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ON RELATIVISTIC OSCILLATORS THROUGH SUPERSYMMETRY AND FOLDY-WOUTHUYSEN TRANSFORMATIONS

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ABSTRACT

A new relativistic oscillator is studied through the general connection between the Dirac formulation and supersymmetric quantum mechanics. Unitary equivalences related to Foldy-Wouthuysen transformations are particularly pointed out. The new F.W. representation is visited.

RESUME

Nous étudions un nouvel oscillateur relativiste à travers la relation générale entre la formulation de Dirac et la mécanique quantique supersymétrique. Des équivalences unitaires reliées aux transformations de Foldy-Wouthuysen sont particulièrement exploitées. La nouvelle représentation de F.W. est analysée.

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1. INTRODUCTION

Relativistic particle descriptions such as the Dirac wave equation have been already connected[1][2] with (non relativistic) supersymmetric considerations[3], mainly when particles are in interaction with constant magnetic fields[1]. This correspondence has been very recently refined and extended[2] to the cases of arbitrary interactions through the use of (generalized) Foldy-Wouthuysen (F.W.) transformations[4]. More specifically, the odd part of an arbitrary Dirac hamiltonian has been identified with anyone of the supercharges leading to the corresponding supersymmetric system. It has also been shown[2] that convenient F.W. transformations expressed in terms of the above mentioned supercharges lead to a diagonalized relativistic hamiltonian essentially seen as the square root of the corresponding supersymmetric hamiltonian.

Many other recent papers[5] have been mainly devoted to the application of these considerations to the harmonic oscillator, so that a relativistic oscillator can now be defined by the Dirac hamiltonian associated with the supersymmetrized version of such a system through a convenient F.W. transformation[6]. It has to be mentioned that spin-orbit coupling terms[7][8] play a prominent role in this supersymmetric point of view. Moreover, all these studies put the accent on only one relativistic oscillator (associated with only one of the two possible supercharges). We propose here to concentrate on the other already mentioned relativistic oscillator[2] and to study its consequences on the corresponding F.W. transformation as well as on its Lie invariance superalgebra, for example.

The contents of this paper are then distributed as follows. In Section 2, we recall the connections between supersymmetric and relativistic formulations and we illustrate such developments on harmonic oscillators. Section 3 is then devoted to the study of the system that we will call the second relativistic oscillator. In particular, we show the unitary equivalence between this system and the already studied one[5]. We conclude by some additional comments in Section 4.

2. SUPERSYMMETRY AND DIRAC HAMILTONIANS

Supersymmetric quantum mechanical systems are described by N odd (i.e. block-antidiagonalized) supercharges Q_j $(j = 1,2,\ldots,N)$ and one even (i.e. block-diagonalized) hamiltonian H_{SS}. These operators have to satisfy the following relations[3]

$$\{Q_j, Q_l\} = 2\delta_{jl}\, H_{SS} \;, \tag{2.1}$$

$$[H, Q_j] = 0 \;, \quad j,l = 1,\ldots,N \;, \tag{2.2}$$

characterizing the superalgebra sqm (N). Restricting our considerations to the $N=2$-context, it is well known that a convenient realization of the two possible supercharges is given in terms of the superpotential $W(\vec{x})$ by[8]

$$Q_1 = \frac{1}{\sqrt{2}} (\vec{p}\cdot\vec{\varphi}^1 + \vec{\nabla}W\cdot\vec{\varphi}^2) \qquad (2.3)$$

and

$$Q_2 = \frac{1}{\sqrt{2}} (\vec{p}\cdot\vec{\varphi}^2 - \vec{\nabla}W\cdot\vec{\varphi}^1) \qquad (2.4)$$

where the fermionic quantities satisfy in general

$$\{\varphi_j^a,\varphi_k^a\} = 2\delta_{jk} \ , \ \{\varphi_j^1,\varphi_k^2\} = 2\Xi_{jk} \ , \ \Xi_{jk} = -\Xi_{kj} \ ; \ j,k = 1,2,3 \ ; \ a = 1,2. \qquad (2.5)$$

We are thus led to the following supersymmetric hamiltonian (see Eq. (2.1))

$$H_{SS} = \frac{1}{2} (\vec{p}^2 + (\vec{\nabla}W)^2) - \frac{i}{4} (\partial_j\partial_k W)[\varphi_j^1,\varphi_k^2] - \frac{1}{2} [(\partial_j W)p_k - (\partial_k W)p_j]\Xi_{jk} \qquad (2.6)$$

where spin-orbit coupling terms explicitely appear if the antisymmetric tensors Ξ_{jk}'s are non vanishing. When three spatial dimensions are concerned, a convenient realization of (2.5) is given by[8]

$$\varphi_j^1 = \sigma_j \otimes \sigma_1 \ , \ \varphi_j^2 = \sigma_j \otimes \sigma_2 . \qquad (2.7)$$

We recognize here, in the standard representation, the Dirac matrices α_j and $i\alpha_j\beta$, respectively.

Applying these considerations to the specific case of the harmonic oscillator characterized by

$$W(\vec{x}) = \frac{1}{2} m\omega\vec{x}^2 , \qquad (2.8)$$

we obtain the following supersymmetric hamiltonian[8]

$$H_{SS}^{HO} = \frac{1}{2} (\vec{p}^2 + m^2\omega^2\vec{x}^2) + \frac{1}{2} m\omega(3\sigma_0 + 2\vec{L}\cdot\vec{\sigma}) \otimes \sigma_3 . \qquad (2.9)$$

It has been shown[2] that the corresponding Dirac hamiltonian writes

$$H_{D,1}^{HO} = \sqrt{2}\, Q_1 + m\beta = \vec{\alpha}\cdot(\vec{p} + im\omega\beta\vec{x}) + m\beta \qquad (2.10)$$

and it is also well known that the associated diagonalized operator is

$$H_{F.W.,1}^{HO} = U_1 H_{D,1}^{HO} U_1^\dagger = \beta(2H_{SS}^{HO} + m^2)^{\frac{1}{2}} \qquad (2.11)$$

with

$$U_1 = \frac{E + m + \sqrt{2}\ \beta Q_1}{[2E(E + m)]^{1/2}} = \exp\ (iS_1)\ ,\ S_1 = -\frac{i}{2}\ \beta Q_1 H^{-1}\theta\ ,$$

$$E = E_p = (\vec{p}^2 + m^2)^{1/2}\ ,\ \tan\theta = \sqrt{2}\ \frac{H}{m}\ ,\ [\theta,\beta] = 0\ . \tag{2.12}$$

The notation H stands here for an <u>even</u> operator defined as the positive square root of the supersymmetric hamiltonian.

3. THE SECOND RELATIVISTIC OSCILLATOR

Let us now propose to study the second possible Dirac hamiltonian[2] describing oscillatorlike systems

$$H^{HO}_{D,2} = \sqrt{2}\ Q_2 + m\beta = \vec{\alpha}\cdot(i\beta\vec{p} - m\omega\vec{x}) + m\beta \tag{3.1}$$

according to Eqs. (2.3), (2.7) and (2.8). Let us first emphasize the <u>covariant</u> formulation of the relativistic equation

$$i\ \frac{\partial\psi}{\partial t} = H^{HO}_{D,2}\ \psi\ . \tag{3.2}$$

It is readily obtained as

$$(\tilde{\gamma}^\mu p_\mu - m - m\omega\eta_\mu\tilde{\sigma}^{\mu\nu}x_\nu)\psi = 0 \tag{3.3}$$

where we have defined

$$\tilde{\gamma}^0 = \beta\ ,\ \tilde{\gamma}^j = i\alpha^j\ ,\ \tilde{\sigma}^{\mu\nu} = \frac{1}{2i}\ [\tilde{\gamma}^\mu,\tilde{\gamma}^\nu]\ ,\ (\eta_\mu) = (1,0,0,0)\ . \tag{3.4}$$

These matrices are such that

$$\{\tilde{\gamma}^\mu,\tilde{\gamma}^\nu\} = 2g^{\mu\nu}\ ,\ g^{00} = 1\ ,\ g^{jj} = -1\ . \tag{3.5}$$

They thus obey characteristic anticommutation relations identical with those satisfied by the usual γ^μ's.

Let us then consider the following (new) F.W. transformation

$$U_2 = \exp\ (iS_2)\ ,\ iS_2 = i\vec{\alpha}\cdot\vec{\pi}\theta = -\ i\vec{\alpha}\cdot(\vec{p} + im\omega\beta\vec{x})\theta\ , \tag{3.6}$$

in order to get the diagonalized form of the hamiltonian (3.1).

Applying this unitary operator on $H^{HO}_{D,2}$, we thus obtain

$$H^{HO}_{F.W.,2} = U_2 H^{HO}_{D,2} U_2^{\dagger} = H^{HO}_{D,2} (U_2^{\dagger})^2 \qquad (3.7)$$

with

$$\{H^{HO}_{D,2}, S_2\} = 0. \qquad (3.8)$$

Straightforward calculations finally lead to

$$H^{HO}_{F.W.,2} = (i\beta\vec{\alpha}\cdot\vec{\pi} + m\beta)(e^{-i\vec{\alpha}\cdot\vec{\pi}\theta})^2$$

$$= i\beta\vec{\alpha}\cdot\vec{\pi}\ (\cos 2H\theta - \frac{m}{H} \sin 2H\theta) + \beta(m \cos 2H\theta + H \sin 2H\theta) \qquad (3.9)$$

if

$$H^2 = (\vec{\alpha}\cdot\vec{\pi})^2 . \qquad (3.10)$$

Asking for the <u>even</u> character of $H^{HO}_{F.W.,2}$, we are led to

$$\tan 2H\theta = \frac{H}{m} . \qquad (3.11)$$

The angle θ is thus the one already appearing in (2.12). The requirement (3.11) finally implies that

$$H^{HO}_{F.W.,2} = \beta(m \cos 2H\theta + H \sin 2H\theta)$$

$$= \beta\sqrt{2H^{HO}_{SS} + m^2} \equiv H^{HO}_{F.W.,1} . \qquad (3.12)$$

Both relativistic oscillators are thus associated with the same (nonrelativistic) supersymmetric hamiltonian. This suggests us to search for a (third) unitary operator connecting the two above mentioned relativistic hamiltonians. We can actually show that

$$H^{HO}_{D,2} = U_3 H^{HO}_{D,1} U_3^{\dagger} \qquad (3.13)$$

with

$$U_3 = C(1 + i\beta) , \qquad (3.14)$$

the condition $|C|^2 = \frac{1}{2}$ ensuring the unitarity of such an operator.

This evidently implies the (physical) equivalence of these two relativistic oscillators. In particular, the Lie invariance superalgebra associated with $H_{D,2}^{HO}$ is isomorphic to the one[2] corresponding to $H_{D,1}^{HO}$, i.e. $so(3) \oplus g_1 \oplus gl(1)$, g_1 being an abelian superalgebra whose order is two.

Let us end this section by characterizing some main operators of the F.W. representation associated with this relativistic oscillator. Through the unitary transformation (3.6) which can also be written on the quotient form

$$U_2 = \frac{E + m + \sqrt{2}\ \beta Q_2}{\sqrt{2E(E + m)}}\ , \tag{3.15}$$

we can determine all the new F.W. operators $O_{F.W.}$ obtained from the Dirac ones O_D by

$$O_{F.W.} = U_2 O_D U_2^{-1}\ . \tag{3.16}$$

Let us only quote the position and momentum operators

$$\begin{aligned}\vec{x}_{F.W.} &= U_2 \vec{x}_D U_2^{-1} \\ &= \left[1 + \frac{m^2\omega^2\vec{x}^2 - m\omega\beta(3 + 2\vec{L}\cdot\vec{\Sigma})}{2E(E + m)}\right]\vec{x} + \frac{\vec{\alpha}}{2E} + i\ \frac{(\vec{\Sigma}\cdot\vec{p})\vec{\Sigma} + im\omega\beta(\vec{\Sigma}\cdot\vec{x})\vec{\Sigma}}{2E(E + m)}\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}\vec{p}_{F.W.} &= U_2 \vec{p}_D U_2^{-1} \\ &= \left[1 + \frac{m^2\omega^2\vec{x}^2 - m\omega\beta(2\vec{L}\cdot\vec{\Sigma} + 3)}{2E(E + m)}\right]\vec{p} - \frac{m\omega}{2E(E + m)}\left[\beta(\vec{\Sigma}\cdot\vec{p})\vec{\Sigma} + im\omega(\vec{\Sigma}\cdot\vec{x})\vec{\Sigma}\right]\end{aligned} \tag{3.18}$$

where $\vec{L} = \vec{r} \wedge \vec{p}$ and $\vec{\Sigma} = \vec{\sigma} \otimes I_2$.

In order to get the right interpretation after Foldy and Wouthuysen[4], we also introduce the <u>mean</u> position operator

$$\vec{X}_{F.W.} = \vec{x}_D = \vec{x} \tag{3.19}$$

leading to

$$\dot{\vec{X}}_{F.W.} = \beta\ \frac{\vec{p} - m\omega\vec{\Sigma} \wedge \vec{x}}{E}\ . \tag{3.20}$$

We immediately observe that, as in the free case ($\omega = 0$), we recover here the continuous character of the spectrum of the <u>mean</u> velocity operator as well as its correspondence with its nonrelativistic analog <u>while</u>, in the Dirac representation, we have $\dot{\vec{x}}_D = i\beta\vec{\alpha}$ admitting only the unacceptable discrete eigenvalues ± 1.

In an analogous way, we can introduce the <u>mean</u> momentum operator

$$\vec{P}_{F.W.} = \vec{p}_D = \vec{p} \tag{3.21}$$

which gives, with respect to the hamiltonian (3.1)

$$\dot{\vec{P}}_{F.W.} = - \beta \frac{m\omega}{E} (\vec{\Sigma} \wedge \vec{p} + m\omega\vec{x}) \ , \tag{3.22}$$

with an interesting interpretation in connection with the nonrelativistic limit.

Both formulas (3.20) and (3.22) also are meaningful when the free case is considered from this interacting context characterizing the second harmonic oscillator.

4. COMMENTS AND CONCLUSIONS

We have exploited the connections between relativistic and (nonrelativistic) supersymmetric formulations by studying a new relativistic oscillator associated with another possible supercharge. It has been shown that this new system corresponds to the nonrelativistic limit already present in a previously considered context, e.g. a supersymmetric oscillator with spin-orbit coupling terms. This result is relevant to the fact that the two relativistic oscillators are (unitarily) equivalent systems as shown through our transformation (3.14). Such an equivalence can then be exploited at different levels making in particular clear that all the information concerning the degeneracies of the associated spectrum[5], as well as the (super)algebras subtended by these degeneracies[5], are still (and directly) available here.

Open further problems in connection with this equivalence on relativistic hamiltonians describing other interactions can be expected to be treated similarly. We plan to come back on this question.

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REFERENCES

[1] R. JACKIW, Phys.Rev. D29,2375 (1984) ; R.J. HUGHES, V.A. KOSTELECKY and M.N. NIETO, Phys.Lett. B171,226 (1986) ; Phys.Rev. D34,1100 (1986) ; S. DURAND, J.M. LINA and L. VINET, in *Group Theoretical Methods in Physics, Proceedings of the XVIIth International Colloquium*, Ste ADELE, Québec,Canada, 1988, ed. by Y. SAINT AUBIN and L. VINET (World Scientific,Singapore,1989) ; Lett.Math. 17, 289 (1989).

[2] J. BECKERS and N. DEBERGH, Phys.Rev. D42,1255 (1990).
[3] E. WITTEN, Nucl.Phys. B188,513 (1981).
[4] L.L. FOLDY and S.A. WOUTHUYSEN, Phys.Rev. 78,29 (1950).
[5] M. MOSHINSKY and A. SZCZEPANIAK, J.Phys. A22,L817 (1989) ; J. BENITEZ, R.P. MARTINEZ y ROMERO, H.N. NUNEZ-YEPEZ and A.L. SALAS-BRITO, Phys.Rev.Lett. 64, 1643 (1990) ; C. QUESNE and M. MOSHINSKY, J.Phys. A23,2263 (1990) ; C. QUESNE, Int.J.Mod.Phys. A6,1567 (1991) ; O.L. de LANGE, J.Math.Phys. 32,1296 (1991) ; J.Phys. A24,667 (1991).
[6] M. MORENO and A. ZENTELLA, J.Phys. A22,L821 (1989).
[7] A.B. BALANTEKIN, Ann.Phys. (N.Y.) 164,277 (1985).
[8] J. BECKERS, D. DEHIN and V. HUSSIN, J.Phys. A20,1137 (1987) ; J.Phys. A21,651 (1988).