Bulletin de la Société Royale des Sciences de Liège, Vol. 71, 3, 2002, pp. 189-192

SIMPLISTIC PHYSICAL AND ECONOMIC ASPECTS OF CONTAMINANT RELEASE INTO COASTAL WATERS

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Abstract. The concentration of a tracer released into coastal waters is calculated from idealised equations in a simple domain. The length of the outfall pipe is found that renders minimum a cost function, which takes into account the impact of pollution and the cost of the release facilities. The optimal length is found to be independent of the cross-shore diffusivity coefficient, which is somewhat counterintuitive.

Introduction: the problem to be solved

Consider an idealised, rectilinear, infinite coastline. Let x and y denote horizontal, Cartesian coordinates (Figure 1), which are defined in such a way that y is the cross-shore coordinate — with the coast being located at y=0. The coastal waters, which are the domain of interest, are defined to be y>0. The depth of the sea is the constant H (>0). The water flows along the coast with constant velocity U (>0). Diffusive processes are neglected in the along-shore direction, but are taken into account in the cross-shore direction and are parameterised by means of a Fourier-Fick formulation involving constant diffusivity K (>0).

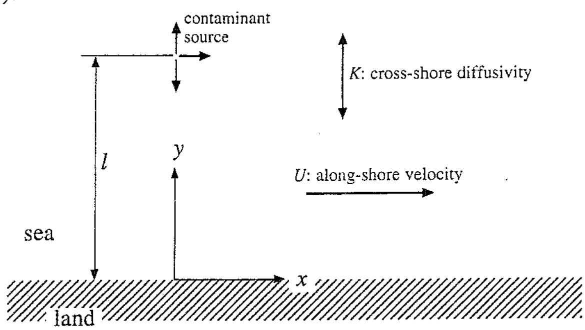

Figure 1. Idealised model setup.

Assume that a contaminant is released at the constant rate Q — expressed in kg s^{-1} — by a point-source located at $(x, y) = (0, l)$, where l (>0) represents the length of the outfall pipe. At a steady state, the depth-averaged concentration[1] $c(x,y)$ of the contaminant satisfies the equation

$$HU\frac{\partial c}{\partial x} = \frac{Q}{\rho}\delta(x-0,y-l) - \gamma Hc + HK\frac{\partial^2 c}{\partial y^2}, \tag{1}$$

where δ denotes the Dirac function; ρ is the water density, which is assumed to be a constant; γ^{-1} is a constant and positive timescale. If the contaminant is passive, γ is set to zero; otherwise, it is assumed that it decays[2] with timescale γ^{-1}. There is no contaminant in the water upstream of the source, i.e. $c(x,y) = 0$ for $x<0$. No contaminant is transported accross the coastline, so that the following boundary condition must be prescribed:

$$\left[HK\frac{\partial c}{\partial y}\right]_{y=0} = 0 . \tag{2}$$

The cost of the pollution is considered to be proportional to the maximum concentration occurring at the coast, i.e. $c_{\max} = \max c(x,0)$. Obviously, $c_{\max}$ decreases as l increases. On the other hand, the cost of the outfall pipe and related facilities is assumed to be proportional to the length of the outfall pipe. The sum of the cost of the pollution and that of the release facilities reads

$$p = ac_{\max} + bl , \tag{3}$$

where a and b are appropriate positive constants, which represent the cost of the pollution per unit concentration and the cost of the release facilities per unit length of the outfall pipe.

The problem to be solved is to *find the optimal length of the pipe, l_{opt}, minimising the total cost p.*

Solution

It is convenient to introduce normalised or dimensionless variables:

$$(x', y', l') = \frac{U}{K}(x, y, l) , \tag{4}$$

$$\gamma' = \frac{K}{U^2}\gamma , \tag{5}$$

$$b' = \frac{\rho HK^2}{aUQ}b , \tag{6}$$

$$c' = \frac{\rho HK}{Q}c , \tag{7}$$

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[1] Herein, the concentration is a dimensionless variable which is defined as the ratio of the mass of the contaminant contained in a arbitrarily small sample of seawater to the mass of the sample under consideration.

[2] Strictly speaking, the expression $-\gamma Hc$ is the parameterisation of a radioactive decay. However, this formulation may be appropriate for representing, in a schematic and idealised way, the transformation of the contaminant under study into another constituent or other constituents by phenomena characterised by timescale $1/\gamma$, be they of a radioactive nature or not.

$$p' = \frac{\rho HK}{aQ} p . \tag{8}$$

Taking into account the definitions above, the equations (1) and (3) may be transformed to

$$\frac{\partial c'}{\partial x'} = \delta(x' - 0, y' - l') - \gamma' c' + \frac{\partial^2 c'}{\partial y'^2}, \tag{9}$$

$$p' = c'_{\max} + b'l' . \tag{10}$$

After some calculations, the normalised concentration may be seen to be equal to

$$c'(x', y') = \frac{e^{-\gamma' x'}}{2\sqrt{\pi x'}} \left\{ \exp\left[-\frac{(y' - l')^2}{4x'} \right] + \exp\left[-\frac{(y' + l')^2}{4x'} \right] \right\} . \tag{11}$$

Thus, the concentration at any location is equal to the product of the concentration of a passive contaminant — for which $\gamma' = 0$ — and the factor $e^{-\gamma' x'}$, which is ≤ 1. Therefore, the concentration of a passive tracer is an upper bound to the concentration of a decaying tracer. This is why it is deemed appropriate to focus on the behaviour of this upper bound, which is equivalent to assume hereinafter that $\gamma' = 0$. The concentration field (11) is illustrated in Figure (2) for $\gamma' = 0$ and $l' = 2$.

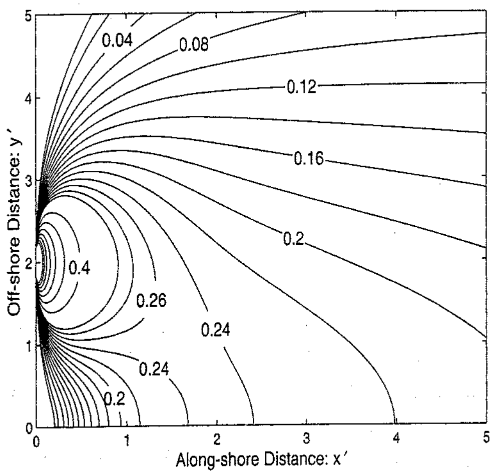

Figure 2. Isolines of the normalised concentration (11) for $\gamma' = 0$ and $l' = 2$.

The maximum of the coastal concentration is attained at

$$x'_{\max} = \frac{l'^2}{2}, \tag{12}$$

so that the maximum concentration is

$$c'_{\max} = \max_{x'>0} c'(x', 0) = c'(x'_{\max}, 0) = \left(\frac{2}{\pi e} \right)^{1/2} \frac{1}{l'}, \tag{13}$$

with $e = \exp(1) \approx 2.7183$. The optimal length of the outfall pipe is then

$$l'_{\text{opt}} = \left(\frac{2}{\pi e} \right)^{1/4} \frac{1}{\sqrt{b'}} . \tag{14}$$

Finally, substituting (13)-(14) into (10) yields the minimum of the total cost:

$$p'_{\min} = \left(\frac{32}{\pi e}\right)^{1/4} \sqrt{b'} \,. \tag{15}$$

It is instructive to re-write results (12)-(15) using dimensional variables:

$$x_{\max} = \frac{Ul^2}{2K}$$

$$c_{\max} = \left(\frac{2}{\pi e}\right)^{1/2} \frac{Q}{\rho U H l}$$

$$l_{\text{opt}} = \left(\frac{2}{\pi e}\right)^{1/4} \left(\frac{Qa}{\rho U H b}\right)^{1/2}$$

$$p_{\min} = \left(\frac{32}{\pi e}\right)^{1/4} \left(\frac{Qab}{\rho U H}\right)^{1/2}$$

The maximum coastal concentration, the optimal length of the outfall pipe and the minimum of the total cost are independent of the diffusivity. This remarkable property has yet to receive a physical explanation.

Acknowledgements. ED is a Research associate with the Belgian National Fund for Scientific Research (FNRS).

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