \\
P_{s,-2} := \varphi_{s,-2}(x) = \varphi_s(x+2) &= (1-x)^4 - \frac{3}{5}(3-x)^5.
\end{aligned}$$

If r_j $(j = 1, \ldots, 6)$ are such that

$$r_1 P_{a,0} + r_2 P_{a,-1} + r_3 P_{a,-2} + r_4 P_{s,0} + r_5 P_{s,-1} + r_6 P_{s,-2} = 0$$

then the coefficients of x^j $(j = 0, \ldots, 5)$ are equal to 0. We get the system

$$\begin{cases} 3r_2 + 3r_3 + 2r_5 - 2r_6 & = 0 \\ 3r_2 - 3r_3 + r_5 + r_6 & = 0 \\ r_5 - r_6 & = 0 \\ -3r_2 + 4r_3 - 5r_5 - 5r_6 & = 0 \\ 3r_1 + 3r_2 - 6r_3 + 4r_4 + 19r_5 + 8r_6 & = 0 \\ -9r_1 + 9r_3 - 11r_4 - 38r_5 - 11r_6 & = 0 \end{cases}$$

which is easy to solve; the unique solution is

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6 = 0.$$

Property 5.2 *For every $f \in V_0$, there are $(c_k)_{k\in\mathbb{Z}}, (d_k)_{k\in\mathbb{Z}} \in l^2$ such that*

$$D^6 f = \lim_{m\to+\infty} \sum_{k=-m}^{m} (c_k \delta_k + d_k \delta_k')$$

in the distribution sense, where δ_k and δ_k' are respectively the Dirac and the derivative of the Dirac distribution at k.

Proof. Let $f \in V_0$ and, for every $k \in \mathbb{Z}$, let $f|_{[k,k+1]} = P_5^{(k)}$ = polynomial of degree at most 5. If $a_0^{(k)}, a_1^{(k)}$ are respectively the coefficients of x^4, x^5 in $P_5^{(k)}$, then

$$D^4 P_5^{(k)}(x) = 5! a_1^{(k)}\ x \ +\ 4! a_0^{(k)}$$

and, for $h \in C_\infty(\mathbb{R})$ with compact support,

$$\begin{aligned} &\int_{\mathbb{R}} f(x) D^6 h(x)\, dx \\ &= 5! \sum_{k\in\mathbb{Z}} \left(a_1^{(k)} - a_1^{(k-1)}\right) h(k) \ +\ \left(4!(a_0^{(k)} - a_0^{(k-1)}) + 5!k(a_1^{(k)} - a_1^{(k-1)})\right) Dh(k). \end{aligned}$$

For every $k \in \mathbb{Z}$, we define

$$\begin{aligned} c_k &= 5!(a_1^{(k)} - a_1^{(k-1)}), \\ d_k &= -4!(a_0^{(k)} - a_0^{(k-1)}) - 5!k(a_1^{(k)} - a_1^{(k-1)}) \\ &= -4!\left((a_0^{(k)} + 5ka_1^{(k)}) - (a_0^{(k-1)} + 5ka_1^{(k-1)})\right) \end{aligned}$$

hence to conclude, it suffices to prove that

$$\left(a_1^{(k)}\right)_{k\in\mathbb{Z}} \in l^2, \quad \left(a_0^{(k)} + 5ka_1^{(k)}\right)_{k\in\mathbb{Z}} \in l^2.$$

Do obtain this, we first remark that, on the linear space of polynomials of degree at most 5, all norms are equivalent. Hence, there are $r, R > 0$ such that

$$r\sum_{j=0}^{5}|A_j|^2 \leq \int_0^1 |P(x)|^2\,dx \leq R\sum_{j=0}^{5}|A_j|^2$$

for every polynomial $P(x) = \sum_{j=0}^{5} A_j x^j$. Next, for $f \in V_0$, using the same notations as just above, we have

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R})} &= \sum_{k=-\infty}^{+\infty}\int_k^{k+1} |P_5^{(k)}(x)|^2\,dx \\ &= \sum_{k=-\infty}^{+\infty}\int_0^1 |P_5^{(k)}(x+k)|^2\,dx.\end{aligned}$$

Moreover, in $P_5^{(k)}(x+k)$, the coefficient of x^5 is $a_1^{(k)}$ and the coefficient of x^4 is $a_0^{(k)} + 5ka_1^{(k)}$. It follows that

$$\sum_{k=-\infty}^{+\infty}(|a_1^{(k)}|^2 + |a_0^{(k)} + 5ka_1^{(k)}|^2) \leq \frac{1}{r}\sum_{k=-\infty}^{+\infty}\int_0^1 |P_5^{(k)}(x+k)|^2\,dx \leq \frac{1}{r}\|f\|^2_{L^2(\mathbb{R})}.$$

Hence the conclusion.□

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Address
Françoise BASTIN
University of Liège, Institute of Mathematics, B37
4000 LIEGE-BELGIUM
email: F.Bastin@ulg.ac.be
Fax: xx32 4 366 95 47