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ENERGY BALANCE AND PROJECTIVE EXTENSIONS OF THE POINCARÉ AND THE GALILEI GROUPS

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ABSTRACT

The relation between energy balance and projective extensions of the Poincaré and the Galilei groups is formulated in a way which leads to further remarks on 5-dimensional momentum spaces and to a possible non-singular metric for the Galilei group.
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INTRODUCTION

The Newtonian energy balance for a free particle of mass m

$$\mathrm{E} = p^2/2m \tag{1}$$

can be obtained from the relativistic expression

$$\mathrm{E}^2 = p^2c^2 + m^2c^4. \tag{2}$$

However in order to do so, it is not sufficient to consider c as a parameter and to let it go to infinity ; it is also necessary to replace the energy E by $\mathrm{E}' = \mathrm{E} - mc^2$ before the limiting process.

This well known result can receive a group theoretical interpretation. It corresponds to the fact that for a particle of mass m, the Newtonian invariance group is a nontrivial projective extension of the Galilei group [1–5] and that such an extension cannot be obtained from a projective extension of the Poincaré group unless the factor of the latter extension is different from zero. This was already pointed out by Saletan [6], [7]. In the present paper, we formulate the same result in a way which leads to further remarks on 5-dimensional momentum spaces and to a possible non-singular metric for the Galilei group different from the one proposed by Pinski [8].

1. FIVE-DIMENSIONAL MOMENTUM SPACES

A. Let us consider relation (2) and write it as follows

$$\sum_{i=1}^{3} p_i^2 - \frac{\mathrm{E}^2}{c^2} + m^2c^2 = [p_1, p_2, p_3, \mathrm{E}, 1]\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & -1/c^2 & \\ 0 & & & & m^2c^2 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \mathrm{E} \\ 1 \end{bmatrix} = 0 \tag{3}$$

This formulation can be interpreted in two ways.

First we have an *affine interpretation* [9]. Let us write

$$[p_1, p_2, p_3, E, 1] = [P_1, P_2, P_3, P_4, P_5] = \underline{P}, \tag{4}$$

and consider the 4-dimensional momentum space V_4^* as the hyperplane π

$$P_5 = 1 \tag{5}$$

of the 5-dimensional vector space V_5^* on R (real field)

$$\{[P_1, \ldots, P_5]\} = \{\underline{P}\}. \tag{6}$$

Then if we put

$$\mathscr{G}^{-1} = \text{diag}\,(1, 1, 1, -1/c^2, m^2c^2) = [g^{\mu\nu}], \tag{7}$$

relation (3) becomes

$$\begin{cases} P_\mu g^{\mu\nu} P_\nu = 0 \qquad \mu, \nu = 1, \ldots, 5 \\ P_5 = 1 \end{cases} \tag{8}$$

and tells us that the states of the particle of mass m are the points of intersection of hyperplane π with the cone of equation

$$\underline{P}\mathscr{G}^{-1}\,{}^t\underline{P} = 0 \tag{9}$$

(t is the transposition symbol).

On the other hand, we have a *projective interpretation* [9]. We map the point $P \equiv {}^t[P_1, P_2, P_3, P_4, 1]$ of π onto the line OP where O is the origin ${}^t[0, 0, 0, 0, 0]$. This mapping is a one-to-one correspondence between the points of π and the lines of V_5^* originating from O {set (O)} and not belonging to the hyperplane π_∞ which contains O and is parallel to π. Thus we may *represent* any point P of π by a line of the set (O), not belonging to π_∞. But this picture suggests that we could complete π by the elements of set (O) which are in π_∞. These elements are the elements at infinity of π, as can be verified. Now we remark that set (O) constitutes a realization of the 4-dimensional projective space $\mathscr{P}_4^*$ on R. Thus we distinguish in this way the points of the affine hyperplane π as the points at finite distance of the projective space realized by the set (O).

In this framework, the affine point P

$${}^t[p_1, p_2, p_3, p_4 = E] \longleftrightarrow {}^t[P_1, P_2, P_3, P_4, 1]$$

can be characterized by any point of the line OP i.e. by any matrix of type

$${}^t\underline{P}' = {}^t[P_1' = P_5'P_1, \ldots, P_4' = P_5'P_4, P_5' = P_5'1] \tag{10}$$

The P_1''s are the homogeneous or projective coordinates of P.

The interpretation of ${}^t\underline{P}'$ with $P_5' \neq 0$ (projective point at finite distance or homogeneous affine point) can be made by going to the inhomogeneous affine point

$$p_i = P_i'/P_5' \qquad i = 1, 2, 3, 4 \tag{11}$$

P_5' is thus seen to be a unit factor for the mass. The present point of view differs from the one often adopted in which the fifth component is taken to be the mass itself.

The equation

$$\underline{P}\mathscr{G}^{-1}\,{}^t\underline{P} = 0 \tag{12}$$

represents then a projective hyperquadric in $\mathscr{P}_4^*$ which is non-degenerate and of hyperbolic type (signature : $+++-+$). Its part at finite distance (i.e. its intersection with π) is characterized by the inhomogeneous equation (hyperquadric of π)

$$\sum p_i^2 - \frac{E^2}{c^2} + m^2c^2 = 0. \tag{13}$$

Now we consider a transformation A_{op}^{-1} belonging to the homogeneous Lorentz group and represented in the space V_4^* by the matrix ${}^tA$ such that

$$A_{op}^{-1} \to {}^tA : A = [a^\mu_\nu] \qquad \mu, \nu = 1, \ldots, 4 \tag{14}$$

with

$$P_\mu g^{\mu\nu} P_\nu = P'_\lambda g^{\lambda K} P'_K = P_\mu a^\mu_\lambda g^{\lambda K} a^\nu_K P_\nu,$$

i.e.

$$G^{-1} = AG^{-1}\,{}^tA \tag{15}$$

where

$$G^{-1} = \mathrm{diag}\,(1, 1, 1, -1/c^2) = [g^{\lambda K}] \qquad \lambda, K = 1, \ldots, 4. \tag{16}$$

If we go to the 5-component formulation, we obtain

$$A_{op}^{-1} \to {}^t\mathscr{A} : \mathscr{A} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = [a^\mu_\nu] \qquad \mu,\nu = 1, \ldots, 5, \tag{17}$$

with

$${}^t\underline{P} \xrightarrow{A_{op}^{-1}} {}^t\underline{P}' : {}^t\underline{P}' = {}^t\mathscr{A}\,{}^t\underline{P}$$

i.e.

$$\underline{P}' = \underline{P}\mathscr{A}.$$

We have

$$\mathscr{G}^{-1} = \mathscr{A}\mathscr{G}^{-1}\,{}^t\mathscr{A}. \tag{18}$$

These transformations constitute a subgroup of the group of automorphism of V_5^* associated with the regular quadratic form

$$\underline{P}\mathscr{G}^{-1}\,{}^t\underline{P}. \tag{19}$$

The transformations belonging to this subgroup are those which transform into itself the hyperplane π and which leave the point ${}^t[0, 0, 0, 0, 1]$ fixed.

B. Now, we perform in V_5^* (or in $\mathscr{P}_4^*$) the active transformation

$${}^t\underline{P}' = \mathscr{S}\,{}^t\underline{P}$$

$$\leftrightarrow \begin{bmatrix} P'_1 \\ \cdot \\ \cdot \\ P'_4 \\ P'_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -mc^2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1 \\ \cdot \\ \cdot \\ P_4 \\ P_5 \end{bmatrix}, \tag{20}$$

such that

$$P'_i = P_i \qquad i = 1, 2, 3, 5$$

$$P'_4 = P_4 - mc^2\,P_5.$$

This transformation leads from the quadratic form

$$\underline{P}\mathscr{G}^{-1}\,{}^t\underline{P}$$

to the form

$$\underline{P}\mathscr{G}'^{-1}\,{}^t\underline{P}, \tag{21}$$

where

$$\mathscr{G}'^{-1} = {}^t\mathscr{S}^{-1}\,\mathscr{G}^{-1}\,\mathscr{S}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1/c^2 & -m \\ 0 & 0 & 0 & -m & 0 \end{bmatrix}. \qquad (22)$$

As to the transformations (17) conserving the form (19), they are mapped onto the transformations ${}^t\mathscr{A}'$

$$ {}^t\mathscr{A}' = \mathscr{S}\,{}^t\mathscr{A}\,\mathscr{S}^{-1} = \left[\begin{array}{c|c} {}^t\mathrm{A} & \begin{matrix} a_1^4 mc^2 \\ a_2^4 mc^2 \\ a_3^4 mc^2 \\ a_4^4 mc^2 - mc^2 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \qquad (23)$$

which conserve the quadratic form (21).

The transformations ${}^t\mathscr{A}'$ constitute a subgroup associated with (21). They transform into itself the hyperplane π and thus the intersection of π with the affine cone

$$\underline{\mathrm{P}}'\,\mathscr{G}'^{-1}\,{}^t\underline{\mathrm{P}}' = 0. \qquad (24)$$

This section is characterized by the equation

$$\begin{cases} \sum \mathrm{P}_i'^2 - \dfrac{1}{c^2}\,\mathrm{P}_4'^2 - 2m\mathrm{P}_4'\mathrm{P}_5' = 0 \\ \mathrm{P}_5' = 1 \end{cases} \qquad (25)$$

i.e., in inhomogeneous coordinates,

$$\sum p_i^2 - \frac{\mathrm{E}'^2}{c^2} - 2m\mathrm{E}' = 0. \qquad (26)$$

The quadratic part of this hyperquadric of V_4^* is given by the matrix

$$\mathrm{diag}\,(1, 1, 1, -1/c^2) \qquad (27)$$

just as in the case of the hyperquadric section by π of the initial cone

$$\underline{\mathrm{P}}\,\mathscr{G}^{-1}\,{}^t\underline{\mathrm{P}} = 0.$$

Both hyperquadrics have the same affine nature ; (26) can be obtained from hyperquadric (13) (the center of which is ${}^t[0, 0, 0, 0, 1]$) by a translation parallel to the fourth axis.

Let us point out that if we consider the transformations (23) as the « valid » transformations in V_5^*, we must change the physical interpretation of $\mathrm{P}_4/\mathrm{P}_5$. The values on the fourth axis are relativistic energies minus mc^2.

C. We consider now c, not any more as a fixed value (in meters/second), but as a parameter and we let it tend to infinity in (22). On the one hand, the quadratic form (22) tends to the form

$$\underline{\mathrm{P}}\,\mathscr{G}''^{-1}\,{}^t\underline{\mathrm{P}} \qquad (28)$$

where

$$\mathscr{G}''^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m \\ 0 & 0 & 0 & -m & 0 \end{bmatrix}. \qquad (29)$$

This form is regular.

On the other hand, the transformations ${}^t\mathscr{A}'$ tend to transformations belonging to the group which conserves this form. They still conserve the hyperplane π and thus the section by π of the cone

$$\underline{\mathrm{P}}'' \mathscr{G}''^{-1} \, {}^t\underline{\mathrm{P}}'' = 0. \qquad (30)$$

This section has the inhomogeneous equation

$$\sum_{i=1}^{3} p_i''^2 - 2m \, p_4'' = 0. \qquad (31)$$

The quadratic part of this hyperquadric of V_4^* is characterized by the matrix

$$\mathrm{diag}\,(1, 1, 1, 0). \qquad (32)$$

This matrix is singular. The hyperquadric is of parabolic type.

We note however that the quadratic form (28) is regular and that we obtain in this way for the homogeneous Galilei group a regular metric which seems to be more natural than the one introduced by Pinski.

Now let us see how we can relate what has been said so far to projective extensions of the Poincaré and the Galilei groups.

2. COORDINATE SPACES

We consider the Abelian 5-parameter group

$$(a'^5, \ldots, a'^1)\,(a^5, \ldots, a^1) = (a'^5 + a^5, \ldots, a'^1 + a^1). \qquad (33)$$

With this group are canonically associated the vector space $(\mathrm{R}^5, \mathrm{R}) \equiv V_5$ and the affine space $(\mathfrak{T}, \mathrm{R}^5, \mathrm{R})$.

In $\mathfrak{T}$, the translation $(a^5, \ldots, a^1)$ is represented by the matrix $\bar{a}$ such that

$$\bar{x} \xleftrightarrow{(a^5, \ldots, a^1)} \bar{y} : \bar{y} = \bar{x} + \bar{a}, \qquad (34)$$

i.e.

$$\begin{bmatrix} y^1 \\ \cdot \\ \cdot \\ \cdot \\ y^5 \end{bmatrix} = \begin{bmatrix} x^1 \\ \cdot \\ \cdot \\ \cdot \\ x^5 \end{bmatrix} + \begin{bmatrix} a^1 \\ \cdot \\ \cdot \\ \cdot \\ a^5 \end{bmatrix}.$$

Let us consider the character space of this group. Its elements can be described by the points

$$[P_1, \ldots, P_5] = \underline{P}$$

of the dual space V_5^* of V_5. In the character $\underline{P}$, the translation $(a^5, \ldots, a^1)$ is represented by

$$\exp(i\, P_\mu a^\mu) = \exp(i\, \underline{P} \,.\, \bar{a}). \tag{35}$$

A. Let us consider the tranformation A_{op}^{-1} of V_5^* given by

$$^t\underline{P} \xrightarrow{A_{op}^{-1}} {}^t\underline{P}' : {}^t\underline{P}' = {}^t\mathscr{A}\, {}^t\underline{P},\ P'_\mu = P_\nu a^\nu_\mu, \tag{36}$$

where $\mathscr{A}$ is the matrix (17).

This transformation in V_5^* can be induced by the automorphism A_{op}

$$A_{op} : \bar{a}' = \mathscr{A}\bar{a} \tag{37}$$

of V_5 defined by

$$\exp(i\, P'_\mu a^\mu) = \exp(i\, P_\nu a'^\nu). \tag{38}$$

We have

$$\underline{P}' \,.\, \bar{a} = \underline{P}\mathscr{A}\bar{a} = \underline{P} \,.\, \bar{a}', \tag{39}$$

and thus

$$\bar{a}' = \mathscr{A}\bar{a}. \tag{40}$$

The group of these operators A_{op} in V_5 is isomorphic to the group of operators A_{op} in V_5^*. We give the same name to corresponding elements.

We thus obtain for the homogeneous Lorentz group the representation

$$A_{op} \to \mathscr{A} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} \tag{41}$$

by automorphisms of V_5 satisfying the orthogonality condition

$$\mathscr{G} = {}^t\mathscr{A}\, \mathscr{G}\, \mathscr{A}, \tag{42}$$

with (in the base adopted here)

$$\mathscr{G} = (g_{\mu\nu}) = \mathrm{diag}\,(1, 1, 1, -c^2, 1/m^2c^2). \tag{43}$$

We turn now to the inhomogeneous transformation $(a^5, \ldots, a^1, A_{op})$ of $\mathcal{C}$ defined by

$$\bar{y} = \mathscr{A}\bar{x} + \bar{a}. \tag{44}$$

The product $(b^5, \ldots, b^1, B_{op})\,(a^5, \ldots, a^1, A_{op})$ of two such transformations is

$$\bar{z} = \mathscr{B}\mathscr{A}\bar{x} + \mathscr{B}\bar{a} + \bar{b}. \tag{45}$$

If we use the symbol $\bar{a}^5$ for the matrix

$$\bar{a}^5 = {}^t[a^1, \ldots, a^4],$$

(45) can be written

$$\bar{z}^5 = BA\bar{x}^5 + B\bar{a}^5 + \bar{b}^5$$
$$z^5 = x^5 + a^5 + b^5$$

We thus obtain the inhomogenous group

$$(b^5, \ldots, b^1, B_{op})\,(a^5, \ldots, a^1, A_{op}) = (b^5 + a^5, \bar{b}^5 + B\bar{a}^5, B_{op}A_{op}). \tag{47}$$

It is a projective extension with zero factor of the Poincaré group

$$(\bar{b}^5, B_{op})\,(\bar{a}^5, A_{op}) = (\bar{b}^5 + B\bar{a}^5, B_{op}A_{op}) \tag{48}$$

B. We proceed in the same way with the transformations A'^{-1}_{op} given by (23). To these correspond the transformation A'_{op} of V_5 characterized by the matrices $\mathscr{A}'$ which satisfy (with supplementary conditions)

$$\mathscr{G}' = {}^t\mathscr{A}'\,\mathscr{G}'\,\mathscr{A}' \tag{49}$$

with

$$\mathscr{G}' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/m \\ 0 & 0 & 0 & -1/m & 1/m^2c^2 \end{bmatrix} = [g_{\mu\nu}] \tag{50}$$

By going to the inhomogeneous transformations

$$(a^5, \bar{a}^5, A'_{op}) \to \bar{y} = \mathscr{A}'\bar{x} + \bar{a}, \tag{51}$$

we obtain the group obeying the law $g_2g_1 = g_3$ given by

$$(b^5, \bar{b}^5, B'_{op})\,(a^5, \bar{a}^5, A'_{op}) = (b^5 + a^5 + B'^{(4)}\bar{a}^5, \bar{b}^5 + B'\bar{a}^5, B'_{op}A'_{op}), \tag{52}$$

where we denote by $B'^{(4)}$ the matrix given by the fourth row of B'.

This group is a projective extension of the Poincaré group (the group $\{A'_{op}\}$ is isomorphic to the group $\{A_{op}\}$, but now the extension factor is different from zero. However this factor

$$\chi'(g_2, g_1) = B'^{(4)}\bar{a}^5 = \sum_{i=1}^{4} b'^4_i a^i \tag{53}$$

is trivial since it can be written as

$$\chi'(g_2, g_1) = \varphi'(g_2) + \varphi'(g_1) - \varphi'(g_2 \cdot g_1) \tag{54}$$

with

$$\varphi'(g) = \varphi'(a^5, a^4, \ldots, a^1, A_{op}) = -mc^2a^4. \tag{55}$$

C. Now we point out that, when we go to the limit $c \to \infty$, this factor χ' tends to a limit which is no longer trivial and which is precisely the characteristic factor of the extended Galilei group for particles of mass m.

Suppose that A_{op} is, in V_5, the product of rotation R by the pure Lorentz transformation $\vec{v}$.

We have, by putting $\gamma = [1 - v^2/c^2]^{-1/2}$ and ${}^t\bar{x} = [\vec{x}, x^4, x^5]$,

$$\bar{x} \xrightarrow{A_{op}} \bar{y} : \vec{y} = R\vec{x} + \frac{\vec{v}}{c}\gamma\left[\frac{\gamma}{\gamma+1}\frac{\vec{v}}{v}R\vec{x} + cx^4\right]$$

$$y^4 = \gamma\frac{\vec{v}}{c^2}R\vec{x} + \gamma x^4 \tag{56}$$

$$y^5 = x^5$$

It results from (56), (23) and (53) that

$$\chi'(g_2, g_1) = \gamma m\vec{v} \cdot R\vec{a} + mc^2(\gamma - 1)a^4 \tag{57}$$

Thus, when $c \to \infty$, the factor χ' tends to the Galilean factor

$$\chi''(g_2, g_1) = m\vec{v} \cdot R\vec{a} + \frac{1}{2}mv^2a^4. \tag{58}$$

Finally we point out that the matrices $\mathscr{A}''$, representing in V_5 the elements of the homogeneous Galilei group, are such that

$$\mathscr{G}'' = {}^t\mathscr{A}''\,\mathscr{G}''\,\mathscr{A}'' \tag{59}$$

where $\mathscr{G}''$ is the regular matrix

$$\mathscr{G}'' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/m \\ 0 & 0 & 0 & -1/m & 0 \end{bmatrix} = [g''_{\mu\nu}]. \tag{60}$$

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