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On the symmetry of the age field of a passive tracer released into a one-dimensional fluid flow by a point-source

by

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Abstract. Tracer is released from a point-source into an incompressible, one-dimensional fluid flow, with constant velocity and diffusivity. The age of a tracer parcel, which is defined as the time elapsed since leaving the source, may be evaluated as the ratio of the age concentration to the tracer concentration. The latter are governed by two partial differential equations. Time-dependent analytical solutions are derived, which show that the age is symmetric with respect to the source. This is astonishing, since it could have been expected that the age would reflect somehow the strong asymmetry of the tracer concentration, which tends to be much larger on the downstream side of the source than on the upstream side. Some finite-difference counterparts of this problem are seen to lead to age fields which, in their steady-state limit, are also symmetric with respect to the source. This is believed to be helpful to interpret the results of numerical models of complex fluid flows in which the age is introduced as a diagnostic variable.

1. Introduction

To understand or interpret a fluid flow the processes taking place in it must be identified and investigated. To do so, appropriate gauges or sufficiently realistic numerical models must provide values of the state variables of the flow under consideration — i.e. velocity components, pressure, density, tracer concentration, etc. — with enough accuracy and space-time resolution so that they can be subsequently pictured by means of appropriate computer graphics software. However, most fluid flows, whether they occur in natural or

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artificial domains, are so complex that examining a number of graphical representations of state variables is not sufficient for gaining a profound insight into their functioning. This is why computer graphics must often be used in conjunction with other interpretation techniques, some of which demand that auxiliary variables be measured or computed. The age is one such auxiliary variable.

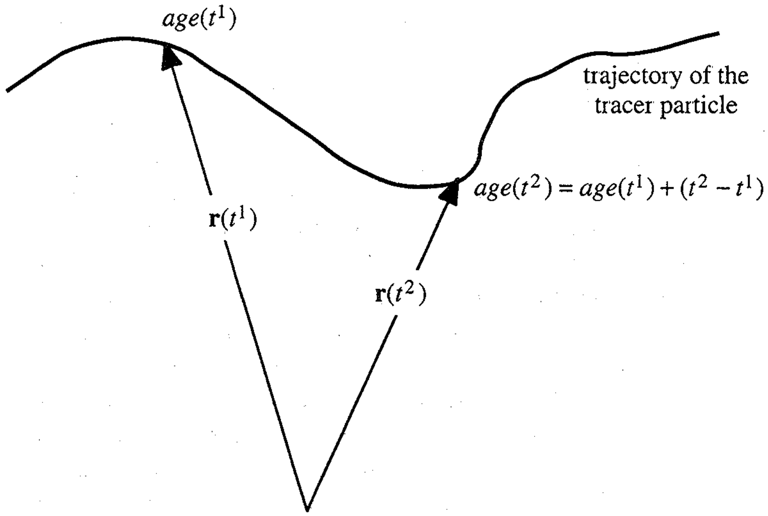

Figure 1. Illustration of the definition of the age. A tracer particle is considered. Its trajectory $\mathbf{r}(t)$ is displayed, where t and $\mathbf{r}$ denote time and the position vector of the particle under study, respectively. The difference between the age of the particle at two successive instant, t^1 and t^2, is defined as the elapsed time, i.e. $t^2 - t^1$.

According to Delhez *et al.* (1999) the age of a parcel of a tracer, i.e. a substance dissolved in a fluid mixture, is defined to be "the time elapsed since the parcel under consideration left the region in which its age is defined to be zero" (Figure 1). The nature of this region, which may be zero- to three-dimensional — i.e. a point, a curve, a surface or a volume — depends obviously on the flow considered and the purposes for which the age is introduced. The general theory outlined in Delhez *et al.* (1999) and set out in Deleersnijder *et al.* (2000) may be resorted to for estimating at any point and time in the domain of interest the age of an individual tracer, as well as that of suitably-defined groups of constituents of the fluid mixture being studied. This theory is intended for numerical models, though some of its aspects are believed to be of use in experimental studies.

The age was used frequently in the realm of oceanography, essentially for estimating the ventilation rate of ocean basins (e.g. Stuiver *et al.* 1983, Broecker *et al.* 1991, England 1995, Campin *et al.* 1999) or inferring the horizontal circulation of shelf seas (e.g. Prandle

1984, Salomon *et al.* 1995, Delhez and Deleersnijder 2000). In these applications various approaches to the concept of age were introduced; all of them may be regarded as deriving, in some cases with severe approximations, from the general theory referred to above. The latter could be resorted to for helping to understand fluid flows others than those occurring in the oceans. To the best of the authors' knowledge, this was hardly ever achieved. It is also regrettable that no effort was ever devoted to the investigation of the behaviour of the age in simple, idealised fluid flow problems. It is doubtless that the results of such studies would have been essential for improving some of the interpretations of measurements or numerical simulations that appealed to the concept of age.

A noticeable exception is the analytical study of Beckers (1999), who sought inspiration in the numerical simulations reported in Prandle (1984), Salomon *et al.* (1995), and Delhez and Deleersnijder (2000). To gain insight into the long-term circulation over the north-western European shelf seas, these authors took advantage of slowly-decaying — i.e. almost passive — tracers released into these waters by the nuclear fuel reprocessing plants of Sellafield, U.K., and Cap de la Hague, France. The concentration of the relevant tracers in the sea was modelled, compared with in situ data, and subsequently used in a diagnostic mode for the purpose of estimating their age as the time elapsed since leaving the vicinity of the outfall pipes of the reprocessing plants. Beckers (1999) tackled a problem ensuing from a drastic idealisation of the three tracer-based studies just mentioned, i.e. the determination of the steady-state concentration and age of a passive tracer released by a constant point-source into a fluid flow with constant velocity and diffusivity. It was shown that the age field is symmetric with respect to the source, a very surprising result since it could have been expected that the age would reflect somehow the strong asymmetry of the tracer concentration, which tends to be much larger on the downstream side of the source than on the upstream side.

The present article may be viewed as a follow-up to the preliminary results of Beckers (1999). A passive tracer is released into a one-dimensional fluid flow by a point-source. The velocity and the diffusivity are constant, whereas the rate of release of the source may vary in time. Analytical, time-dependent expressions of the concentration of the tracer, its age concentration and its age are sought to determine whether or not the age is symmetric with respect to the source. Nonetheless, before actually tackling this particular problem, it is necessary that the key aspects of the general theory of the age be recalled.

2. Summary of the general theory of the age

A fluid mixture is made up of I constituents that can be identified by the index i ($1 \leq i \leq I$). Let x, y and z denote Cartesian co-ordinates such that the position vector of any point in the domain of interest reads $\mathbf{x}=(x,y,z)$. According to Delhez *et al.* (1999) the concentration distribution function of the i-th constituent, $c_i(t,\mathbf{x},\tau)$, is defined as follows: at time t, the mass of the i-th constituent contained in the volume

$$(x-\Delta x/2,\, y-\Delta y/2,\, z-\Delta z/2) \;\leq\; (x',\, y',\, z') \;\leq\; (x+\Delta x/2,\, y+\Delta y/2,\, z+\Delta z/2) \qquad (2.1)$$

with an age lying in the interval

$$\tau-\Delta\tau/2 \;\leq\; \tau' \;\leq\; \tau+\Delta\tau/2 \qquad (2.2)$$

tends to $\rho c_i(t,\mathbf{x},\tau)\Delta x\,\Delta y\,\Delta z\,\Delta\tau$, as Δx, Δy, Δz, and $\Delta\tau$ tend to zero, where ρ is the density of the fluid. For the sake of simplicity, it is assumed that the variations of the latter are negligible, i.e. the Boussinesq approximation may be made. Further hypothesising that the age is positive definite, the concentration at time t and location $\mathbf{x}$ of the i-th constituent is thus

$$C_i(t,\mathbf{x}) \;=\; \int_0^\infty c_i(t,\mathbf{x},\tau)\; d\tau. \qquad (2.3)$$

Therefore, at the same time and location, the mean age of the i-th constituent is given by

$$a_i(t,\mathbf{x}) \;=\; \frac{1}{C_i(t,\mathbf{x})}\int_0^\infty \tau c_i(t,\mathbf{x},\tau)\; d\tau. \qquad (2.4)$$

As will be seen, it is convenient to introduce an additional variable, namely the age concentration, which is defined to be

$$\alpha_i(t,\mathbf{x}) \;=\; C_i(t,\mathbf{x})a_i(t,\mathbf{x}). \qquad (2.5)$$

From the mass budget of every constituent, Delhez *et al.* (1999) showed that the concentration distribution function obeys the following partial differential equation:

$$\frac{\partial c_i}{\partial t} + \mathbf{u}\bullet\nabla c_i \;=\; p_i \;-\; d_i \;+\; \nabla\bullet(\mathbf{K}\bullet\nabla c_i) \;-\; \frac{\partial c_i}{\partial \tau}, \qquad (2.6)$$

where ∇ and $\nabla\bullet$ represent the gradient and divergence operators, respectively; p_i (≥ 0) and d_i (≥ 0) denote the rate of production and destruction of the i-th constituent, which may be due to radioactive decay, chemical reactions, etc.; $\mathbf{u}$ represents the fluid velocity resolved in the model considered, while $\mathbf{K}$ is the diffusivity tensor needed to parameterise in a Fourier-Fick manner the unresolved transport of the constituent under study, which is due to turbulent fluctuations and molecular-scale processes.

If relevant initial and boundary conditions are available, (2.6) may be solved so as to eventually obtain, from (2.3) and (2.4), the concentration and the mean age of any constituent of the fluid mixture under study. In most applications, carrying out this task is unlikely to be easy, since the concentration distribution function depends on 5 independent variables, i.e. t, x, y, z, and τ. If no information is required about the distribution of the mass in the age direction, it is not necessary to solve the equation governing the age distribution. It is probably more straightforward to estimate the concentration and the mean age from the equations they obey, which may be derived from (2.6), essentially by integration over τ. The associated mathematical manipulations are set out in Delhez *et al.* (1999) without any simplifying hypotheses. In this study it is sufficient to outline the method for obtaining the age and age concentration equations for a passive tracer within the assumptions underlying the simple, idealised, one-dimensional fluid flow problem to be dealt with.

3. The one-dimensional problem

Only one tracer is considered: the subscript "i" will be omitted from here on so as to simplify the notations. The fluid flow is one-dimensional, so that one space co-ordinate and one velocity component, namely x and u, need to be retained. The diffusivity tensor reduces to the diffusivity coefficient k. Both u and k are assumed to be positive constants. The tracer is passive, implying that its production and destruction rates are zero. Under all these hypotheses, the fundamental equation (2.6) transforms to

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} = k\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial \tau}. \quad (3.1)$$

The age of the tracer parcels is assumed to be zero at the very moment they are released by the source into the fluid flow. Hence, their age represents the time elapsed since leaving the source. Accordingly, if $q(t)$ represents the rate of tracer release of the point-source, which is located at x=0, and if δ denotes the Dirac function, the concentration distribution function must satisfy the boundary condition

$$c(t,x,\tau=0) = \rho^{-1}q(t)\delta(x-0). \quad (3.2)$$

As was argued in Delhez *et al.* (1999), c must also be such that

$$\lim_{\tau\to\infty}[\tau c(t,x,\tau)] = 0. \quad (3.3)$$

Therefore, integrating (3.1) over τ and taking (3.2)-(3.3) into account leads to the equation governing the tracer concentration:

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \rho^{-1}q\delta(x-0) + k\frac{\partial^2 C}{\partial x^2}. \quad (3.4)$$

Then, multiplying (3.1) by τ, integrating over τ, and using the boundary conditions above, it is readily seen that the age concentration obeys

$$\frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} = C + k\frac{\partial^2 \alpha}{\partial x^2}. \quad (3.5)$$

Given that the mean age, a, is the ratio of the age concentration to the concentration, (3.4) and (3.5) may be combined to give the equation governing the evolution of the mean age, i.e.

$$\frac{\partial a}{\partial t} + w\frac{\partial a}{\partial x} = 1 - \rho^{-1}q\delta(x-0)\frac{a}{C} + k\frac{\partial^2 a}{\partial x^2}, \quad (3.6)$$

where w may be interpreted as an "equivalent velocity" defined to be

$$w = u - \frac{2k}{C}\frac{\partial C}{\partial x}. \quad (3.7)$$

The space-time domain of interest is defined as

$$0 \le t < \infty \quad \text{and} \quad -\infty < x < \infty. \tag{3.8}$$

At the initial instant, C, α, and a are prescribed to be zero:

$$[C(t=0,x),\ \alpha(t=0,x),\ a(t=0,x)] = (0,\ 0,\ 0). \tag{3.9}$$

Finally, none of the unknowns is allowed to grow exponentially as $|x| \to \infty$.

The tracer present in the vicinity of the source is a mixture of tracer that has just left the source, of which the age tends to zero, and tracer that has travelled the domain of interest as a result of advective and diffusive transport since it was released by the source, implying that its age is greater than zero. Therefore, the mean age at x=0, being the average of the age of all tracer parcels present at this point, is likely to be larger than zero. For diagnostic purposes it may be equally relevant to prescribe that the age be zero at x=0. In this case, the age represents the time elapsed since leaving the point x=0, rather than the time elapsed since leaving the tracer source. Obviously, this has no impact on the tracer concentration. In addition, the equation governing the evolution of the age concentration is unchanged. However, the domain of interest in which this equation must be solved is modified, since, in this alternative approach to the determination of the age, the point x=0 belongs to the boundary of the domain. At this point the following boundary conditions must be satisfied:

$$\alpha^*(t,x=0) = 0 = a^*(t,x=0), \tag{3.10}$$

where asterisked variables are those associated with the alternative age definition.

4. Exact solution to the problem

To simplify the notations and render the solutions more general, it is convenient to introduce the following dimensionless independent variables:

$$\tilde{t} = \frac{t}{4k/u^2} \quad \text{and} \quad \tilde{x} = \frac{x}{4k/u}. \tag{4.1}$$

The equivalent velocity w and the rate of release of the source may be scaled as

$$\tilde{w} = \frac{w}{u} \quad \text{and} \quad \tilde{q} = \frac{q}{Q}, \tag{4.2}$$

where Q represents the typical order of magnitude of q. Finally, the dimensionless counterparts of the unknowns of the problem,

$$\tilde{C} = \frac{C}{Q/(\rho u)} \quad \text{and} \quad \tilde{\alpha} = \frac{\alpha}{4kQ/(\rho u^3)} \quad \text{and} \quad \tilde{a} = \frac{a}{4k/u^2}, \tag{4.3}$$

are defined in such a way that

$$\tilde{\alpha}(\tilde{t},\tilde{x}) = \tilde{C}(\tilde{t},\tilde{x})\tilde{a}(\tilde{t},\tilde{x}). \tag{4.4}$$

Substituting the dimensionless variables into the partial differential problem and dropping the tildes yields the dimensionless equations to be solved

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial x} = q\delta(x-0) + \frac{1}{4}\frac{\partial^2 C}{\partial x^2}, \tag{4.5}$$

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha}{\partial x} = C + \frac{1}{4}\frac{\partial^2 \alpha}{\partial x^2}, \tag{4.6}$$

$$\frac{\partial a}{\partial t} + w\frac{\partial a}{\partial x} = 1 - q\delta(x-0)\frac{a}{C} + \frac{1}{4}\frac{\partial^2 a}{\partial x^2}, \tag{4.7}$$

with

$$w = 1 - \frac{1}{2C}\frac{\partial C}{\partial x}. \tag{4.8}$$

Similar transformations apply to α^* and a^*, as well as the equations they obey. This is rather straightforward, so that no details need to be given. From here on, unless otherwise stated, only dimensionless quantities will be dealt with, but, for the sake of simplicity, their usual appellations — i.e. time, velocity, age, etc. — will still be used.

Being non-linear, the age equation (4.7) is likely to be uneasy to solve. Fortunately, it is possible to obtain the age without having recourse to (4.7): it is sufficient to determine the tracer concentration and the age concentration from coupled linear equations (4.5) and (4.6), and, then, derive the age according to (4.4) as the ratio of the age concentration to the tracer concentration. This does not imply, however, that the equation obeyed by the age should always be disregarded: this relation may turn out to be helpful to investigate certain properties of the age, as is seen below and in Deleersnijder *et al.* (2000).

For solving the partial differential problem above, it is convenient to resort to the temporal Laplace transform. Applying the latter to the tracer concentration and the age concentration equations, and taking into account the initial conditions (3.9) leads to the following differential equations

$$s\mathcal{L}(C) + \frac{\partial \mathcal{L}(C)}{\partial x} = \mathcal{L}(q)\delta(x-0) + \frac{1}{4}\frac{\partial^2 \mathcal{L}(C)}{\partial x^2}, \tag{4.9}$$

$$s\mathcal{L}(\alpha) + \frac{\partial \mathcal{L}(\alpha)}{\partial x} = \mathcal{L}(C) + \frac{1}{4}\frac{\partial^2 \mathcal{L}(\alpha)}{\partial x^2}, \tag{4.10}$$

where the operator $\mathcal{L}$ represents the Laplace transform, which is such that

$$\mathcal{L}(C,\alpha,q) = \int_0^\infty (C,\alpha,q)e^{-st}\, dt. \tag{4.11}$$

Obviously, the Laplace transform of the alternative age concentration obeys an equation similar to (4.10).

Applying the boundary conditions set out above, it is readily seen that

$$\mathcal{L}(C,\alpha,\alpha^*) = \left[\frac{1}{(1+s)^{1/2}}, \frac{1}{2(1+s)^{3/2}} + \frac{|x|}{1+s}, \frac{|x|}{1+s}\right] e^{2x-2|x|(1+s)^{1/2}}. \tag{4.12}$$

By using the Laplace transform properties listed in Abramowitz and Stegun (1964) and the table of Laplace transform originals compiled in this treatise, the originals of transforms (4.12) are obtained after a few manipulations:

$$C(t,x) = \pi^{-1/2}\, e^{2x} \int_0^t q(t')\,(t-t')^{-1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt', \tag{4.13}$$

$$\alpha(t,x) = \pi^{-1/2}\, e^{2x} \int_0^t q(t')\,(t-t')^{1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt', \tag{4.14}$$

$$\alpha^*(t,x) = |x|\, e^{2x} \int_0^t q(t')\, \mathrm{erfc}\frac{|x|}{(t-t')^{1/2}}\, e^{-(t-t')}\, dt', \tag{4.15}$$

where "erfc" is the complementary error function,

$$\mathrm{erfc}(\upsilon) = 2\pi^{-1/2} \int_\upsilon^\infty e^{-\theta^2}\, d\theta. \tag{4.16}$$

5. Symmetry of the age

From (4.13), it is readily demonstrated that the tracer concentration tends to be much larger on the downstream side ($x>0$) of the source:

$$C(t,x) = e^{4x} C(t,-x). \tag{5.1}$$

This is in agreement with physical intuition. The age, however, which is given by

$$a(t,x) = \frac{\alpha(t,x)}{C(t,x)} = \frac{\int_0^t q(t')\,(t-t')^{1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt'}{\int_0^t q(t')\,(t-t')^{-1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt'}, \tag{5.2}$$

$$a^*(t,x) = \frac{\alpha^*(t,x)}{C(t,x)} = \frac{|x| \int_0^t q(t')\, \mathrm{erfc}\frac{|x|}{(t-t')^{1/2}}\, e^{-(t-t')}\, dt'}{\int_0^t q(t')\,(t-t')^{-1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt'}, \tag{5.3}$$

is symmetric with respect to the point at which the source is located, i.e.

$$a(t,x) = a(t,-x), \tag{5.4}$$

$$a^*(t,x) = a^*(t,-x). \tag{5.5}$$

This property is counterintuitive, since the age field could have been expected to reflect somehow the strong asymmetry of the concentration field.

Table 1 The tracer concentration, age concentration, and the age obtained with a unit rate of release of the source ($q(t)=1$). The incomplete gamma function, γ, is defined in relation (5.10). Integral (5.11) is crucial for deriving the limits $t \to \infty$ of the functions exhibited here.

	$0 \le t < \infty$, $-\infty < x < \infty$	$t \to \infty$, $-\infty < x < \infty$	$0 \le t < \infty$, $x = 0$
$C(t,x)$:	$\pi^{-1/2} e^{2x} \int_0^t \theta^{-1/2} e^{-\frac{x^2}{\theta}-\theta} d\theta$	$e^{2(x-\lvert x \rvert)}$	$\pi^{-1/2}\gamma(1/2,t)$
$\alpha(t,x)$:	$\pi^{-1/2} e^{2x} \int_0^t \theta^{1/2} e^{-\frac{x^2}{\theta}-\theta} d\theta$	$(1/2+\lvert x \rvert) e^{2(x-\lvert x \rvert)}$	$\pi^{-1/2}\gamma(3/2,t)$
$a(t,x)$:	$\dfrac{\int_0^t \theta^{1/2} e^{-\frac{x^2}{\theta}-\theta} d\theta}{\int_0^t \theta^{-1/2} e^{-\frac{x^2}{\theta}-\theta} d\theta}$	$1/2+\lvert x \rvert$	$\dfrac{\gamma(3/2,t)}{\gamma(1/2,t)}$
$\alpha^*(t,x)$:	$\lvert x \rvert e^{2x} \int_0^t e^{-\theta} \operatorname{erfc}\frac{\lvert x \rvert}{\theta^{1/2}} d\theta$	$\lvert x \rvert e^{2(x-\lvert x \rvert)}$	0
$a^*(t,x)$:	$\pi^{1/2} \lvert x \rvert \dfrac{\int_0^t e^{-\theta} \operatorname{erfc}\frac{\lvert x \rvert}{\theta^{1/2}} d\theta}{\int_0^t \theta^{-1/2} e^{-\frac{x^2}{\theta}-\theta} d\theta}$	$\lvert x \rvert$	0

It must be pointed out that the latter property can also be deduced by analysing the equation governing the age, (4.7), provided the tracer concentration field is known. Substituting (4.13) into (4.8) provides the following expression of the equivalent velocity:

$$w(t,x) = x \frac{\int_0^t q(t')\,(t-t')^{-3/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt'}{\int_0^t q(t')\,(t-t')^{-1/2}\, e^{-\frac{x^2}{t-t'}-(t-t')}\, dt'}. \tag{5.6}$$

Thus, w is an odd function in x, i.e. $w(t,x) = -w(t,-x)$, which implies that the equation (4.7) may be transformed to

$$\frac{\partial a}{\partial t} + w(t,\lvert x \rvert)\frac{\partial a}{\partial \lvert x \rvert} = 1 - q\delta(x-0)\frac{a}{C} + \frac{1}{4}\frac{\partial^2 a}{\partial \lvert x \rvert^2}. \tag{5.7}$$

As a consequence, owing to the initial condition $a(t=0,x)=0$, the age must obey the symmetry property (5.4). Obviously, this discussion applies to the alternative age a^* too.

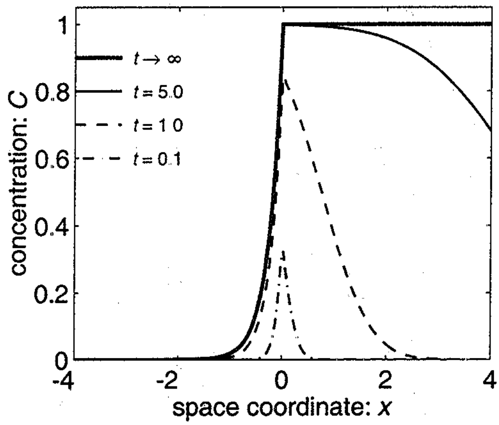

Figure 2. The tracer concentration field at different instants, obtained with a unit rate of release of the source ($q(t)$=1).

According to Abramowitz and Stegun (1964), the complementary error function admits the following asymptotic expansion

$$\mathrm{erfc}(\upsilon) \sim \pi^{-1/2}\upsilon^{-1}e^{-\upsilon^2}\left(1-\frac{1}{2\upsilon^2}+\frac{3}{4\upsilon^4}\right), \quad \upsilon \to \infty, \tag{5.8}$$

Taking advantage of the latter, it may be seen that the difference between the two ages considered herein decreases as the distance to the source increases:

$$a(t,x) - a^*(t,x) = \frac{1}{2|x|^2}\frac{\int_0^t q(t')\,(t-t')^{3/2}\,e^{-\frac{x^2}{t-t'}-(t-t')}\left[1+\mathrm{O}\left(\frac{t-t'}{|x|^2}\right)\right]dt'}{\int_0^t q(t')\,(t-t')^{-1/2}\,e^{-\frac{x^2}{t-t'}-(t-t')}\,dt'} \tag{5.9}$$

To illustrate the properties of the age established above, it is appropriate to examine the particular case where the rate of release $q(t)$ of the source is constant. With no loss of generality, it may be assumed that $q(t)$=1 at any time $t\geq 0$. The ensuing tracer concentration, age concentration, and age are collected in Table 1, along with their values in the limit $t\to\infty$ and at x=0. These solutions are also displayed in Figures 2 and 3. Figure 4 shows that the relative difference between the two ages introduced here, a and a^*, decreases as the distance to the point-source increases. In other words, it is only in the vicinity of the source that the treatment of the age at the source point has a significant impact.

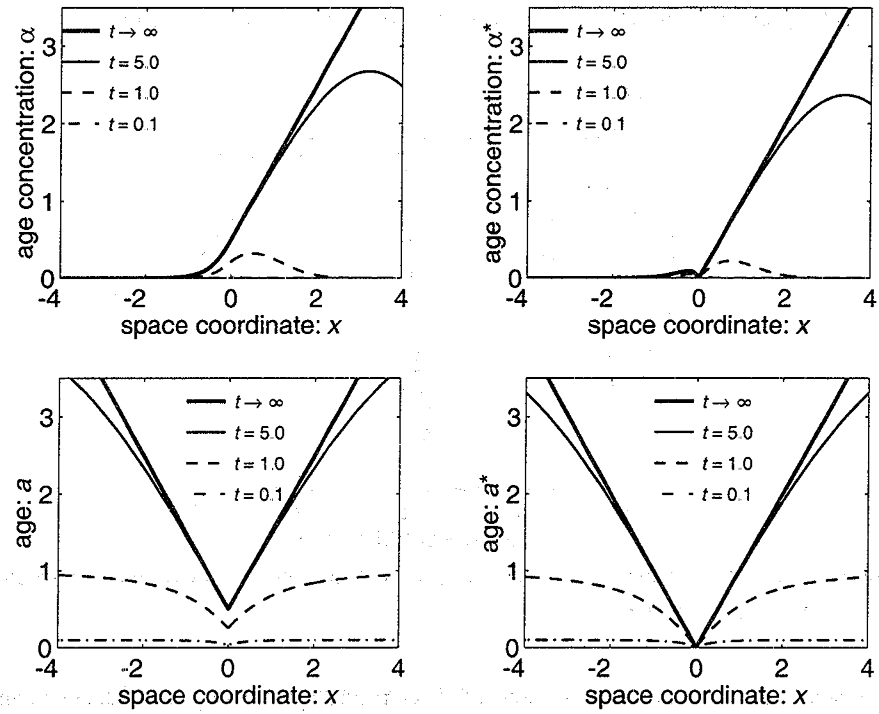

Figure 3. The age concentration and age fields at different instants, obtained with a unit rate of release of the source ($q(t)$=1). The alternative age concentration and age are displayed in the panels in the right-hand column.

In Table 1 use is made of the incomplete gamma function, which is defined as

$$\gamma(\varsigma, \upsilon) = \int_0^{\upsilon} \theta^{\varsigma-1} e^{-\theta} \, d\theta. \tag{5.10}$$

On the other hand, to obtain to obtain the limit $t \to \infty$ of the relevant variables, it is necessary to have recourse to the following integral (see Gradshteyn and Ryzhik, 1965):

$$\int_0^{\infty} \theta^{\nu-1} e^{-\frac{\beta}{\theta}-\gamma\theta} \, d\theta = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu(2\sqrt{\beta\gamma}), \tag{5.11}$$

where K_ν denotes a modified Bessel function.

For the constant source case, the age in the limit $t \to \infty$ may be expressed in dimensional variables as follows:

$$a(t \to \infty, x) = \frac{2k}{u^2} + \frac{|x|}{u} \quad \text{(dimensional variables).} \tag{5.12}$$

So, though the age at x=0 depends on the diffusivity k, the rate at which the age grows as the distance to the source increases is independent of the diffusivity. This seems rather natural on the downstream side of the source, since, in this region, the tracer concentration in the limit $t \to \infty$ is determined by advection only:

$$C(t \to \infty, x > 0) = \frac{Q}{\rho u} \quad \text{(dimensional variables).} \tag{5.13}$$

On the upstream side of the source, however, the concentration,

$$C(t \to \infty, x < 0) = \frac{Q}{\rho u} e^{-\frac{u|x|}{k}} \quad \text{(dimensional variables),} \tag{5.14}$$

clearly depends on diffusive processes, which is why it is surprising that the growth rate of the age is independent of the diffusivity in this region too. The expression of the alternative age,

$$a^*(t \to \infty, x) = \frac{|x|}{u} \quad \text{(dimensional variables),} \tag{5.15}$$

is even more surprising, since it does not involve the diffusion coefficient at all. The physical explanation of these rather strange behaviours is yet to be found.

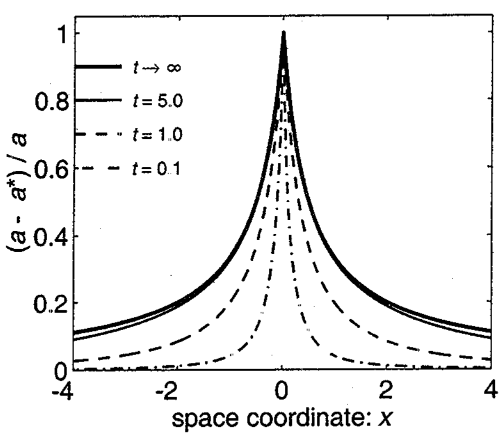

Figure 4. The relative difference between the two types of age, $(a - a^*)/a$, at different instants, obtained with a unit rate of release of the source ($q(t)$=1).

6. Discrete solution

The age, as devised in Delhez *et al.* (1999) is mostly intended for numerical models, i.e. systems for providing discrete approximate values of the solutions of continuous equations. This is why it is desirable that it be examined whether or not discrete approximations of the age are symmetric with respect to the point-source. Obviously, the answer depends on the numerical method resorted to. In other words, no universally-valid analysis of the symmetry of discrete solutions can be made. Nevertheless, it is appropriate to solve at least one discrete problem, in order to see if there is any potential for discrete approximations of the age to be symmetric with respect to the source.

A steady-state situation with $q(t)=1$ is considered. A discrete approximation to equations (4.5) and (4.6) must be worked out and solved under the hypothesis that the solutions are time-independent. The finite-difference method is selected for obtaining the discrete values of the discrete tracer concentration $\overline{C}_j$ and age concentration $\overline{\alpha}_j$ at points $x_j=j\Delta x$ ($j=0,\pm1,\pm2,\ldots$), where Δx is the space increment and the overbar is used to distinguish the discrete variables from their continuous counterparts. The diffusive terms, i.e. the second derivatives appearing in (4.5) and (4.6), are discretised by means of the classical, three-point, second-order accurate technique. The first order derivatives representing advective transport are approximated by a weighted average of the first-order, upwind scheme and the second-order, centred scheme, which is a highly simplified version of the sophisticated, non-linear advection schemes based on the concept of flux limiter. Accordingly, if μ ($0\leq\mu\leq1$) represents the upwinding rate, the steady-state finite difference analogues of continuous equations (4.5) and (4.6) read

$$\mu\frac{\overline{C}_j-\overline{C}_{j-1}}{\Delta x} + (1-\mu)\frac{\overline{C}_{j+1}-\overline{C}_{j-1}}{2\Delta x} = \overline{q}_j + \frac{1}{4}\frac{\overline{C}_{j+1}+\overline{C}_{j-1}-2\overline{C}_j}{\Delta x^2} \tag{6.1}$$

and

$$\mu\frac{\overline{\alpha}_j-\overline{\alpha}_{j-1}}{\Delta x} + (1-\mu)\frac{\overline{\alpha}_{j+1}-\overline{\alpha}_{j-1}}{2\Delta x} = \overline{C}_j + \frac{1}{4}\frac{\overline{\alpha}_{j+1}+\overline{\alpha}_{j-1}-2\overline{\alpha}_j}{\Delta x^2}. \tag{6.2}$$

where

$$\overline{q}_0 = \frac{1}{\Delta x} \quad \text{and} \quad \overline{q}_j = 0 \quad \text{for } j=\pm1,\pm2,\ldots . \tag{6.3}$$

Needless to say that the alternative age concentration obeys a relation similar to (6.2) and that the discrete ages can be evaluated as

$$(\overline{a}_j,\ \overline{a}_j^*) = \frac{(\overline{\alpha}_j,\ \overline{\alpha}_j^*)}{\overline{C}_j}. \tag{6.4}$$

Let

$$\kappa = \frac{1}{4} + \frac{\mu\Delta x}{2} \tag{6.5}$$

be viewed as an equivalent — dimensionless — diffusivity, consisting of the sum of the actual diffusivity, 1/4, and the artificial diffusivity due to upwinding. If the discrete variables are determined in such a way that their limits $|x|\to\infty$ is equal to that of their continuous counterparts, the solution of the problem (6.1)-(6.3) is

$$(\overline{C}_j,\ \overline{\alpha}_j,\ \overline{\alpha}_j^*) = \left[1,\ 2\kappa + |j|\Delta x,\ |j|\Delta x\right] r^{\frac{j-|j|}{2}}, \tag{6.6}$$

where

$$r = \frac{2\kappa + \Delta x}{2\kappa - \Delta x}. \tag{6.7}$$

For the concentration to remain positive, as is highly desirable, the upwinding rate μ must be sufficiently large that r is positive, i.e.

$$\mu > 1 - \frac{1}{2\Delta x}. \tag{6.8}$$

Nonetheless, whether or not the constraint above is met, the age,

$$(\overline{a}_j,\ \overline{a}_j^*) = \left[2\kappa + |j|\Delta x,\ |j|\Delta x\right], \tag{6.9}$$

is symmetric with respect to the source:

$$(\overline{a}_j,\ \overline{a}_j^*) = (\overline{a}_{-j},\ \overline{a}_{-j}^*). \tag{6.10}$$

In addition, it is readily seen that, as expected, the discrete age converges towards its continuous counterpart as the space increment Δx decreases:

$$\lim_{\substack{\Delta x \to 0 \\ j\Delta x = \text{const.}}} \overline{a}_j = a(t \to \infty, x_j). \tag{6.11}$$

It is somewhat surprising, however, that, whatever the value of Δx, the discrete alternative age is equal to its continuous counterpart:

$$\overline{a}_j^* = a^*(t \to \infty, x_j). \tag{6.12}$$

Finally, it must be realised that the age cannot be symmetric with respect to the source if the upwinding rate used in (6.1) is not equal to that employed in (6.2).

7. Conclusion

The age of a tracer released by a point-source, i.e., roughly speaking, the time elapsed since leaving the source, may be estimated in two ways, according to whether the age is prescribed to be zero or not at the location of the source. A one-dimensional fluid flow with constant velocity and diffusivity was considered. It was shown that the difference between these two age fields decreases as the distance to the source increases and that the age, whatever the approach selected, is symmetric with respect to the point-source. These results were obtained from the analytical, time-dependent solutions of the partial differential problem from which the age may be derived — according to the general theory of the age developed in Delhez *et al* (1999) and Deleersnijder and Delhez (2000). Finally, simple, steady-state, finite difference analogues of this problem were also shown to exhibit the symmetry property, implying that a numerical model, provided appropriate discretisation schemes are resorted to, can reproduce the symmetry of the age field.

The aforementioned analytical results, though obtained in a highly idealised fluid flow problem, are believed to be helpful, in a heuristic manner, to assess numerical simulations of complex fluid flows in which the age is introduced for diagnostic purposes. It is often tempting to ascribe the supposedly excessive smoothness or symmetry of a scalar field to the numerical diffusion being too large or the grid being too coarse. As far as the age is concerned, this temptation must be resisted, at least in circumstances that bear some similarity with the problem dealt with above. In other words, that a numerical model produces an age field rather symmetric with respect to a point-source should not be interpreted *a priori* as a numerical artefact.

In another study (Beckers *et al.* 2000) the properties of the age were investigated in multi-dimensional fluid flows, with a particular emphasis on the symmetry in the vicinity of a point-source. The multi-dimensional nature of the problem to be dealt with prevented the authors from generalising some key results obained herein. In particular, it was not possible to examine the alternative strategy in which the age is precribed to be zero at the point-source and no difference equation could be solved.

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