On a Deformation of the Dirac Hamiltonian

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Abstract

We propose a specific quantum deformation of the well known Dirac Hamiltonian leading to the (expected) undeformed relativistic context when the deformation parameter κ tends to infinity. The non-relativistic limit is also considered.

1 Introduction

The Dirac equation is well known as being a relativistic equation describing non-zero rest mass and spin $\frac{1}{2}$ particles. It has been extensively studied in the free case but also when interactions are involved such as the Coulomb problem or the oscillator case [1].

The Dirac equation has also been related to non-relativistic supersymmetric quantum mechanics [2].

Recently, the Dirac equation has been revisited in connection with the quantum deformation theories [3]-[5]. In ref. [4] a covariant equation invariant with respect to the quantum Poincaré algebra, the so-called κ-Dirac equation, has been put in evidence. However, the proposed equation doesn't lead to an explicit Hamiltonian formulation. In the present paper, we develop a new formulation based on a κ-Dirac Hamiltonian. We search for its non-relativistic limit and we show that the usual Schrödinger Hamiltonian can be recovered when the deformation parameter tends to infinity.

2 Deformation of the Dirac Hamiltonian

Before entering into details, let us recall that the Dirac Hamiltonian writes

$$H_D = c\,\vec{\alpha}\,\vec{P} + m_0c^2\beta, \tag{1}$$

where as usual, m_0 is the non zero rest mass of the particle. As a consequence, from the usual four-momentum relation, we know that

$$P_0 = \sqrt{m_0^2c^4 + c^2\vec{P}^2} \tag{2}$$

which can be developped as follows

$$P_0 = \lambda + \mu\vec{P}^2 + \gamma(\vec{P}^2)^2 + \ldots \tag{3}$$

Here the coefficients λ, μ and γ are given by

$$\lambda = m_0c^2\ ,\ \mu = \frac{1}{2m_0}\ ,\ \gamma = -\frac{1}{8m_0^3c^2}, \ldots \tag{4}$$

As usual, let us define the rest mass of the particle as $\frac{E_0}{c^2}$ and start with the first Casimir operator of the q-Poincaré algebra [4]

$$C_1 = c^2\vec{P}^2 + 2\kappa^2(1 - ch\frac{P_0}{\kappa}),\ \ \frac{1}{\kappa} = Rlnq. \tag{5}$$

We then propose to consider the equation

$$c^2\vec{P}^2 + 2\kappa^2(1 - ch\frac{P_0}{\kappa}) = 2\kappa^2(1 - ch\frac{m_0c^2}{\kappa}), \tag{6}$$

which evidently implies (2) when $\kappa \to \infty$, i.e. in the undeformed context.

Equation (6) can be written on the form

$$ch\frac{P_0}{\kappa} = \frac{c^2}{2\kappa^2}\vec{P}^2 + ch\frac{m_0c^2}{\kappa}, \tag{7}$$

that is

$$1 + \frac{P_0^2}{2\kappa^2} + \frac{P_0^4}{4!\kappa^4} + \ldots = \frac{c^2}{2\kappa^2}\vec{P}^2 + ch\frac{m_0c^2}{\kappa}. \tag{8}$$

Now let us start with this equation and develop the operator P_0 according to eq. (3). After some rearrangements, we obtain

$$\begin{aligned}&(1 + \frac{\lambda^2}{2!\kappa^2} + \frac{\lambda^4}{4!\kappa^4} + \frac{\lambda^6}{6!\kappa^6} + \ldots) + (\frac{\lambda\mu}{\kappa^2} + \frac{\lambda^3\mu}{3!\kappa^4} + \frac{\lambda^5\mu}{5!\kappa^6} + \ldots)\vec{P}^2\\ &+(\frac{\mu^2 + 2\lambda\gamma}{2!\kappa^2} + \frac{6\lambda^2\mu^2 + 4\lambda^3\gamma}{4!\kappa^4} + \frac{15\lambda^4\mu^2 + 6\lambda^5\gamma}{6!\kappa^6} + \ldots)(\vec{P}^2)^2 + \ldots\\ &= \frac{c^2}{2\kappa^2}\vec{P}^2 + ch\frac{m_0c^2}{\kappa}.\end{aligned} \tag{9}$$

This last equation can be put in the following compact form

$$ch\frac{\lambda}{\kappa} + \frac{\mu}{\kappa}\vec{P}^2 sh\frac{\lambda}{\kappa} + (\vec{P}^2)^2(\frac{\mu^2}{2\kappa^2}ch\frac{\lambda}{\kappa} + \frac{\gamma}{\kappa}sh\frac{\lambda}{\kappa}) = \frac{c^2}{2\kappa^2}\vec{P}^2 + ch\frac{m_0c^2}{\kappa} \tag{10}$$

when we omit all the terms in $(\vec{P}^2)^n (n \geq 3)$. Inside this proposal, the coefficients λ, μ, γ are then easily determined and are

$$\lambda = m_0 c^2 \ , \ \mu = \frac{c^2}{2\kappa sh \frac{m_0 c^2}{\kappa}} \ , \ \gamma = -\frac{\mu^2}{2\kappa} cth \frac{\lambda}{\kappa} = -\frac{c^4 ch \frac{m_0 c^2}{\kappa}}{8\kappa^3 sh^3 \frac{m_0 c^2}{\kappa}} \tag{11}$$

We finally obtain a new "mass-energy" relation

$$P_0 = m_0 c^2 + \frac{c^2}{2\kappa sh \frac{m_0 c^2}{\kappa}} \vec{P}^2 - \frac{c^4 ch \frac{m_0 c^2}{\kappa}}{8\kappa^3 sh^3 \frac{m_0 c^2}{\kappa}} (\vec{P}^2)^2 + O(\vec{P}^2)^3, \tag{12}$$

extending (2) to the context of the quantum deformation considered in Ref. [4].

With this new result, we are ready to search for a κ-Dirac Hamiltonian H_D^κ such that

$$(H_D^\kappa)^2 = P_0^2 \tag{13}$$

where P_0 is given by the expression (12).

We propose the form

$$H_D^\kappa = f \ \vec{\alpha}.\vec{P} + \beta g + \beta h (\vec{P}^2)^2 \tag{14}$$

Here f, g and h are some functions depending on the deformation parameter κ and are such that

$$f \to c \ , \ g \to m_0 c^2 \ , \ h \to 0 \tag{15}$$

when $\kappa \to \infty$.

Using equations (11)-(14), we easily obtain

$$\begin{aligned} f &= c \sqrt{\frac{m_0 c^2}{\kappa sh \frac{m_0 c^2}{\kappa}}}, \\ g &= m_0 c^2, \\ h &= \frac{c^2}{8 m_0 \kappa^2 sh^2 \frac{m_0 c^2}{\kappa}} \left(1 - \frac{m_0 c^2}{\kappa} cth \frac{m_0 c^2}{\kappa}\right) \end{aligned} \tag{16}$$

It is straightforward to verify that (16) obey the constraint (13). The resulting κ-Dirac Hamiltonian writes

$$H_D^\kappa = c c \sqrt{\frac{m_0 c^2}{\kappa sh \frac{m_0 c^2}{\kappa}}} \vec{\alpha}.\vec{P} + \beta \left(m_0 c^2 + \frac{c^2}{8 m_0 \kappa^2 sh^2 \frac{m_0 c^2}{\kappa}} (\vec{P}^2)^2 \left(1 - \frac{m_0 c^2}{\kappa} cth \frac{m_0 c^2}{\kappa}\right)\right) \tag{17}$$

One immediately observes that $H_D^\kappa \to H_D$ when $\kappa \to \infty$ and H_D^κ is invariant with respect to the parity operator.

3 Determination of the non relativistic limit

We start with the Hamiltonian (17) in the standard realization of the Dirac matrices

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \ , \ \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \tag{18}$$

where σ_j are the usual Pauli matrices and σ_0 is the 2 by 2 unit matrix. The time independent wave equation writes

$$H_D^\kappa \psi = E\psi \ , \ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{19}$$

that is

$$(f\vec{\sigma}.\vec{p})\psi_2 = (E - m_0c^2 - (\vec{p}^2)^2 h)\psi_1, \tag{20}$$

$$(f\vec{\sigma}.\vec{p})\psi_1 = (E + m_0c^2 + (\vec{p}^2)^2 h)\psi_2, \tag{21}$$

where the coefficients f and h are given by formulas (16).

The system (20)-(21) is easily separated. Indeed, we obtain

$$f^2\vec{p}^2\psi_1 = (E^2 - m_0^2c^4 - 2m_0c^2(\vec{p}^2)^2 h)\psi_1 \tag{22}$$

where, once more, we neglect terms of more than second order in $\vec{p}^2$.

The eigenvalue equation is then

$$E^2 = m_0^2c^4 + \frac{m_0c^4}{\kappa sh\frac{m_0c^2}{\kappa}}\vec{p}^2 + \frac{c^4}{4\kappa^2 sh^2\frac{m_0c^2}{\kappa}}(1 - \frac{m_0c^2}{\kappa}cth\frac{m_0c^2}{\kappa})(\vec{p}^2)^2 \tag{23}$$

and the usual expression of the non relativistic energies ϵ takes here the form

$$\epsilon = \frac{E^2 - m_0^2c^4}{2m_0c^2} = \frac{c^2}{2\kappa sh\frac{m_0c^2}{\kappa}}\vec{p}^2 + \frac{c^2}{8m_0\kappa^2 sh^2\frac{m_0c^2}{\kappa}}(1 - \frac{m_0c^2}{\kappa}cth\frac{m_0c^2}{\kappa})(\vec{p}^2)^2 \tag{24}$$

The usual Schrödinger Hamiltonian $\frac{\vec{p}^2}{2m}$ is obtained when $\kappa \to \infty$.

4 Conclusions

The main result of this paper is Eq. (14) with the coefficients f, g, h determined by (16). We obtained a Hamiltonian for one q-deformed Dirac equation by performing an expansion of the relevant operator P_0 in powers of the operator $(\vec{P}^2)^2$. Throughout the calculations, terms of order $(\vec{P}^2)^3$ and higher powers have been neglected.

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