A NOTE ON NONDISTINGUISHED KÖTHE SPACES OF INFINITE TYPE

Carmen FERNÁNDEZ

Departamento de Análisis Matemático, Facultad de Matemáticas
Universidad de Valencia, Dr. Moliner 50, E-46100 Burjassot (Valencia) Spain

ABSTRACT: In this note it is proved that if \( \lambda_\infty(A) \) is any nondistinguished Köthe echelon space of infinite order, then there is a linear form on its strong dual \( (\lambda_\infty(A))^\prime \), which is locally bounded (i.e. bounded on the bounded sets) but not continuous.

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A Fréchet space \( F \) is distinguished if its strong dual \( F^\prime \) is barrelled or, equivalently, homological. This means that the canonical representation of \( F \) as the (reduced) projective limit \( \text{proj } F_n \) leads to the representation \( \text{ind } F_n^\prime \) of \( F^\prime \) as the inductive limit of the dual spectrum \( (F_n^\prime)_{n=1}^\infty \). Distinguished Fréchet spaces were introduced by Dieudonné, Schwartz and Grothendieck.

The first example of a nondistinguished Fréchet space was given by Köthe

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and Grothendieck and it was the Köthe echelon space of order one $\lambda_1(A)$ for the Köthe matrix $A=(a_{n,n})_{n\in\mathbb{N}}$ on the index set $\mathbb{N}\times\mathbb{N}$ given by $a_{n,n}(k,j)=j$ if $k\leq n$ and $a_{n,n}(k,j)=1$ if $k\geq n+1$. For this matrix Grothendieck even proved that there is a linear form on $(\lambda_1(A))^*_b$ which is locally bounded but not continuous. Distinguished echelon spaces of order one were characterized in terms of the Köthe matrix $A$ in [3]. In [2] it is shown that all nondistinguished Köthe echelon spaces of order one share the bad behaviour of the Köthe-Grothendieck example. On the other hand, an example of Komura (see e. g. [6] p.292) shows that there are nondistinguished Fréchet spaces such that every locally bounded linear form on $F^*_b$ is continuous. More examples of both types of nondistinguished Fréchet spaces can be seen in [5], where one can also find the following characterization of the quasibarrelled spaces $E$ such that on the strong dual $E^*_b$, there exists a noncontinuous locally bounded linear form

Lemma ([5]): Let $E$ be a quasibarrelled l.c.s., Then the following statements are equivalent:
(i) there exists a noncontinuous locally bounded linear form on $E^*_b$
(ii) there is a filter $\mathcal{F}$ in $E$ such that
(a) for every 0-neighbourhood $U$ there is $\rho_U>0$ with $\rho_U U \in \mathcal{F}$
(b) for every bounded set $B$ there exists a closed 0-neighbourhood in $(E,\sigma(E,E^*))$, $V_B$, such that $E(B+V_B) \in \mathcal{F}$

On the other hand, distinguished Köthe echelon spaces $\lambda_{\infty}(A)$ were characterized in terms of the Köthe matrix $A$ in [1]. Making use of these two results we show that all nondistinguished echelon spaces $\lambda_{\infty}(A)$ present exactly the same behaviour as in the case of echelon spaces of order one.

Notations for Köthe spaces are as in [4]. We recall that a Köthe space $\lambda_{\infty}(A)$ has a fundamental system of bounded sets of the form 
$$\tilde{v}(\lambda_{\infty})=(z \in \mathcal{K}^I: z=\tilde{v}z'; \sup_{i \in I} |z_i| \leq 1),$$
where $\tilde{v}=\inf_{m} \rho_{m} A^{-1}$ for some sequence of positive numbers $\rho_{m}$.

Theorem: Let $\lambda_{\infty}(A)$ be a nondistinguished Köthe echelon space of infinite order. Then there is a locally bounded noncontinuous linear form on $(\lambda_{\infty}(A))^*_b$.
Proof: First, we assume that $A$ is a Köthe matrix on $\mathbb{N} \times \mathbb{N}$ satisfying

1. $a_{n}(k,j)=1 \quad \forall k \geq n, \forall j \in \mathbb{N}$
2. $\lim_{j \to \infty} \frac{1}{a_{n+1}(n,j)} = 0 \quad \forall n \in \mathbb{N}$.

Let $\mathcal{G}$ be the family of all the subsets $B$ of $\mathbb{N} \times \mathbb{N}$ of the form $B = \bigcup_{k \geq p} \{ k \} \times B_k$, where $p = p(B) \geq h$, and $B_k = \{ 1, 2, ..., n_k \}$, $n_k \in \mathbb{N}$.

Now, given $h \in \mathbb{N}$ and $(j_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, we put

$$M(h,(j_k)_{k \geq h}) := \{ \chi_B : B \in \mathcal{G} \text{ and } j_k \in B_k \text{ for } k \geq p(B) \},$$

where $\chi_B$ denotes the characteristic function of $B$.

Hence $\{ M(h,(j_k)_{k \geq h}) : h \in \mathbb{N}, (j_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}} \}$ generates a filter $\mathcal{F}$ in $\lambda_{\infty}(A)$. We will see that $\mathcal{F}$ satisfies the conditions (a) and (b) in Lemma above.

Given $n \in \mathbb{N}$, since $a_{n}(k,j)=1$ for all $k \geq n$, we have that

$$M(h,(j_k)_{k \geq n}) \subseteq U_n := \{ (x_{k,j}) : \sup_{k,j} a_{n}(k,j) \left| x_{k,j} \right| \leq 1 \}$$

for all $h \geq n$ and arbitrary $(j_k)_{k \geq h}$. Therefore, condition (a) holds.

To check the other condition, we take a bounded set in $\lambda_{\infty}(A)$. We may assume that it is of the form $\tilde{v}(l_1)$. Using (2), we obtain that

$$\forall k \in \mathbb{N} \exists j_k \in \mathbb{N} \forall j \geq j_k : \left| x_{k,j} \right| < 1/4 \forall x = (x_{k,j}) \in \tilde{v}(l_1),$$

On the other hand, if $x$ belongs to the convex hull of $M(1,(j_k)_{k \in \mathbb{N}})$, $x$ can be expressed as a convex sum $x = \sum_{r=1}^{m} \chi_{r} \cdot \chi_{r}$, where $\chi_{r} \in M(1,(j_k)_{k \in \mathbb{N}})$, $r > 0$, $1 \leq r \leq m$, and $\chi_{1} + ... + \chi_{m} = 1$. By the definition of $M(1,(j_k)_{k \in \mathbb{N}})$, we may find $h \geq 1$ such that $j \in B_{p}$ for all $p \geq h$, and $1 \leq r \leq m$, whence, $x(p,j) = 1$ for all $p \geq h$. Therefore,

$$(2\tilde{v}(l_1) + \frac{1}{4} U_1(<1)) \cap (\text{co}(M(1,(j_k)_{k \in \mathbb{N}}))) + \frac{1}{4} U_1(<1)) = \emptyset$$

where $U_1(<1)$ denotes the open unit ball for $a_1$. Since these are disjoint open
sets, of which the first one is absolutely convex and the second one convex, we can find \( u \in (\lambda(\mathcal{A}))' \) such that \( u \in (2\tilde{\nu}(l_\infty)_1 + \frac{1}{4}U_1(<1))^o \) and \( |u(x)| > 1 \) for each \( x \in co(M(1,(j_k)_{k \in \mathbb{N}})) + \frac{1}{4}U_1(<1)) \). Therefore,

\[
M(1,(j_k)_{k \in \mathbb{N}}) \cap (\tilde{\nu}(l_\infty)_1 + \frac{1}{2}\{u\}^o) = \emptyset,
\]

Now we apply the former Lemma to obtain that there is a noncontinuous locally bounded linear form on \( (\lambda_\infty(\mathcal{A}))' \).

In the general case, if \( \lambda_\infty(\mathcal{A}) \) is not distinguished, according with [1] and [2], it has a sectional subspace isomorphic to \( \lambda_\infty(\mathcal{B}) \) where \( \mathcal{B} \) is a Köthe matrix on \( \mathbb{N} \times \mathbb{N} \) satisfying (1) and (2) above. Since sectional subspaces are complemented, we are done.

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**References**


