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The Beurling-Roumieu space $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega')$

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Abstract

In [10], we started the study of the countable intersections of non quasi-analytic classes of ultradifferentiable functions. In particular the Beurling and the Roumieu intersections coincide as vector spaces. We then studied the countable Beurling-Beurling intersections in [11] and [12], up to tensor product characterizations and kernel theorems. In this paper, we present a study of the countable Beurling-Roumieu intersections.

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1 Introduction

In [10], we introduced countable intersections of non quasi-analytic classes of ultradifferentiable functions of Beurling and of Roumieu type. In particular, we proved that as vector spaces, they coincide but are in general new spaces, developed some general properties (about denseness, for example) and gave a condition under which these spaces are nuclear.

In [11] and [12], we introduced the countable Beurling-Beurling intersections

$$\mathcal{E}_{(\mathfrak{M}\times\mathfrak{M}')}(\Omega\times\Omega'), \mathcal{D}_{(\mathfrak{M}\times\mathfrak{M}')}(\Omega\times\Omega') \text{ and } \mathcal{D}_{(\mathfrak{M}\times\mathfrak{M}')}(K\times K'),$$

of non quasi-analytic classes of ultradifferentiable functions and presented a study of their properties up to tensor product characterizations and kernel theorems.

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In this paper we study the remaining Beurling-Roumieu case, i.e. we study the spaces

$$\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K \times K') \text{ and } \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega').$$

It is appropriate to mention that a huge literature exists on non quasi-analytic classes of ultradifferentiable functions, and on their duals, the spaces of ultradistributions. Basic references are [4] and [1]. Moreover kernel theorems have already been obtained in this setting, such as [5], [6] and [7].

2 Notations

All functions we consider are complex-valued and all vector spaces are $\mathbb{C}$-vector spaces. The euclidean norm of $x \in \mathbb{R}^n$ is designated by $|x|$. If f is a function on $A \subset \mathbb{R}^n$, then $\|f\|_A$ is defined by $\|f\|_A = \sup_{x\in A} |f(x)|$.

If E is a Hausdorff locally convex topological vector space (in short, a locally convex space), we designate by E' its topological dual endowed with the strong topology $\beta(E', E)$. We refer to [3] and [8] for properties of locally convex spaces.

Whenever $\boldsymbol{m}$ is a sequence $(m_p)_{p\in\mathbb{N}_0}$ of real numbers, the notation $\boldsymbol{M}$ designates as usual the sequence $(M_p)_{p\in\mathbb{N}_0}$ where $M_p = m_0 \ldots m_p$ for every $p \in \mathbb{N}_0$. Such a sequence $\boldsymbol{m}$ is:
(a) *normalized* if $m_0 = 1$ and $m_p \geq 1$ for every $p \in \mathbb{N}$;
(b) *non quasi-analytic* if $\sum_{p=0}^{\infty} 1/m_p < \infty$.

A *semi-regular matrix* is a matrix of the type $\mathfrak{m} = (m_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$ of real numbers such that, for every $j \in \mathbb{N}$, the sequence $\boldsymbol{m}_j = (m_{j,p})_{p\in\mathbb{N}_0}$ is increasing, normalized, non quasi-analytic and such that:
(a) $m_{j,p} \geq m_{j+1,p}$ for every $p \in \mathbb{N}_0$;
(b) $\lim_{p\to\infty} m_{j+1,p}/m_{j,p} = 0$.

From now on and unless explicitely stated,
a) r and s are positive integers;
b) Ω and Ω' are non empty open subsets of $\mathbb{R}^r$ and $\mathbb{R}^s$ respectively;
c) $\boldsymbol{m}' = (m'_p)_{p\in\mathbb{N}_0}$ is an increasing, normalized and non quasi-analytic sequence and we set $\boldsymbol{M}' = (M'_p)_{p\in\mathbb{N}_0}$. Moreover, from Paragraph 6 on, we require that $\boldsymbol{M}'$ *is stable under differential operators*, i.e. $\boldsymbol{M}'$ verifies the condition (M.2)' of [4]: there are constants A, $H > 1$ such that $M'_{p+1} \leq AH^p M'_p$ for every $p \in \mathbb{N}_0$.
d) $\mathfrak{m}$ is a semi-regular matrix and we set $\boldsymbol{M}_j = (M_{j,p})_{p\in\mathbb{N}_0}$ for every $j \in \mathbb{N}$ and $\mathfrak{M} = (M_{j,p})_{j\in\mathbb{N},p\in\mathbb{N}_0}$.

3 The spaces $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K\times K')$ and $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega\times\Omega')$

Definition. For every $j\in\mathbb{N}$, $h>0$ and $k>0$, $\mathcal{B}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(\mathbb{R}^r\times\mathbb{R}^s)$ is the Banach space of the C^∞-functions f on $\mathbb{R}^r\times\mathbb{R}^s$ such that

$$\|f\|_{j,h,k} := \sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s} \frac{\|\mathrm{D}^{(\alpha,\beta)}f\|_{\mathbb{R}^r\times\mathbb{R}^s}}{h^{|\alpha|}k^{|\beta|}M_{j,|\alpha|}M'_{|\beta|}} < \infty$$

endowed with the norm $\|.\|_{j,h,k}$.

For every $h>0$ and $k>0$, the Fréchet space $\mathcal{B}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(\mathbb{R}^r\times\mathbb{R}^s)$ is the projective limit of the spaces $\mathcal{B}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(\mathbb{R}^r\times\mathbb{R}^s)$.

Definition. Let us introduce our main spaces $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(K\times K')$ and $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega\times\Omega')$.

With the notations $j\in\mathbb{N}$, $h>0$, $k>0$, K a non void compact subset of $\mathbb{R}^r$ and K' a non void compact subset of $\mathbb{R}^s$, we successively set:
a) $\mathcal{D}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(K\times K')$ is the Banach subspace of $\mathcal{B}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(\mathbb{R}^r\times\mathbb{R}^s)$ the elements of which have their support contained in $K\times K'$;
b) $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(K\times K') = \underset{j\in\mathbb{N}}{\underleftarrow{\lim}}\mathcal{D}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(K\times K')$;
c) $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K\times K') = \underset{h>0}{\underleftarrow{\lim}}\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(K\times K')$;
d) $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(K\times K') = \underset{k>0}{\underrightarrow{\lim}}\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K\times K')$;
e) $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega\times\Omega') = \underset{K\Subset\Omega,K'\Subset\Omega'}{\underrightarrow{\lim}}\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(K\times K')$.

Definition. A subset B of $\mathbb{R}^n$ *has the local displacement property* if every $x\in B$ has a neighbourhood W such that, for every $\varepsilon>0$, there is $a\in\mathbb{R}^n$ such that $|a|\le\varepsilon$ and $a+(B\cap W)\subset B^\circ$.

If $B_1,\ldots,B_q$ are a finite number of closed balls of $\mathbb{R}^n$ such that $B_j\cap B_k\neq\emptyset$ implies $B_j^\circ\cap B_k^\circ\neq\emptyset$, one can chek that their union has the local displacement property. Moreover if the compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$ have the local displacement property, it is clear that $K\times K'$ also has this property.

Therefore, we may consider exhaustions $(K_n)_{n\in\mathbb{N}}$ and $(K'_n)_{n\in\mathbb{N}}$ of Ω and Ω' respectively, made of non void compact sets having the local displacement property and such that $K_n\subset K_{n+1}^\circ$ and $K'_n\subset K'^\circ_{n+1}$ for every $n\in\mathbb{N}$. In particular, we have

$$\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega\times\Omega') = \underset{n\in\mathbb{N}}{\underrightarrow{\lim}}\,\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(K_n\times K'_n) = \underset{n\in\mathbb{N}}{\underrightarrow{\lim}}\,\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),n}(K_n\times K'_n).$$

Proposition 9.1 of [9] states that, for every $j \in \mathbb{N}$, $0 < h < h'$, $0 < k < k'$ and compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$, the canonical injection

$$J \colon \mathcal{D}^{(\boldsymbol{M}_j \times \boldsymbol{M}'),h,k}(K \times K') \to \mathcal{D}^{(\boldsymbol{M}_j \times \boldsymbol{M}'),h',k'}(K \times K')$$

is a compact linear map. Therefore
a) $\mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'),k}(K \times K')$ *is a* (FS)-*space*;
b) $\mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'\}}(K \times K')$ *and* $\mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'\}}(\Omega \times \Omega')$ *are* (LFS)-*spaces*.

For the sake of clarity, let us recall the following facts (cf. [4]).

If $\boldsymbol{m}$ is an increasing, normalized and non quasi-analytic sequence, then
a) for every $h > 0$ and non void compact subset K of $\mathbb{R}^n$, $\mathcal{D}^{(\boldsymbol{M}),h}(K)$ is *the Banach space* of the functions $f \in \mathcal{E}(\mathbb{R}^n)$ having their support contained in K and such that

$$\|f\|_h := \sup_{\alpha \in \mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_{\mathbb{R}^n}}{h^{|\alpha|} M_{|\alpha|}} < \infty$$

endowed with the norm $\|.\|_h$;
b) *the* (FS)-*space* $\mathcal{D}^{(\boldsymbol{M})}(K)$ is the projective limit of the spaces $\mathcal{D}^{(\boldsymbol{M}),h}(K)$;
c) *the* (DFS)-*space* $\mathcal{D}^{\{\boldsymbol{M}\}}(K)$ is the inductive limit of the spaces $\mathcal{D}^{(\boldsymbol{M}),h}(K)$.
d) the (DFS)-space $\mathcal{D}^{\{\boldsymbol{M}\}}(\Omega)$ is the inductive limit of the spaces $\mathcal{D}^{\{\boldsymbol{M}\}}(K_n)$ hence of the spaces $\mathcal{D}^{(\boldsymbol{M}),n}(K_n)$.

This leads to the following definitions:
e) *the* (FS)-*space* $\mathcal{D}^{(\mathfrak{M})}(K)$ is the projective limit of the spaces $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$ and, in order to avoid any confusion, we shall denote by $\|.\|_{j,h}$ the norm $\|.\|_h$ of the space $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$;
f) *the* (LFS)-*space* $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ is the strict inductive limit of the spaces $\mathcal{D}^{(\mathfrak{M})}(K_n)$.

4 First properties

Proposition 4.1 *For every $f \in \mathcal{D}^{(\mathfrak{M})}(K)$ and $g \in \mathcal{D}^{(\boldsymbol{M}'),k}(K')$, it is immediate that $f \otimes g$ belongs to $\mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'),k}(K \times K')$ and verifies $\|f \otimes g\|_{j,h,k} = \|f\|_{j,h} \|g\|_k$ for every $j \in \mathbb{N}$ and $h > 0$.*

Therefore

$$\otimes \colon \mathcal{D}^{(\mathfrak{M})}(K) \times \mathcal{D}^{(\boldsymbol{M}'),k}(K') \to \mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'),k}(K \times K'); \quad (f,g) \mapsto f \otimes g$$

is a well defined continuous bilinear map and the canonical injection from $\mathcal{D}^{(\mathfrak{M})}(K) \otimes_\pi \mathcal{D}^{(\boldsymbol{M}'),k}(K')$ into $\mathcal{D}^{(\mathfrak{M} \times \boldsymbol{M}'),k}(K \times K')$ is continuous.∎

Proposition 4.2 *The multiplication maps*

$$\Lambda \colon \mathcal{B}^{(\mathfrak{M} \times \boldsymbol{M}'),h,k}(\mathbb{R}^r \times \mathbb{R}^s)^2 \to \mathcal{B}^{(\mathfrak{M} \times \boldsymbol{M}'),2h,2k}(\mathbb{R}^r \times \mathbb{R}^s)$$

and

$$\Lambda\colon \mathcal{B}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(\mathbb{R}^r\times\mathbb{R}^s)\times\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(K\times K')\to\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),2h,2k}(K\times K')$$

defined by $\Lambda(f,g)=fg$ *are well defined, continuous and bilinear.*

Proof. This is a direct consequence of the Leibniz formula.∎

Definition. We designate by $\mathcal{E}^{(p!q!)}(\mathbb{R}^r\times\mathbb{R}^s)$ the space of the functions $f\in\mathcal{E}(\mathbb{R}^{r+s})$ such that

$$|f|_{K\times K',h}:=\sup_{(\alpha,\beta)\in\mathbb{N}_0^r\times\mathbb{N}_0^s}\frac{\|\mathrm{D}^{(\alpha,\beta)}f\|_{K\times K'}}{h^{|\alpha|+|\beta|}\alpha!\beta!}<\infty$$

for every $h>0$ and non void compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$, endowed with the system of norms $\{|.|_{K\times K',h} : K\Subset\mathbb{R}^r, K'\Subset\mathbb{R}^s, h>0\}$. It clearly is a Fréchet space but more can be said.

We let moreover $\mathcal{H}(\mathbb{C}^n)$ designate the Fréchet space of the holomorphic functions on $\mathbb{C}^n$ endowed with the topology of uniform convergence on the compact subsets of $\mathbb{C}^n$.

Then standard holomorphic properties provide directly that *the restriction map* $R\colon\mathcal{H}(\mathbb{C}^{r+s})\to\mathcal{E}^{(p!q!)}(\mathbb{R}^r\times\mathbb{R}^s)$ *defined by* $f\mapsto f|_{\mathbb{R}^r\times\mathbb{R}^s}$ *is a topological isomorphism.*

Proposition 4.3 *The multiplication map*

$$\lambda\colon\mathcal{E}^{(p!q!)}(\mathbb{R}^r\times\mathbb{R}^s)\times\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(K\times K')\to\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),2h,2k}(K\times K')$$

defined by $\Lambda(f,g)=fg$ *is well defined, continuous and bilinear.*

Proof. Direct consequence of Proposition 10.2 of [9].∎

5 Approximation and denseness

Notation. For every $m\in\mathbb{N}$, ψ_m is the function defined on $\mathbb{R}^r\times\mathbb{R}^s$ by

$$\psi_m(u,v):=m^{r+s}\pi^{-(r+s)/2}e^{-m^2|(u,v)|^2},\quad\forall(u,v)\in\mathbb{R}^r\times\mathbb{R}^s.$$

Let us note that, for every $f\in\mathcal{D}(\mathbb{R}^r\times\mathbb{R}^s)$ and $m\in\mathbb{N}$, the convolution product $f\star\psi_m$ belongs to $\mathcal{E}^{(p!q!)}(\mathbb{R}^r\times\mathbb{R}^s)$ since it has a holomorphic extension on $\mathbb{C}^{r+s}$.

Proposition 5.1 *For every $f \in \mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(K \times K')$, $(f \star \psi_m)_{m\in\mathbb{N}}$ is a sequence converging to f in $\mathcal{B}^{(\mathfrak{M}\times\boldsymbol{M}'),2h,2k}(\mathbb{R}^r \times \mathbb{R}^s)$.*

Proof. This is a direct consequence of Proposition 10.1 of [9] stating that, for every $j \in \mathbb{N}$ and $g \in \mathcal{D}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),h,k}(K \times K')$, $(g \star \psi_m)_{m\in\mathbb{N}}$ is a sequence converging to g in $\mathcal{B}^{(\boldsymbol{M}_j\times\boldsymbol{M}'),2h,2k}(\mathbb{R}^r \times \mathbb{R}^s)$.∎

Proposition 5.2 *Let the compact subsets H, K of $\mathbb{R}^r$ and H', K' of $\mathbb{R}^s$ be such that $H \subset K^\circ$ and $H' \subset K'^\circ$.*

a) *The closure of $\mathcal{D}^{(\mathfrak{M})}(K)\otimes\mathcal{D}^{(\boldsymbol{M}')}(K')$ in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),4h,4k}(K \times K')$ contains $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),h,k}(H \times H')$.*

b) *The closure of $\mathcal{D}^{(\mathfrak{M})}(K) \otimes \mathcal{D}^{(\boldsymbol{M}')}(K')$ in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),4k}(K \times K')$ contains $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(H \times H')$.*

Proof. One has just to proceed as in the proof of Proposition 5.2 of [9].∎

Notation. Given $c \in \mathbb{R}^n$ and a function f defined on $\mathbb{R}^n$, $\tau_c f$ designates the function defined on $\mathbb{R}^n$ by $\tau_c f(.) = f(. - c)$.

Proposition 5.3 *For every $c \in \mathbb{R}^r \times \mathbb{R}^s$, the map τ_c is an isometry of $\mathcal{B}^{(\mathfrak{M},\boldsymbol{M}'),h,k}(\mathbb{R}^r \times \mathbb{R}^s)$ onto itself.*

Moreover we have $\lim_{|c|\to 0} \tau_c f = f$ in $\mathcal{B}^{(\mathfrak{M},\boldsymbol{M}'),2h,2k}(\mathbb{R}^r \times \mathbb{R}^s)$ for every $f \in \mathcal{B}^{(\mathfrak{M},\boldsymbol{M}'),h,k}(\mathbb{R}^r \times \mathbb{R}^s)$.

Proof. This is a direct consequence of Proposition 10.4 of [9].∎

Proposition 5.4 *If the compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$ have the local displacement property, then*

a) *the closure of $\mathcal{D}^{(\mathfrak{M})}(K) \otimes \mathcal{D}^{(\boldsymbol{M}')}(K')$ in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),8k}(K \times K')$ contains the space $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K \times K')$;*

b) *$\mathcal{D}^{(\mathfrak{M})}(K) \otimes \mathcal{D}^{(\boldsymbol{M}')}(K')$ is dense in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(K \times K')$;*

c) *$\mathcal{D}^{(\mathfrak{M})}(\Omega) \otimes \mathcal{D}^{(\boldsymbol{M}')}(\Omega')$ is dense in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega \times \Omega')$.*

Proof. One has just to proceed as in the proof of Proposition 5.4 of [9].∎

6 Structure of the elements of the space $\mathcal{D}^{(\mathfrak{M},\boldsymbol{M}'\}}(K \times K')$

Let us recall that, from now on, we require that the sequence $\boldsymbol{M}'$ is *stable under differential operators*. This implies in particular that, for every $l \in \mathbb{N}$, there is a constant $A_l > 1$ such that $M'_{p+l} \leq A_l H^{lp} M'_p$ for every $p \in \mathbb{N}$ (in fact $A_l = A^l H^{l(l-1)/2}$ is suitable).

Proposition 6.1 *For every $k > 0$ and $\beta \in \mathbb{N}_0^s$, the differential map $D^{(0,\beta)}$ is a well defined continuous linear map from $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k/H^{|\beta|}}(K \times K')$ into $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')$.*

Therefore, for every $\beta \in \mathbb{N}_0^s$, $D^{(0,\beta)}$ is a well defined continuous linear map from $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K \times K')$ into itself.

Proof. This is clear since, for every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\},k}(K \times K')$, $j \in \mathbb{N}$ and $h > 0$, we easily get

$$\|\mathrm{D}^{(0,\beta)} f\|_{j,h,k} \leq A_{|\beta|} k^{|\beta|} H^{-|\beta|^2} \, \|f\|_{j,h,k/H^{|\beta|}} . \blacksquare$$

Proposition 6.2 a) *For every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')$, the function*

$$g \colon \mathbb{R}^s \to \mathcal{D}^{(\mathfrak{M})}(K); \quad y \mapsto f(.,y)$$

is well defined, C^∞ and such that $[D^\beta g(y)](.) = D^{(0,\beta)} f(.,y)$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$.

b) *Therefore, for every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K \times K')$, the function*

$$g \colon \mathbb{R}^s \to \mathcal{D}^{(\mathfrak{M})}(K); \quad y \mapsto f(.,y)$$

is well defined, C^∞ and such that $[D^\beta g(y)](.) = D^{(0,\beta)} f(.,y)$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$.

Proof. a) For every $j \in \mathbb{N}$, f belongs to $\mathcal{D}^{(\boldsymbol{M}_j,\boldsymbol{M}'),k}(K \times K')$. The proof of Proposition 11.1 of [9] then applies. $\blacksquare$

Proposition 6.3 *Let S belong to $\mathcal{D}^{(\mathfrak{M})}(K)'$.*

a) *If $k > 0$ is fixed, then, for every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')$, the function $\langle S, f(.,y)\rangle$ belongs to the space $\mathcal{D}^{(\boldsymbol{M}'),k}(K')$ and verifies the equality $D^\beta \langle S, f(.,y)\rangle = \langle S, D^{(0,\beta)} f(.,y)\rangle$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$. Moreover*

$$\Gamma_S \colon \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K') \to \mathcal{D}^{(\boldsymbol{M}'),k}(K'); \quad f \mapsto \langle S, f(.,y)\rangle$$

is a well defined continuous linear map.

b) *Therefore, for every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K \times K')$, the function $\langle S, f(.,y)\rangle$ belongs to $\mathcal{D}^{(\boldsymbol{M}')}(K')$ and verifies $D^\beta \langle S, f(.,y)\rangle = \langle S, D^{(0,\beta)} f(.,y)\rangle$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$. Moreover*

$$\Gamma_S \colon \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(K \times K') \to \mathcal{D}^{\{\boldsymbol{M}'\}}(K'); \quad f \mapsto \langle S, f(.,y)\rangle$$

is a well defined continuous linear map.

Proof. a) By the previous result, we know that $\langle S, f(.,y)\rangle$ is a C^∞-function with support contained in K', that verifies the announced equalities. As S belongs to $\mathcal{D}^{(\mathfrak{M})}(K)'$, for every $j \in \mathbb{N}$, there are $h(j) > 0$ and $C_j > 0$ such that $|\langle S, .\rangle| \leq C_j \|.\|_{j,h(j)}$ on $\mathcal{D}^{(\boldsymbol{M}_j)}(K)$, which leads to

$$\|\langle S, f(.,y)\rangle\|_k \leq C_j \|f\|_{j,h(j),k} < \infty$$

and we conclude at once.∎

Proposition 6.4 *For every $k > 0$, the map*

$$\Delta\colon \mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K \times K') \times \mathcal{D}^{(\mathfrak{M})}(K)' \to \mathcal{D}^{(\boldsymbol{M}'),k}(K'); \quad (f, S) \mapsto \langle S, f(.,y)\rangle$$

is well defined, bilinear and hypocontinuous.

Proof. As $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K \times K')$ is a Fréchet space and as $\mathcal{D}^{(\mathfrak{M})}(K)'$ is the strong dual of an (FS)-space, we only need to prove that the map Δ is well defined and sequentially continuous. This easily follows from the following argument.

Let the sequences $(f_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ converge to 0 in respectively $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'),k}(K \times K')$ and $\mathcal{D}^{(\mathfrak{M})}(K)'$. As $\{S_n : n \in \mathbb{N}\}$ is an equicontinuous subset of $\mathcal{D}^{(\mathfrak{M})}(K)'$, there are $j \in \mathbb{N}$, $h > 0$ and $C > 0$ such that

$$|\langle S_n, g\rangle| \leq C \|g\|_{j,h}, \quad \forall g \in \mathcal{D}^{(\mathfrak{M})}(K),$$

and this leads to

$$\|\langle S_n, f_n(.,y)\rangle\|_k \leq C \|f_n\|_{j,h,k}, \quad \forall n \in \mathbb{N}.∎$$

7 Structure of the elements of the space $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}')}(\Omega \times \Omega')$

Let us recall that $\boldsymbol{M}'$ is supposed stable under differential operators.

Proposition 7.1 *For every $\beta \in \mathbb{N}_0^s$, $D^{(0,\beta)}$ is a well defined continuous linear map from $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega \times \Omega')$ into itself.*

Proof. Direct consequence of Proposition 6.1.∎

Proposition 7.2 *For every $f \in \mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M}'\}}(\Omega \times \Omega')$, the function*

$$g\colon \mathbb{R}^s \to \mathcal{D}^{(\mathfrak{M})}(\Omega); \quad y \mapsto f(.,y)$$

is well defined, C^∞ and such that $[D^\beta g(y)](.) = D^{(0,\beta)} f(.,y)$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$.

Proof. Direct consequence of Proposition 6.2.∎

Proposition 7.3 *Let S belong to $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$.*

a) *For every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega')$, the function $\langle S, f(.,y)\rangle$ belongs to the space $\mathcal{D}^{\{\boldsymbol{M}'\}}(\Omega')$ and verifies the equality $D^{\beta}\langle S, f(.,y)\rangle = \langle S, D^{(0,\beta)} f(.,y)\rangle$ for every $\beta \in \mathbb{N}_0^s$ and $y \in \mathbb{R}^s$.*

b) *The linear map*

$$\Gamma_S \colon \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega') \to \mathcal{D}^{\{\boldsymbol{M}'\}}(\Omega'); \quad f \mapsto \langle S, f(.,y)\rangle$$

is continuous.

Proof. Direct consequence of Proposition 6.3.∎

Proposition 7.4 *The bilinear map*

$$\Delta \colon \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega') \times \mathcal{D}^{(\mathfrak{M})}(\Omega)' \to \mathcal{D}^{\{\boldsymbol{M}'\}}(\Omega'); \quad (f, S) \mapsto \langle S, f(.,y)\rangle$$

is hypocontinuous.

Proof. As the space $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega')$ and $\mathcal{D}^{(\mathfrak{M})}(\Omega)'$ are barrelled, it suffices to establish that the bilinear map Δ is separately continuous.

By the previous Proposition, we know that, for every $S \in \mathcal{D}^{(\mathfrak{M})}(\Omega)'$, $\Delta(., S)$ is a continuous linear map.

For every $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'\}}(\Omega \times \Omega')$, there are $k_0 > 0$, $K_0 \Subset \Omega$ and $K_0' \Subset \Omega'$ such that $f \in \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k_0}(K_0 \times K_0')$. As every $S \in \mathcal{D}^{(\mathfrak{M})}(\Omega)'$ belongs to $\mathcal{D}^{(\mathfrak{M})}(K_0)'$, we get

$$\langle S, f(.,y)\rangle \in \mathcal{D}^{(\boldsymbol{M}'),k_0}(K_0') \subset \mathcal{D}^{\{\boldsymbol{M}'\}}(K_0')$$

with

$$\|\langle S, f(.,y)\rangle\|_{k_0} = \sup_{\beta \in \mathbb{N}_0^s} \frac{\left\|\mathrm{D}^{\beta}\langle S, f(.,y)\rangle\right\|_{K_0'}}{k_0^{|\beta|} M'_{|\beta|}} = \sup_{g \in B} |\langle S, g\rangle|$$

where

$$B := \{\frac{\mathrm{D}^{(0,\beta)} f(.,y)}{k_0^{|\beta|} M'_{|\beta|}} : \beta \in \mathbb{N}_0^s, y \in K_0'\}$$

is a bounded subset of $\mathcal{D}^{(\mathfrak{M})}(K_0)$ hence of $\mathcal{D}^{(\mathfrak{M})}(\Omega)$ since, for every $j \in \mathbb{N}$ and $h > 0$, we have

$$\sup_{g \in B} \|g\|_{j,h} = \sup_{\substack{\beta \in \mathbb{N}_0^s \\ y \in K_0'}} \sup_{\alpha \in \mathbb{N}_0^r} \frac{\left\|\mathrm{D}^{(\alpha,\beta)} f(.,y)\right\|_{K_0}}{h^{|\alpha|} M'_{j,|\alpha|}} = \|f\|_{j,h,k_0} < \infty. ∎$$

8 Tensor product characterizations

Let us recall that $\boldsymbol{M}'$ is supposed stable under differential operators.

Definition. Given $k \in \mathbb{N}$, $S \in \mathcal{D}^{(\mathfrak{M})}(K)'$ and $T \in \mathcal{D}^{(\boldsymbol{M}'),k}(K')'$, we just obtained that the *semi-tensor product*

$$S \oslash T \colon \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K') \to \mathbb{C}; \quad f \mapsto \langle T, \langle S, f(.,y)\rangle\rangle$$

is a continuous linear functional.

Proposition 8.1 *For every $k \in \mathbb{N}$, the bilinear map*

$$\oslash \colon \mathcal{D}^{(\mathfrak{M})}(K)' \times \mathcal{D}^{(\boldsymbol{M}'),k}(K')' \to \mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')'; \quad (S,T) \mapsto S \oslash T$$

is continuous.

If the sets $\mathcal{P} \subset \mathcal{D}^{(\mathfrak{M})}(K)'$ and $\mathcal{Q} \subset \mathcal{D}^{(\boldsymbol{M}'),k}(K')'$ are equicontinuous, then $\mathcal{P} \oslash \mathcal{Q} = \{S \oslash T \colon S \in \mathcal{P}, T \in \mathcal{Q}\}$ is an equicontinuous subset of $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')'$.

Proof. The space $\mathcal{D}^{(\mathfrak{M})}(K)'$ is barrelled and $\mathcal{D}^{(\boldsymbol{M}'),k}(K')'$ is a Banach space. So it suffices to note that, for every closed absolutely convex neighbourhood U of 0 in $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')'$,

$$\bigcap_{\|T\|\leq 1} \{S \in \mathcal{D}^{(\mathfrak{M})}(K)' \colon S \oslash T \in U\}$$

is a barrel.∎

Notation. For every $k \in \mathbb{N}$, we designate by $G_k(K \times K')$ the topological vector subspace $\mathcal{D}^{(\mathfrak{M})}(K) \otimes \mathcal{D}^{(\boldsymbol{M}'),k}(K')$ of $\mathcal{D}^{(\mathfrak{M}\times \boldsymbol{M}'),k}(K \times K')$.

Definition. As in [10], we say that $\mathfrak{m}$ (or equivalently $\mathfrak{M}$) is *regular* if, for every $j \in \mathbb{N}$, there are constants $A(j)$, $H(j) > 1$ such that

$$M_{j+1,p+1} \leq A(j)H(j)^p M_{j,p}, \quad \forall p \in \mathbb{N}.$$

Proposition 8.2 *For every $k \in \mathbb{N}$,*
a) *the canonical injection $I \colon G_k(K \times K') \to \mathcal{D}^{(\mathfrak{M})}(K) \otimes_\epsilon \mathcal{D}^{(\boldsymbol{M}'),k}(K')$ is continuous.*
b) *if $\mathfrak{M}$ is regular, then the spaces $G_k(K \times K')$, $\mathcal{D}^{(\mathfrak{M})}(K) \otimes_\pi \mathcal{D}^{(\boldsymbol{M}'),k}(K')$ and $\mathcal{D}^{(\mathfrak{M})}(K) \otimes_\epsilon \mathcal{D}^{(\boldsymbol{M}'),k}(K')$ coincide algebraically and topologically.*

Proof. a) is a direct consequence of the second part of Proposition 8.1.
b) is then a direct consequence of Proposition 4.1 and of the fact that, by Proposition 8.2 of [10], as $\mathfrak{M}$ is regular, $\mathcal{D}^{(\mathfrak{M})}(K)$ is a nuclear space.∎

As $\mathcal{D}^{(\mathfrak{M})}(K)$ is a Fréchet space and $\mathcal{D}^{(\boldsymbol{M'}),k}(K')$ a Banach space, every separately continuous bilinear functional on $\mathcal{D}^{(\mathfrak{M})}(K) \times \mathcal{D}^{(\boldsymbol{M'}),k}(K')$ is continuous. Therefore, by [2], we have

$$\mathcal{D}^{(\mathfrak{M})}(K) \otimes_\pi \mathcal{D}^{(\boldsymbol{M'}),k}(K') = \mathcal{D}^{(\mathfrak{M})}(K) \otimes_i \mathcal{D}^{(\boldsymbol{M'}),k}(K')$$

where $\otimes_i$ denotes the inductive tensor product. So designating by $\widehat{G}_k(K\times K')$ the completion of $G_k(K \times K')$, i.e. the closure of the space $G_k(K \times K')$ in $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}),k}(K \times K')$, we obtain

$$\widehat{G}_k(K \times K') = \mathcal{D}^{(\mathfrak{M})}(K)\widehat{\otimes}_\pi\mathcal{D}^{(\boldsymbol{M'}),k}(K') = \mathcal{D}^{(\mathfrak{M})}(K)\widehat{\otimes}_i\mathcal{D}^{(\boldsymbol{M'}),k}(K')$$

if $\mathfrak{M}$ is regular.

Theorem 8.3 *If $\mathfrak{M}$ is regular,*
a) *and if the compact subsets K of $\mathbb{R}^r$ and K' of $\mathbb{R}^s$ have the local displacement property, then we have*

$$\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}\}}(K \times K') = \lim_{\rightarrow} \mathcal{D}^{(\mathfrak{M})}(K)\widehat{\otimes}_\pi\mathcal{D}^{(\boldsymbol{M'}),n}(K').$$

b) *we have*

$$\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}\}}(\Omega \times \Omega') = \lim_{\rightarrow} \mathcal{D}^{(\mathfrak{M})}(K_n)\widehat{\otimes}_\pi\mathcal{D}^{(\boldsymbol{M'}),n}(K'_n).$$

Proof. a) For every $n \in \mathbb{N}$, $\widehat{G}_n(K\times K')$ is a topological vector subspace of $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}),n}(K \times K')$ which, by Proposition 5.4, is continuously embedded in $\widehat{G}_{8n}(K \times K')$ hence

$$\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}\}}(K \times K') = \lim_{\rightarrow} \mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}),n}(K \times K') = \lim_{\rightarrow} \widehat{G}_n(K \times K').$$

b) As we have

$$\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}\}}(\Omega \times \Omega') = \lim_{\rightarrow} \mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}),n}(K_n \times K'_n),$$

the same procedure applies since $\widehat{G}_n(K_n \times K'_n)$ is a topological vector subspace of $\mathcal{D}^{(\mathfrak{M}\times\boldsymbol{M'}),n}(K_n \times K'_n)$ which is continuously embedded in the space $\widehat{G}_{8n}(K_n \times K'_n)$, a topological vector subspace of $\widehat{G}_{8n}(K_{8n} \times K'_{8n})$.∎

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