Tietze - Type Theorem for Locally Nonconical Convex Sets

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Abstract

A convex subset $Q$ of a real topological linear space $L$ is called locally nonconical at a point $q \in Q$ if and only if there exists a relative neighbourhood $Q_q$ of $q$ in $Q$ such that for every two points $x, y \in Q_q$ there is a relative neighbourhood $Q_x$ of $x$ in $Q$ such that $Q_x + \frac{1}{2}(y - x) \subseteq Q$. $Q$ is called locally nonconical if and only if this condition is satisfied for every two points $x, y \in Q$. It is proved that $Q$ is locally nonconical if and only if it is locally nonconical at every boundary point belonging to $Q$. This contributes to a recent work of Shell.

Key words: locally nonconical convex set, Tietze-type theorem.


Let $Q$ be a convex subset of a real topological linear space $L$. $Q$ is called locally nonconical at a point $q \in Q$ if and only if there exists a relative neighbourhood $Q_q$ of $q$ in $Q$ such that for every two points $x, y \in Q_q$, there is a relative neighbourhood $Q_x$ of $x$ in $Q$ such that $Q_x + \frac{1}{2}(y - x) \subseteq Q$. $Q$ is called locally nonconical if and only if the above condition is satisfied for every two points $x, y \in Q$. A motivation for studying such sets has been given in [5] and [6].

Tietze’s famous local characterization of convexity states in the general form that a closed connected subset $S$ of $L$ consisting exclusively of weak local convexity points is convex [7, Th 4.4]. In [1, Cor. 2.3] the author obtained Tietze’s theorem for an open connected subset $S$ of $L$. Further generalizations of this theorem can be found in [2] and [4], and its variant for starshaped sets in [3]. The purpose of this note is to prove a new local versus global result for a recently introduced class of locally nonconical convex sets. Observe that the analogous theorem for the subclass of strictly convex sets is trivial.

Theorem. A convex subset $Q$ of a real topological linear space $L$ is locally nonconical if and only if it is locally nonconical at every boundary point belonging to $Q$.

Proof. Observe that, by [6, Remark 1.1], every open convex set in $L$ is locally nonconical and it is easily seen that in this case the second part of the desired equivalence is vacuously true, so that we are done and assume in the sequel that $Q \cap \text{bdry}Q \neq \emptyset$. The necessity of the condition is obvious, so that we prove its sufficiency. The following three claims will be useful.
Firstly, let three pairwise distinct points \( x_1, x_2, x_3 \) in \( Q \) be given. Suppose that there is a relative neighbourhood \( Q_1 \) of \( x_1 \) in \( Q \) such that \( Q_1 + (x_2 - x_1) \subset Q \) and a relative neighbourhood \( Q_2 \) of \( x_2 \) in \( Q \) such that \( Q_2 + (x_3 - x_2) \subset Q \). We claim that then there is a relative neighbourhood \( Q_0 \) of \( x_1 \) in \( Q \) such that \( Q_0 + (x_3 - x_1) \subset Q \). By assumption, \( Q_1 = Q \cap U_{x_1} \) and \( Q_2 = Q \cap U_{x_2} \) for neighbourhoods \( U_{x_1} \) and \( U_{x_2} \) of points \( x_1 \) and \( x_2 \), respectively. Put \( Q_0 = Q \cap U_{x_1} \cap (U_{x_2} + (x_1 - x_2)) \) as the desired relative neighbourhood of \( x_1 \) in \( Q \). In fact, on one hand \( Q_0 + (x_3 - x_1) = (Q_0 + (x_2 - x_1)) + (x_3 - x_2) \subset U_{x_2} + (x_3 - x_2) \) and on the other hand \( Q_0 + (x_3 - x_1) \subset (Q_1 + (x_2 - x_1)) + (x_3 - x_2) \subset Q + (x_3 - x_2) \), so that \( Q_0 + (x_3 - x_1) \subset Q \cap U_{x_2} + (x_3 - x_2) = Q_2 + (x_3 - x_2) \subset Q \), as desired.

Secondly, select arbitrarily distinct points \( x, y \in Q \) with the property that for every point \( z \in [x, y] \) there is a relative neighbourhood \( Q_z \) of \( z \) in \( Q \) such that \( Q_z + \frac{1}{2}(y - x) \subset Q \). We claim that then for every natural number \( n \) there exists a relative neighbourhood \( Q_{x,n} \) of \( x \) in \( Q \) such that \( Q_{x,n} + (1 - \frac{1}{2^n})(y - x) \subset Q \). We apply the induction on \( n \). The case \( n = 1 \) is obvious, so that assume the truth for \( n - 1 \geq 1 \) and consider the case of \( n \).

Hence, \( Q_{x,n-1} + (1 - \frac{1}{2^{n-1}})(y - x) \subset Q \) for a relative neighbourhood \( Q_{x,n-1} \) of \( x \) in \( Q \). Put \( y_n = x + (1 - \frac{1}{2^n})(y - x) \). By assumption, there is a relative neighbourhood \( Q_{y_{n-1}} \) of \( y_{n-1} \) in \( Q \) such that \( Q \supseteq Q_{y_{n-1}} + \frac{1}{2}(y - y_{n-1}) = Q_{y_{n-1}} + (y - y_{n-1}) \) and, by induction hypothesis, \( Q_{x,n-1} + (y_{n-1} - x) \subset Q \). Consequently, the first claim above implies the existence of a relative neighbourhood \( Q_{x,n} \) of \( x \) in \( Q \) for which \( Q_{x,n} + (y_n - x) = Q_{x,n} + (1 - \frac{1}{2^n})(y - x) \subset Q \), as desired.

Thirdly, select arbitrarily distinct points \( x, y \in Q \). Observe that since \( Q \) is convex, it must be locally nonconvex at every interior point. Hence, \( Q \) is locally nonconvex at every point of \([x, y]\). For every point \( z \in [x, y] \) there exists, by [7, Th.1.4], a neighbourhood \( U_z \) of \( z \) in \( Q \cap U_{x} \) such that for every two points \( r, s \in Q \cap U_z \) there is a relative neighbourhood \( Q_z \) of \( r \) in \( Q \) such that \( Q_z + \frac{1}{2}(s - r) \subset Q \). Since \([x, y]\) is compact, it can be covered by finitely many such neighbourhoods. Denote by \( V_1, \ldots, V_m \) line segments relatively open in \([x, y]\) being their intersections with \([x, y]\). Without loss of generality, assume that none of them is covered by a union of others and that they are so arranged that \( V_1 \cap V_2 \neq \emptyset, V_1 \cap V_2 \neq \emptyset, \ldots, V_{m-1} \cap V_m \neq \emptyset \). We claim that for every \( j = 1, \ldots, m \) there exist a point \( v_j \in V_j \) and a relative neighbourhood \( Q_{z,j} \) of \( z \) in \( Q \) such that \( Q_{z,j} + (v_j - x) \subset Q \). We apply the induction on \( j \). The case \( j = 1 \) is clear when we take \( v_1 = x \). Suppose the truth for \( j - 1 \geq 1 \) and consider the case of \( j \). Select any point \( w \in V_{j-1} \cap V_j \) and, by virtue of induction hypothesis, a point \( v_{j-1} \) and a set \( Q_{z,j-1} \) as described above. For nontriviality, let \( v_{j-1} \neq V_j \). There is a natural number \( N \) such that \( v_j = v_{j-1} + (1 - \frac{1}{2^N})(w - v_{j-1}) \in V_j \). By the second claim above for the line segment \( V_{j-1} \) there exists a relative neighbourhood \( Q_{v_{j-1}} \) of \( v_{j-1} \) in \( Q \) such that \( Q \supseteq Q_{v_{j-1}} + (1 - \frac{1}{2^N})(w - v_{j-1}) = Q_{v_{j-1}} + (v_j - v_{j-1}) \). By induction hypothesis, there exists a relative neighbourhood \( Q_{z,j-1} \) of \( z \) in \( Q \) such that \( Q_{z,j-1} + (v_{j-1} - x) \subset Q \). The first claim above yields now a relative neighbourhood \( Q_{z,j} \) of \( z \) in \( Q \) such that \( Q_{z,j} + (v_j - x) \subset Q \), as desired.

To finish the proof select arbitrarily distinct points \( x, y \in Q \) and assume that \( \frac{1}{2}(x + y) \in V_k \) for some index \( 1 \leq k \leq m \). By the third claim above, there is a point \( v_k \in V_k \) and a relative neighbourhood \( Q_{z,k} \) of \( z \) in \( Q \) for which \( Q_{z,k} + (v_k - x) \subset Q \). Select a natural number \( K \) such that \( v_k = v_k + (1 - \frac{1}{2^K})v_k \). It is seen that \( v_k = v_k + (1 - \frac{1}{2^K})v_k \).
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\frac{1}{2}\gamma(v_k - u_k), \text{ so that, by the second claim above, there is a relative neighbourhood } Q_{x_k} \text{ of } u_k \\
\text{in } Q \text{ such that } Q \supseteq Q_{x_k} + (1 - \frac{1}{2}\gamma)(v_k - u_k) = Q_{x_k} + (\frac{1}{2}(y) - u_k). \text{ But } Q_{x, k} + (v_k - z) \subseteq Q, \\
\text{so that, by the first of above claims, there exists a relative neighbourhood } Q_x \text{ of } x \text{ in } Q \\
\text{for which } Q_x + (\frac{1}{2}(x + y) - x) = Q_x + \frac{1}{2}(y - x) \subseteq Q, \text{ as desired.} \\
\text{The proof is complete.} \Box
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References

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