On a Deformation of the Dirac Hamiltonian

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Abstract

We propose a specific quantum deformation of the well known Dirac Hamiltonian leading to the (expected) undeformed relativistic context when the deformation parameter $\kappa$ tends to infinity. The non-relativistic limit is also considered.

1. Introduction

The Dirac equation is well known as being a relativistic equation describing non-zero rest mass and spin $\frac{1}{2}$ particles. It has been extensively studied in the free case but also when interactions are involved such as the Coulomb problem or the oscillator case [1].

The Dirac equation has also been related to non-relativistic supersymmetric quantum mechanics [2].

Recently, the Dirac equation has been revisited in connection with the quantum deformation theories [3]-[5]. In ref. [4] a covariant equation invariant with respect to the quantum Poincaré algebra, the so-called $\kappa$-Dirac equation, has been put in evidence. However, the proposed equation doesn't lead to an explicit Hamiltonian formulation. In the present paper, we develop a new formulation based on a $\kappa$-Dirac Hamiltonian. We search for its non-relativistic limit and we show that the usual Schrödinger Hamiltonian can be recovered when the deformation parameter tends to infinity.
2 Deformation of the Dirac Hamiltonian

Before entering into details, let us recall that the Dirac Hamiltonian writes

$$H_D = c \vec{\sigma} \vec{P} + m_0 c^2 \beta,$$

where as usual, $m_0$ is the non-zero rest mass of the particle. As a consequence, from the usual four-momentum relation, we know that

$$P_0 = \sqrt{m_0^2 c^4 + c^2 \vec{P}^2},$$

which can be developed as follows

$$P_0 = \lambda + \mu \vec{P}^2 + \gamma (\vec{P}^2) + \cdots$$

Here the coefficients $\lambda, \mu$ and $\gamma$ are given by

$$\lambda = m_0 c^2, \mu = \frac{1}{2m_0}, \gamma = -\frac{1}{8m_0^3 c^2},$$

As usual, let us define the rest mass of the particle as $\frac{P_0}{\kappa}$ and start with the first Casimir operator of the $q$-Poincaré algebra [4]

$$C_1 = c^2 \vec{P}^2 + 2\kappa^2 (1 - c\frac{\kappa}{\kappa}), \quad \frac{1}{\kappa} = Rllqf.$$ (5)

We then propose to consider the equation

$$c^2 \vec{P}^2 + 2\kappa^2 (1 - c\frac{\frac{P_0}{\kappa}}{\kappa}) = 2\kappa^2 (1 - c\frac{m_0 c^2}{\kappa}),$$

which evidently implies (2) when $\kappa \to \infty$, i.e. in the undeformed context.

Equation (6) can be written on the form

$$c\frac{P_0}{\kappa} = \frac{c^2}{2\kappa^2} \vec{P}^2 + c\frac{m_0 c^2}{\kappa},$$

that is

$$1 + \frac{P_0^2}{2\kappa^2} + \frac{P_0^4}{4\kappa^4} + \cdots = \frac{c^2}{2\kappa^2} \vec{P}^2 + c\frac{m_0 c^2}{\kappa}.$$ (7)

Now let us start with this equation and develop the operator $P_0$ according to eq. (3).

After some rearrangements, we obtain

$$\begin{align*}
(1 + \frac{\lambda^2}{2\kappa^2} + \frac{\lambda^4}{4\kappa^4} + \frac{\lambda^6}{6\kappa^6} + \cdots) + (\frac{\mu^2}{\kappa^2} + \frac{\mu^4}{3\kappa^4} + \frac{\mu^6}{5\kappa^6} + \cdots) \vec{P}^2 \\
+ \left(\frac{\gamma^2 + 2\lambda^2}{2\kappa^2} + \frac{3\lambda^4 + 4\lambda^6 + 18\lambda^6 \mu^2 + 6\lambda^8 \gamma}{6\kappa^6} + \cdots \right. \\
\left.\frac{\gamma^2}{4\kappa^2} + \frac{\gamma^4}{4\kappa^4} + \cdots \right) \vec{P}^2 \\
= \frac{c^2}{2\kappa^2} \vec{P}^2 + c\frac{m_0 c^2}{\kappa},
\end{align*}$$

This last equation can be put in the following compact form

$$c\frac{P_0^4}{\kappa} + \frac{\mu^2}{\kappa} \vec{P}^2 \frac{\lambda}{\kappa} + (\vec{P}^2)^2 \left(\frac{\mu^2}{2\kappa^2} c\frac{\lambda}{\kappa} + \frac{\gamma^2}{\kappa} + \frac{\lambda}{\kappa} \right) = \frac{c^2}{2\kappa^2} \vec{P}^2 + c\frac{m_0 c^2}{\kappa}.$$ (9)

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when we omit all the terms in \((\vec{P}^2)^n(n \geq 3)\). Inside this proposal, the coefficients \(\lambda, \mu, \gamma\) are then easily determined and are

\[
\lambda = m_0 c^2, \quad \mu = \frac{c^2}{2\kappa \hbar^2 \frac{m_0 c^2}{\kappa}}, \quad \gamma = \frac{\mu^2}{2\kappa} c^2 \frac{\lambda}{\kappa} = -\frac{c^4 \hbar^4 m_0^2}{8\kappa^3 \hbar^3 m_0 c^2 \kappa}.
\]  

\[(11)\]

We finally obtain a new “mass-energy” relation

\[
P_0 = m_0 c^2 + \frac{c^2}{2\kappa \hbar^2 \frac{m_0 c^2}{\kappa}} \vec{P}^2 - \frac{c^4 \hbar^4 m_0^2}{8\kappa^3 \hbar^3 m_0 c^2 \kappa}(\vec{P}^2)^2 + O(\vec{P}^2)^3,
\]

\[
\text{(12)}
\]

extending (2) to the context of the quantum deformation considered in Ref. [4].

With this new result, we are ready to search for a \(\kappa\)-Dirac Hamiltonian \(H_D^\kappa\) such that

\[
(H_D^\kappa)^2 = P_0^2
\]

\[(13)\]

where \(P_0\) is given by the expression (12).

We propose the form

\[
H_D^\kappa = f \vec{\alpha} \cdot \vec{P} + \beta g + \eta h (\vec{P}^2)^2
\]

\[(14)\]

Here \(f, g\) and \(h\) are some functions depending on the deformation parameter \(\kappa\) and are such that

\[
f \to c, \quad g \to m_0 c^2, \quad h \to 0
\]

\[(15)\]

when \(\kappa \to \infty\).

Using equations (11)-(14), we easily obtain

\[
f = c \sqrt{\frac{m_0 c^2}{\kappa \hbar^2 \frac{m_0 c^2}{\kappa}}},
\]

\[
g = m_0 c^2,
\]

\[
h = \frac{c^2}{8m_0^2 \kappa^2 \hbar^3 \frac{m_0 c^2}{\kappa}}(1 - \frac{m_0 c^2}{\kappa} c^2 \frac{m_0 c^2}{\kappa})
\]

\[
(16)
\]

It is straightforward to verify that (16) obey the constraint (13). The resulting \(\kappa\)-Dirac Hamiltonian writes

\[
H_D^\kappa = c \sqrt{\frac{m_0 c^2}{\kappa \hbar^2 \frac{m_0 c^2}{\kappa}}} \vec{\alpha} \cdot \vec{P} + \beta (m_0 c^2 + \frac{c^2}{8m_0^2 \kappa^2 \hbar^3 \frac{m_0 c^2}{\kappa}}(\vec{P}^2)^2(1 - \frac{m_0 c^2}{\kappa} c^2 \frac{m_0 c^2}{\kappa}))
\]

\[
(17)
\]

One immediately observes that \(H_D^\kappa \to H_D\) when \(\kappa \to 0\) and \(H_D^\kappa\) is invariant with respect to the parity operator.

3 Determination of the non relativistic limit

We start with the Hamiltonian (17) in the standard realization of the Dirac matrices

\[
\vec{\alpha} = \begin{pmatrix} 0 & \vec{\beta} \\ \vec{\beta} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},
\]

\[
(18)
\]


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where $\sigma_j$ are the usual Pauli matrices and $\sigma_0$ is the 2 by 2 unit matrix. The time independent wave equation writes

$$H_0^\prime \psi = E \psi, \quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

(19)

that is

$$(f \sigma_1 \vec{p}) \psi_2 = (E - mc^2 - (\vec{p}^2)h)\psi_1,$$  (20)

$$(f \sigma_\mu \vec{p}) \psi_1 = (E + mc^2 + (\vec{p}^2)h)\psi_2,$$  (21)

where the coefficients $f$ and $h$ are given by formulas (16).

The system (20)-(21) is easily separated. Indeed, we obtain

$$f^2 \vec{p}^2 \psi_1 = (E^2 - mc^4 - 2mc^2(\vec{p}^2)h)\psi_1$$  (22)

where, once more, we neglect terms of more than second order in $\vec{p}^2$.

The eigenvalue equation is then

$$E^2 = mc^4 + \frac{m_0c^4}{\kappa h^2 mc^2} \vec{p}^2 + \frac{c^4}{4\kappa h^2 mc^2} (1 - \frac{mc^2}{\kappa} cth \frac{mc^2}{\kappa}) (\vec{p}^2)^2$$  (23)

and the usual expression of the non relativistic energies $\epsilon$ takes here the form

$$\epsilon = \frac{E^2 - mc^4}{2m_0c^2} = \frac{c^2}{2\kappa h^2 mc^2} \vec{p}^2 + \frac{c^2}{8\kappa h^2 mc^2} (1 - \frac{mc^2}{\kappa} cth \frac{mc^2}{\kappa}) (\vec{p}^2)^2$$  (24)

The usual Schrödinger Hamiltonian $\frac{\vec{p}^2}{2m}$ is obtained when $\kappa \to \infty$.

4 Conclusions

The main result of this paper is Eq. (14) with the coefficients $f$, $g$, $h$ determined by (16). We obtained a Hamiltonian for one $q$-deformed Dirac equation by performing an expansion of the relevant operator $P_0$ in powers of the operator $(\vec{P}^2)^2$. Throughout the calculations, terms of order $(\vec{P}^2)^3$ and higher powers have been neglected.

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References


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