# Low Levels of Visibility

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**Abstract**

Points with different types of visibility restrictions appear frequently in the literature on Starshapedness and Visibility Theory. In this note three distinct levels of low visibility are defined, compared and characterized.

**Key words:** visibility, starshaped set, visibility restrictions.

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## 1 Introductory definitions.

Unless otherwise stated, all the points and sets considered here are included in $E^d$, real d-dimensional euclidean space. $\Omega_d$ is the unit spherical surface in $E^d$. The interior, closure, boundary, complement, affine hull and convex hull of a set $S$ are denoted by *int* $S$, *cl* $S$, $\partial S$, $S^C$, *lin* $S$ and *conv* $S$, respectively. The open segment joining $x$ and $y$ is denoted $(x\ y)$, and the substitution of one or both parentheses by square brackets indicates the adjunction of the corresponding endpoints. $R(x \to y)$ denotes the ray issuing from $x$ and going through $y$. $B(x, \varepsilon)$ and $U(x, \varepsilon)$ are, respectively, the closed and the open balls centered at $x$ and having radius $\varepsilon$. If $p$ is a point and $M$ is a set such that $p \notin M$ , the *join* of $p$ and $M$ is the set $J(p, M) = \bigcup_{q \in M} [p\ q]$.

We say that $x$ *sees* $y$ *via* $S$ if $[x\ y] \subset S$. The *star* of $x$ in $S$ is the set *st* $(x, S)$ of all the points of $S$ that see $x$ via $S$. A *star-center* of $S$ is a point $c \in S$ such that *st* $(c, S) = S$. The *kernel* (*convex kernel, mirador*) of $S$ is the set $\ker S$ of all the star-centers of $S$. $S$ is *convex* if $\ker S = S$, and $S$ is *starshaped* if $\ker S$ is not empty. We say that $x$ *sees clearly* $y$ *via* $S$ if $y$ admits a neighborhood $\mathcal{U}_y$ such that $x$ sees the whole $(\mathcal{U}_y \cap S)$ via $S$. The notion of *clear visibility* was introduced by N. Stavrakas in [1]. The *nova* (or *clear star*) of $p$ in $S$ is the set *nova* $(p, S)$ of all the points of $S$ that see $p$ clearly via $S$. A point $x \in S$ is a *point of local convexity* if there

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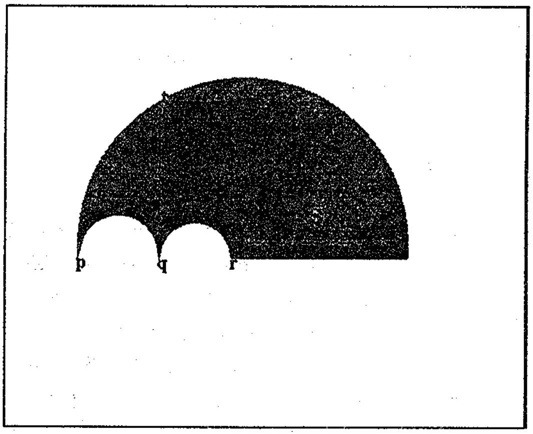
exists a neighborhood $\mathcal{U}_x$ of $x$ such that $\mathcal{U}_x \cap S$ is convex. Otherwise we say that $x$ is a *point of local nonconvexity of* $S$. The set of all points of local convexity of $S$ is denoted $lc\ S$, and that of all points of local nonconvexity $lnc\ S$. A *hunk* is a closed connected set $M$ such that $int\ M$ is connected and $M = cl\ int\ M$.
A point $x \in S$ is *totally blind* if $st\ (x, S) = \{x\}$, i.e. a totally blind point cannot see any other point of $S$. A point $y \in S$ is *blind* if $\forall\ t \in S \quad y \notin nova\ (t, S)$, i. e. a blind point cannot see clearly any point of $S$. A point $p \in S$ is a *Magoo point* if $p \notin nova\ (p, S)$, i. e. a Magoo point cannot see itself clearly. The last category of points is named after a shortsighted cartoons character. The following result is obvious.

**Proposition 1.** *A totally blind point is blind. A blind point is a Magoo point.*




In this picture, $p$ is totally blind, $q$ is blind but not totally blind, and $r$ is a Magoo point but not a blind one.

## 2 Characterization of blind and totally blind points.

A point $x \in cl\ S$ is *accessible from* $S$ if there exists another point $t \in S$ such that $[t\ ;\ x) \subset S$. A *convex component* $K$ of $S$ is a maximal convex subset of $S$. A convex set $M$ is *plump* if $int\ M$ is not empty.

**Theorem 2.** *A point $x \in S$ is totally blind if and only if $x$ is not accessible from $S$.*

*Proof.* If $x$ is a totally blind point of $S$, $st(x, S) = \{x\}$. Hence, there is no $t \in S$, $t$ distinct from $x$ such that $[t\ ;\ x] \subset S$. This means that $x$ is not accessible from S. The converse implication follows the same line of thought. It is important to remark that the total blindness of a point does not implies its isolation in the topological sense, as the counterexample at the end of last section shows.■

**Theorem 3.** *Let $S$ be a hunk in $E^d$. A point $x \in S$ is a blind point if and only if no plump convex component of $S$ includes $x$.*

*Proof.* Assume that $x$ is not a blind point of $S$. Then there should exist $p \in S$ such that $x$ sees $p$ clearly via $S$. This implies the existence of a neighborhood $\mathcal{U}_p$ of $p$ such that $(\mathcal{U}_p \cap S) \subset st(x, S)$. Hence $J = conv\ (\{x\} \cup (\mathcal{U}_p \cap S)) = \bigcup_{t \in \mathcal{U}_p \cap S} [x\ \ t] \subset S$. If we consider the family of all convex subsets of $S$ that include $J$, and apply to it Zorn's Maximal Lemma, we obtain a plump convex component of $S$ that includes $x$. Conversely, assume that $K$ is a plump convex component of $S$ that includes $x$, and let $y \in int\ K$. Then $x$ sees $y$ clearly via $K$, and, *a fortiori*, via $S$. Hence, $x$ is not a blind point.■

# 3 Magoo points.

In this section we will prove that the existence of Magoo points implies a particular distribution of points of local nonconvexity.

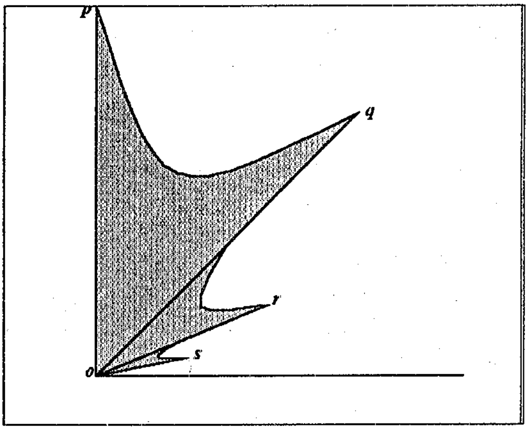
**Lemma 4.** *Let $S$ be a hunk in $E^d$ and $x \in \partial S$. If $x$ is a Magoo point of $S$ then $x \in lnc\, S$.*

*Proof.* Assume that $x \in lc\, S$. Then there should exist a neighborhood $\mathcal{U}_x$ of $x$ such that $\widehat{U} = \mathcal{U}_x \cap S$ be convex. Hence $x$ would be able to see each point of $\widehat{U}$ in contradiction with the definition of Magoo point.■

**Lemma 5.** *Let $S$ be a hunk in $E^d$, and $x \in \partial S$. If $x$ is an isolated point of $lnc\, S$, then $x$ is not a Magoo point of $S$.*

*Proof.* Assume that $x$ is an isolated point of $lnc\, S$. Then there exists a compact neighborhood $\mathcal{U}_x$ of $x$ such that $\mathcal{U}_x \cap lnc\, S = \{x\}$. Denote $U = \mathcal{U}_x \cap S$, hence, it is clear that the only point of local nonconvexity of $U$ is precisely $x$. Since $U$ is compact, Corollary 2.2 of [3] implies that $x$ is included into each convex component of $U$. Hence, according with a well known result of [2], $x \in ker\, U$. Consequently, $x$ is not a Magoo point of $S$.■

The conjunction of this two lemmas imply that any Magoo point is a non-isolated point of local nonconvexity of $S$. This statement is not a characterization of Magoo points, since the reciprocal implication is false, as the following example shows.




**Example 1.** *Consider the planar set in the picture above. Points p, q, r, s, ... are points of local nonconvexity of the shaded set, that converge to o. But o is not a Magoo point since it can see the whole set.*

**Theorem 6.** *Let $S$ be a hunk in $E^d$, and $x \in \partial S$. The following statements are equivalent:*

*(1) $x$ is a Magoo point of $S$.*

*(2) $\forall \mathcal{U}_x$ neighborhood of $x$ there exists $y \in \mathcal{U}_x \cap lnc\, S$ that is not clearly visible from $x$.*

*Proof.* (1)⇒(2). Assume the existence of a neighborhood $\mathcal{U}_x$ of $x$ such that $x$ can see clearly every point of $\mathcal{U}_x \cap lnc\, S$. If we denote $U = \mathcal{U}_x \cap S$, the previous condition would imply that $x$ can see clearly every point of $lnc\, U$, whence a well known result of Stavrakas (cf. [1]) would yield that $x \in ker\, U$. But this contradicts (1).

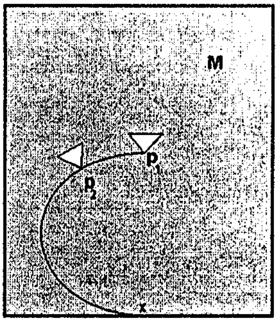
(2)⇒(1). Assume that $y \in \mathcal{U}_x \cap lnc\, S$ and $y$ is not clearly visible. Hence, there should exist a neighborhood $\mathcal{V}_y$ of $y$ such that $\mathcal{V}_y$ is not entirely included in $st(x, S)$. Furthermore, we can choose $\mathcal{V}_y \subset \mathcal{U}_x$, whence part of $\mathcal{U}_x$ must be invisible from $x$. Since this construction is feasible for every neighborhood $\mathcal{U}_x$, $x$ is a Magoo point of $S$. ■

In the previous theorem it is impossible to substitute *"invisible"* instead of *"not clearly visible"*, as the following counterexample shows.

**Example 2.** *Consider a closed rectangle $Q$ and let $x$ be the midpoint of one of its edges. Consider an arc $\Gamma$ of ellipse included in the rectangle and having $x$ as one of its endpoints. Let $\{p_i\}$ be a sequence of distinct points of $\Gamma$ that converges monotonically towards $x$. For each point $p_i$ let $T_i$ be a small isosceles triangle located inside $Q$ but "outside" the curve $\Gamma$, and such* *that $p_i$ is its principal vertex and the ray $R(x \to p_i)$ its axis of symmetry. Furthermore, we can take these triangles small enough so $x$ can see all of its vertices without entering other triangle. Denote*

$$M = Q \sim \left( \bigcup_{i=1}^{\infty} int\, T_i \right).$$

*Then, $M$ is a planar hunk. The set $lnc\, M$ is constituted precisely by all the vertices of the triangles, and $x$ can see all these points. Nevertheless, $x$ is a Magoo point of $M$ and it cannot see clearly most of these vertices.*

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