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A FOX INTEGRAL TRANSFORMATION OF GENERALIZED FUNCTIONS

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ABSTRACT.

In this paper an integral transformation introduced by Ch. Fox is extended to a certain spaces of generalized functions. Boundedness, smoothness and inversion theorems are established for the generalized transformation.

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1. Introduction . The integral transformation defined by

$$F(y) = H_E(f(x))(y) = \int_0^\infty f(x)(xy)^{1/2}\int_0^\infty \frac{u}{E(u)} J_\mu(ux)J_\mu(uy)dudx$$

where $E(u)=\prod_{n=1}^{\infty}(1+u^2a_n^{-2})$, a_n is real for all nonnegative integer values of n, and $\sum_{n=1}^{\infty} a_n^{-2}$ is convergent, was introduced by Ch. Fox [2] who established the following inversion formula.

Theorem 1 ([2]) : Let $\mu\geq-\frac{1}{2}$. If $f(x)\in L_1(0,\infty)$ and it is of bounded variation in a neighborhood of x=y, then the integral transform

$$F(y) = \int_0^\infty f(x)(xy)^{1/2}\int_0^\infty \frac{u}{E(u)} J_\mu(ux)J_\mu(uy)dudx$$

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is inverted by the differential operator

$$\prod_{n=1}^{\infty} (1-a_n^{-2}S_{\mu,y})F(y) = \tfrac{1}{2}(f(y+0)+f(y-0))$$

where $S_{\mu,y}$ denotes the differential operator $y^{-\mu-(1/2)}Dy^{2\mu+1}Dy^{-\mu-(1/2)}$.

Our main objetive in this paper is to extend the classical H_E transformation to generalized functions. We extend the inversion formula due to Ch. Fox [2] (Theorem 1) to certain spaces of generalized functions. Other form of Hankel convolution was defined on distributions by J.N. Pandey [8]. The spaces of J.N. Pandey are different to the ones introduced by us herein.

The notation and terminology of this work will follow that [9] and [15]. I denotes the open interval $(0,\infty)$ and all testing functions herein are defined on I. If f is a generalized function on I, the notation f(t) simply indicates that the testing functions on which f is define have t as their independent variable. $\langle f(t),\phi(t)\rangle$ denotes the number assigned to some element $\phi(t)$ in a testing function space by a member f of the dual space. Finally D(I) is the space of infinitely differentiable functions defined on I having compact support. The topology of D(I) is that which makes its dual the space D'(I) of Schwartz distributions.

2. The testing function space T_μ and its dual T'_μ. Let μ be a fixed real number. We define T_μ as the collection of infinitely differentiable complex valued functions $\phi(t)$ defined on I such that

$$\gamma_k(\phi) = \sup_{x\in I} |\xi(x)S_\mu{}^k\phi(x)| < \infty$$

for each k=0,1,2,... , where $\xi(x)$ is a nonnegative regular function defined on I such that $\lim_{x\to 0} \xi(x) = \lim_{x\to\infty} \xi(x) = 0$.

We assign to T_μ the topology generated by the seminorms $\{\gamma_k\}_{k=0}^{\infty}$, thereby making it a countably multinormed space [15]. The dual space T'_μ consists of all continuous linear functionals on T_μ. T'_μ is also a linear space to which we endowed with the weak topology generated by the multinorm $\{\eta_\phi\}$, where $\eta_\phi(f)=|<f,\phi>|$ and ϕ ranges through T_μ.

It is obvious that the space D(I) is contained in T_μ, and the topology of D(I) is stronger than that induced on it by T_μ. Hence the restriction of any f $\in T'_\mu$ to D(I) is in D'(I).

Next, we give a structure formula for the restriction of an element of T'_μ to D(I).

Proposition 1 : Let f be in T'_μ. Then there exist essentially bounded measurable functions $g_i(x)$ defined for $x>0$, for $i=0,1,2,...,r$, where r is some nonnegative integer depending on f such that for an arbitrary $\phi \in D(I)$ we have

$$<f,\phi> = <\sum_{m=0}^{r} S_\mu{}^m(\xi(x)(-D)g_m(x)),\phi(x)>$$

PROOF: By virtue of [15, Theorem 1.8-1] there exist a constant $C>0$ and a nonnegative integer r depending on f such that for all $\phi \in D(I)$

$$<f,\phi> \le C \max_{0\le k\le r} \gamma_k(\phi) \le C \max_{0\le k\le r} \sup_{x\in I} |\xi(x)S_\mu{}^k\phi(x)| \le$$

$$\le C \max_{0\le k\le r} \sup_{x\in I} \int_0^x |D_t(\xi(t)S_\mu{}^k\phi(t))|dt \le C \max_{0\le k\le r} \| D_t(\xi(t)S_\mu{}^k\phi(t))\|_1 \qquad (1)$$

where $\| \ \|_1$ denotes the norm in the space $L_1(0,\infty)$.

Hence, if we define the mapping

$$T:D(I) \longrightarrow TD(I)\subset L_1(0,\infty)x\overset{r+1}{\dots}xL_1(0,\infty)=(L_1(0,\infty))^{r+1}$$

$$\phi \longrightarrow T(\phi) = (D_t(\xi(t)S_\mu{}^k\phi(t)))_{k=0}^r$$

according to (1), the mapping

$$J: TD(I) \longrightarrow \mathbb{C}$$

$$(D_t(\xi(t)S_\mu{}^k\phi(t)))_{k=0}^r \longrightarrow <f,\phi>$$

is linear and continuous when TD(I) is endowed with the topology induced in it by $(L_1(0,\infty))^{r+1}$. By using the Hahn-Banach theorem J can be continuously extended to $(L_1(0,\infty))^{r+1}$. Moreover, by taking into account that the dual of $L_1(0,\infty)$ is equivalent to $L_\infty(0,\infty)$ (see F. Treves [11]) there exist essentially bounded measurable functions $g_m(x)$ defined on I, m=0,1,2,...,r satisfying

$$\langle f,\phi\rangle = \sum_{m=0}^{r} \langle g_m(x), D(\xi(x)S_\mu{}^m\phi(x))\rangle =$$

$$= \langle \sum_{m=0}^{r} S_\mu{}^m(\xi(x)(-D)g_m(x)), \phi(x)\rangle, \text{ for every } \phi \in D(I).$$

This completes the proof of Proposition 1.

One can easily check that if f(x) is a function on I such that

$$\int_0^\infty \frac{|f(x)|}{\xi(x)}dx < \infty$$

then f(x) generates a regular generalized function on T_μ defined by

$$\langle f,\phi\rangle = \int_0^\infty f(x)\phi(x)dx,\ \phi \in T_\mu.$$

Proposition 2 : If $\mu\geq-\frac{1}{2}$, then

$$K(x,y) = (xy)^{1/2}\int_0^\infty \frac{u}{E(u)}J_\mu(ux)J_\mu(uy)du$$

as function of x, is in T_μ, for every $y \in I$.

PROOF : By using the relation

$$S_\mu z^{1/2}J_\mu(z) = -z^{1/2}J_\mu(z) \qquad (2)$$

we can deduce that

$$S_{\mu,x}^m K(x,y) = S_{\mu,x}^m \int_0^\infty \frac{1}{E(u)}(ux)^{1/2}J_\mu(ux)(uy)^{1/2}J_\mu(uy)du =$$

$$= (-1)^m \int_0^\infty \frac{u^{2m}}{E(u)}(ux)^{1/2}J_\mu(ux)(uy)^{1/2}J_\mu(uy)du$$

for every $m \in \mathbb{N}$.

Hence, since $z^{1/2}J_\mu(z)$ is a bounded function for $z\in(0,\infty)$ provided that $\mu\geq-\frac{1}{2}$, one has

$$\sup_{x\in I}|\xi(x)S_{\mu,x}^m K(x,y)| \leq C\int_0^\infty \frac{u^{2m}}{E(u)}\,du < \infty$$

for every $m \in \mathbb{N}$, where C is a suitable positive constant.

Therefore $K(x,y) \in T_\mu$, for $y \in I$, when $\mu \geq -\frac{1}{2}$.

3. The generalized H_E transformation. Throughout this section we assume that $\mu \geq -\frac{1}{2}$. For $f \in T'_\mu$ we define the generalized H_E transform by the relation

$$F(y) = (H_E f)(y) = \langle f(x), K(x,y) \rangle \ , \ y>0 \tag{3}$$

Here $K(x,y)$ is defined as in Proposition 2.

Notice that (3) is well defined according to Proposition 2. Moreover, if f generates a regular distribution then the generalized H_E transform of f reduces to the classical H_E transform of f.

In the sequel we establish boundedness and smoothness properties for the generalized H_E transformation.

Proposition 3 : Let $f \in T'_\mu$. The generalized H_E transform $H_E f$ of f is bounded on I.

PROOF: By virtue of [15, Theorem 1.8-1] we have that

$$|(H_E f)(y)| \leq C \max_{0 \leq k \leq r} \gamma_k(K(x,y)) \ , \ y>0$$

for certain nonnegative integer r and $C>0$.

Therefore,

$$|(H_E f)(y)| \leq C \max_{0 \leq k \leq r} \sup_{x \in I} \left|\xi(x) S_{\mu,x}^{k}((xy)^{1/2} \int_0^\infty \frac{u}{E(u)} J_\mu(ux) J_\mu(uy)du)\right| \leq$$

$$\leq C_1 \max_{0 \leq m \leq r} \int_0^\infty \frac{u^{2m}}{E(u)} du, \text{ for } y>0$$

where C_1 is a positive constant.

Proposition 4 : Let F(y) be the generalized H_E transform of f. Then F(y) is infinitely differentiable on I and

$$\frac{d^n}{dy^n} F(y) = \langle f(x), \frac{\partial^n}{\partial y^n} K(x,y) \rangle \ , \quad \text{for } y \in I \text{ and } n \in \mathbb{N}.$$

PROOF: We only prove the assertion for $n=1$. The proof for other values of n can be done in a similar way.

Let h be an arbitrary increment in y. Without any loss of generality assume $0<|h|<\frac{y}{2}$. Now

$$\frac{F(y+h)-F(y)}{h} = <f(x), \frac{K(x,y+h)-K(x,y)}{h}>$$

Let $\vartheta_h(x,y)$ denote the expression

$$\frac{K(x,y+h)-K(x,y)}{h} - \frac{\partial}{\partial y}K(x,y)$$

We will show that $\vartheta_h(x,y)$ converges to zero in T_μ as $h\to 0$. Our result will then follow from the continuity of f(x). Now, for any nonnegative integer k

$$\xi(x)S_{\mu,x}^{\ k}\vartheta_h(x,y) =$$

$$= (-1)^k\xi(x)\frac{1}{h}\int_y^{y+h}du\int_y^u \frac{\partial^2}{\partial t^2}((tx)^{1/2}\int_0^\infty \frac{1}{E(v)}v^{2k+1}J_\mu(vx)J_\mu(vt)dv)dt \tag{4}$$

By using wellknown properties of Bessel functions we can deduce from (4) that $\lim_{h\to 0} \xi(x)S_{\mu,x}^{\ k}\vartheta_h(x,y) = 0$ uniformly in $x\in(0,\infty)$.

We now extend the inversion formula stated in Theorem 1 ([2]) to T'_μ interpreting convergence in the weak distributional sense.

Theorem 2 : Let $f \in T'_\mu$ and let F(y) be the generalized H_E transform of f. Then for each $\phi \in D(I)$,

$$\lim_{n\to\infty} <\prod_{n=1}^{\infty} (1-a_n^{-2}S_{\mu,y})F(y), \phi(y)> = <f(x),\phi(x)>$$

PROOF: Let ϕ be in D(I). According to standard definitions, one has

$$<\prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})F(y), \phi(y)> = <F(y),\prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})\phi(y)>$$

for every $n \in \mathbb{N}$.

By virtue of Proposition 3, F(y) generates a regular distribution in D'(I), hence we can write

$$<\prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})F(y), \phi(y)> = \int_a^b F(y)\prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})\phi(y)dy$$

where $0<a<b<\infty$ and the support of ϕ is contained in [a,b].

We now prove by making use of Riemann sums that

$$\int_a^b F(y)\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})\phi(y)dy = \tag{5}$$
$$= \langle f(x), \int_a^b \prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})(K(x,y))\phi(y)dy\rangle$$

In effect, if $\{y_{\nu,m}\}_{\nu=0}^{m}$ are partitions of the interval (a,b) such that $d_m=y_{\nu,m}-y_{\nu-1,m}$ ($\nu=1,2,\ldots,m$) tends to zero as $m\to\infty$, then

$$\int_a^b \prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})(F(y))\phi(y)dy =$$
$$= \lim_{m\to\infty} d_m \sum_{\nu=1}^{m}\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y_{\nu,m}})(F(y_{\nu,m}))\phi(y_{\nu,m}) =$$
$$= \lim_{m\to\infty} \langle f(x), d_m \sum_{\nu=1}^{m}\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y_{\nu,m}})(K(x,y_{\nu,m}))\phi(y_{\nu,m})\rangle$$

Therefore we have to see that

$$\lim_{m\to\infty} d_m \sum_{\nu=1}^{m}\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y_{\nu,m}})(K(x,y_{\nu,m}))\phi(y_{\nu,m}) =$$
$$= \int_a^b \prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})(K(x,y))\phi(y)dy$$

in the sense of convergence in T_μ.

Notice that for every $\ell\in\mathbb{N}$,

$$|\xi(x)(\int_a^b \prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})S_{\mu,x}^{\ell}(K(x,y))\phi(y)dy - \tag{6}$$
$$- d_m \sum_{\nu=1}^{m}\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y_{\nu,m}})S_{\mu,x}^{\ell}(K(x,y_{\nu,m}))\phi(y_{\nu,m}))| \leq C\xi(x)$$

for a certain C constant. Hence, given an $\varepsilon>0$ there exist two real numbers X_1 and X_2 ($X_1<X_2$) such that the left hand side of (6) is less than ε provided that $x\in(0,X_1)\cup(X_2,\infty)$.

Moreover

$$\xi(x)d_m\sum_{\nu=1}^{m}\prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y_{\nu,m}})S_{\mu,x}^{\ell}(K(x,y_{\nu,m}))\phi(y_{\nu,m})$$

converges to

$$\xi(x)\int_a^b \prod_{k=1}^{n}(1-a_k^{-2}S_{\mu,y})S_{\mu,x}^{\ell}(K(x,y))\phi(y)dy \tag{7}$$

uniformly on every compact subset of I, as $m\to\infty$, because the integrand in (7) is continuous, and therefore uniformly continuous, on each compact.

Hence (5) is proved.

To finish the proof of this theorem we must prove that

$$\lim_{n\to\infty} \int_a^b \prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})(K(x,y))\phi(y)dy = \phi(x) \qquad (8)$$

in the sense of convergence in T_μ.

For every $\ell \in \mathbb{N}$, we get

$$S_{\mu,x}^{\ell} \int_a^b \prod_{k=1}^{n} (1-a_k^{-2}S_{\mu,y})(K(x,y))\phi(y)dy =$$

$$= \int_a^b (xy)^{1/2}\left(\int_0^\infty \frac{u}{E_n(u)} J_\mu(xu)J_\mu(uy)du\right)S_{\mu,y}^{\ell}\phi(y)dy$$

where $E_n(u)=\prod_{k=n+1}^{\infty}(1+u^2a_k^{-2})$, because according to (2) $S_{\mu,x}K(x,y)=S_{\mu,y}K(x,y)$.

By Hankel's theorem [12,Ξ14.4],

$$S_{\mu,x}^{\ell}\phi(x) = \int_0^\infty (xu)^{1/2}J_\mu(xu)du\int_a^b (yu)^{1/2}J_\mu(yu)S_{\mu,y}^{\ell}(\phi(y))dy$$

for every $x \in I$.

We now define as in [2, pp. 884],

$$P(x,u) = (xu)^{1/2}J_\mu(xu)\int_a^b (yu)^{1/2}J_\mu(yu)S_{\mu,y}^{\ell}(\phi(y))dy$$

and

$$P_n(x,u) = \frac{1}{E_n(u)}(xu)^{1/2}J_\mu(xu)\int_a^b (yu)^{1/2}J_\mu(yu)S_{\mu,y}^{\ell}(\phi(y))dy$$

Then, to prove (8) it is sufficient to see that

$$\sup_{x\in I}\left|\xi(x)\int_0^\infty (P(x,u)-P_n(x,u))du\right| \longrightarrow 0 \qquad (9)$$

as $n\to\infty$.

By invoking [15, p. 139], if $\Phi \in D(I)$ then

$$\int_0^\infty (xy)^{1/2}J_\mu(xy)\Phi(x)dx$$

is an absolutely integrable function on I and one has

$$\int_0^\infty |P(x,u)|du \le C \int_0^\infty |\int_0^\infty (uy)^{1/2} J_\mu(uy) S'_{\mu,y}(\phi(y))dy|du$$

for a certain C constant independent of x. Hence, given an $\varepsilon>0$ there exists $U_0>0$ such that

$$\int_Y^Z |P(x,u)|du < \frac{\varepsilon}{3} \text{, for every } Z>Y>U_0 \text{ and } x \in I \tag{10}$$

On the other hand, by applying the second mean value theorem and by (10) we can deduce

$$|\int_Y^Z P_n(x,u)du| < \frac{\varepsilon}{3} \tag{11}$$

provided that $U_0<Y<Z$, $x \in I$ and $n \in \mathbb{N}$.

Finally, for every $n \in \mathbb{N}$

$$|\int_0^{U_0} (P_n(x,u)-P(x,u))du| \le$$

$$\le \int_0^{U_0} |\frac{1}{E_n(u)} - 1| \int_0^\infty |(uy)^{1/2} J_\mu(uy) S'_{\mu,y}\phi(y)dy|du$$

and, by virtue of Lebesgue dominated convergence theorem, there exists $n_0 \in \mathbb{N}$ such that

$$|\int_0^{U_0} (P_n(x,u)-P(x,u))du| < \frac{\varepsilon}{3} \tag{12}$$

for every $n>n_0$ and for $x \in I$.

By combining (10), (11) and (12) we can establish (9).

Thus, the generalized inversion formula is proved.

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