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TIETZE-TYPE THEOREM FOR PARTIALLY CONVEX PLANAR SETS

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Abstract

Let S be a nonempty subset of R^2 and $\mathcal{V} \subseteq R^2$ a set of directions. S is called *$\mathcal{V}$-convex* or *partially convex relative to $\mathcal{V}$ at a point* $s \in \text{cl}S$ if and only if there exists a neighbourhood N of s in R^2 such that the intersection of any straight line parallel to a vector in $\mathcal{V}$ with $S \cap N$ is connected or empty. S is called *$\mathcal{V}$-convex* or *partially convex relative to* $\mathcal{V}$ if and only if the intersection of any straight line parallel to a vector in $\mathcal{V}$ with S is connected or empty. It is proved that if $\mathcal{V}$ is open, S is connected and open or polygonally connected and closed, and $\mathcal{V}$-convex at every boundary point, then it is $\mathcal{V}$-convex. This contributes to a recent work of Rawlins, Wood, Metelskij and others.

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Let S be a nonempty subset of R^2 and $\mathcal{V} \subseteq R^2$ a set of directions. $\mathcal{V}$ may be understood as a set of endpoints of unit vectors issuing from the origin. S is called *$\mathcal{V}$-convex* or *partially convex relative to $\mathcal{V}$ at a point* $s \in \text{cl}S$ if and only if there exists a neighbourhood N of s in R^2 such that the intersection of any straight line parallel to a vector in $\mathcal{V}$ with $S \cap N$ is connected or empty. S is called *$\mathcal{V}$-convex* or *partially convex relative to* $\mathcal{V}$ if and only if the intersection of any straight line parallel to a vector in $\mathcal{V}$ with S is connected or empty [7],[9],[10]. (xy) will represent the straight line determined by two distinct points x, y, and $(xy)_z$ the closed half-plane determined by (xy) and containing the point $z \notin (xy)$.

The application of convexity theory to various practical problems led to the exploration of nontraditional notions of convexity such as orthogonal convexity [5], finitely oriented convexity [2]-[4],[6] and, finally, partial convexity relative to an arbitrary set of directions [7]-[10]. These nontraditional convexities have been used in digital picture processing, locked transaction systems, VLSI design, motion planning and other areas. Basic properties of partially convex sets such as connectedness, simple connectedness and separation properties have been recently systematically investigated in [6],[7],[9] and [10]. It is the purpose of this paper to add yet another property to that list. We prove, roughly speaking, that a planar set which is locally $\mathcal{V}$-convex at every boundary point is necessarily globally $\mathcal{V}$-convex. The first such local versus global result involving usual convexity is due to Tietze [1,Th.4.4]. For further recent developments in Tietze's theorem the reader is referred to [11,Cor.2.3],[12] and [13].

Theorem. *Let $\mathcal{V}$ be an open set of directions in R^2. If S is an open connected or closed polygonally connected subset of R^2, $\mathcal{V}$-convex at every boundary point, then S is $\mathcal{V}$-convex.*

Proof. Select an arbitrary straight line $\mathfrak{l}$ of direction in $\mathcal{V}$ having a nonempty intersection with S. Let x and y be arbitrarily chosen distinct points in $\mathfrak{l} \cap S$. We claim that the line segment $[x, y] \subseteq S$ in both specified cases.

Let us consider first S open and connected. An easy argument reveals that S is polygonally connected. Hence, there exists in S a simple polygonal path $\mathcal{P} = [x_0, x_1] \cup \dots \cup [x_n, x_{n+1}] (x_0 = x, x_{n+1} = y)$ with the minimal number N of nondegenerate line segments not parallel to $\mathfrak{l}$. If $N = 0$, then we are done, so that let in the sequel $N \geq 1$. As compact $\mathcal{P}$ has a point p farthest from $\mathfrak{l}$. Of course, $p \notin \mathfrak{l}$. Suppose first that $p \in [x_k, x_{k+1}]$, where $1 \leq k < n$, with $[x_k, x_{k+1}]$ parallel to $\mathfrak{l}$. The other case when $p = x_l$ for some $1 \leq l \leq n$, with none of segments $[x_{l-1}, x_l], [x_l, x_{l+1}]$ parallel to $\mathfrak{l}$ will be briefly discussed as second. One can assume, without loss of generality, that $[x_{k-1}, x_k]$ and $[x_{k+1}, x_{k+2}]$ are line segments not parallel to $\mathfrak{l}$. To fix attention assume that $\text{dist}(x_{k-1}, \mathfrak{l}') \leq \text{dist}(x_{k+2}, \mathfrak{l}')$, where $\mathfrak{l}' = (x_k x_{k+1})$. If the converse inequality holds, the argument proceeds similarly. We claim that $[x_{k-1}, x'_{k-1}] \subseteq S$, where $x'_{k-1} = [x_{k+1}, x_{k+2}] \cap (x_{k-1} - x_k + \mathfrak{l}')$. Define a set $Q = \{z \in [x_{k-1}, x_k] : \text{conv}\{z, x_k, x_{k+1}, z'\} \subseteq S\}$, where $z' = [x_{k+1}, x_{k+2}] \cap (z - x_k + \mathfrak{l}')$. We will show that Q is a set simultaneously open and closed in $[x_{k-1}, x_k]$. That Q is relatively open in $[x_{k-1}, x_k]$ follows immediately from the openness of S and the compactness of $[z, z']$. To prove that it is relatively closed in $[x_{k-1}, x_k]$ suppose that $(z_0, x_k] \subseteq Q$ for some point $z_0 \in (x_{k-1}, x_k)$. We have to show that $[z_0, z'_0] \subseteq S$. Let $[z_0, w_0)$ be the longest subsegment of $[z_0, z'_0]$ contained in S. By assumption, S is $\mathcal{V}$-convex at $w_0 \in \text{bdry}S$, so that one can select an open ball B_{w_0} centered at w_0 such that $B_{w_0} \cap (z_0 z'_0)_p \subseteq \text{conv}\{z_0, x_k, x_{k+1}, z'_0\}$ and the nonempty intersection of any straight line parallel to a vector in $\mathcal{V}$ with $S \cap B_{w_0}$ is connected. Moreover, there is an open ball $B_{u_0} \subseteq S \cap B_{w_0}$ centered at a point $u_o \in (z_0, w_0) \cap B_{w_0}, w_0 \notin B_{u_0}$. Let B_{v_0} be a ball symmetrical to B_{u_0} with respect to w_0. Consider the set of straight lines determined by w_0 and points of $B_{v_0} \cap \text{conv}\{z_0, x_k, x_{k+1}, z'_0\} \subseteq S$. Since $\mathcal{V}$ is open and the direction of $(z_0 z'_0)$ belongs to $\mathcal{V}$, there must be a point $v \in B_{v_0} \cap \text{conv}\{z_0, x_k, x_{k+1}, z'_0\}$ such that the direction of $(w_0 v)$ lies in $\mathcal{V}$. Consequently, $w_0 \in S$, contradictory to the choice of $[z_0, w_0)$, by virtue of the openness of S. Hence, $[z_0, z'_0] \subseteq S$, that is Q is simultaneously relatively open and closed in $[x_{k-1}, x_k]$. As nonempty, since $x_k \in Q$, it must coincide with $[x_{k-1}, x_k]$. Now replace $\mathcal{P}$ by the polygonal path $\mathcal{P}' = [x_0, x_1] \cup \dots \cup [x_{k-2}, x_{k-1}] \cup [x_{k-1}, x'_{k-1}] \cup [x'_{k-1}, x_{k+2}] \cup \dots \cup [x_n, x_{n+1}]$ It joins x with y and may not be simple, but contains at most $N-1$ line segments not parallel to $\mathfrak{l}'$. Moreover, it contains a simple polygonal subpath $\mathcal{P}''$ joining x with y also with at most $N-1$ line segments not parallel to $\mathfrak{l}$, contradictory with the choice of $\mathcal{P}$. It remains to consider the case $p = x_l$ for some $1 \leq l \leq n$, with none of segments $[x_{l-1}, x_l], [x_l, x_{l+1}]$ parallel to $\mathfrak{l}$. Again, assume that $\text{dist}(x_{l-1}, \mathfrak{l}') \leq \text{dist}(x_{l+1}, \mathfrak{l}')$, where $\mathfrak{l}'$ is the straight line parallel to $\mathfrak{l}$ through x_l. By the openness of S, there is a line segment $[z_0, z'_0] \subseteq S$ with

$\mathrm{conv}\{z_0, x_l, z_0'\} \subseteq S$, where $z_0 \in [x_{l-1}, x_l), z_0' \in (x_l, x_{l+1}]$, which is parallel to $\mathfrak{l}$. Arguing as above we indicate a simple polygonal path $\mathcal{P}''$ joining x with y and made of at most $N-1$ line segments not parallel to $\mathfrak{l}$, again a contradiction. This proves the case of S open and connected.

Now let us consider S closed and polygonally connected. As before let $\mathcal{P} = [x_0, x_1] \cup \ldots \cup [x_n, x_{n+1}] (x_0 = x, x_{n+1} = y)$ be a simple polygonal path in S with the minimal number $N \geq 1$ of nondegenerate line segments not parallel to $\mathfrak{l}$, p a point of $\mathcal{P}$ farthest from $\mathfrak{l}$. First, let $p \in [x_k, x_{k+1}]$, where $1 \leq k < n$, with $[x_k, x_{k+1}]$ parallel to $\mathfrak{l}$ and, without loss of generality, $[x_{k-1}, x_k]$ and $[x_{k+1}, x_{k+2}]$ not parallel to $\mathfrak{l}$, and $\mathrm{dist}(x_{k-1}, \mathfrak{l}') \leq \mathrm{dist}(x_{k+2}, \mathfrak{l}')$, where $\mathfrak{l}' = (x_k x_{k+1})$. Define a set $Q = \{z \in [x_{k-1}, x_k] : \mathrm{conv}\{z, x_k, x_{k+1}, z'\} \subseteq S\}$, where $z' = [x_{k+1}, x_{k+2}] \cap (z - x_k + \mathfrak{l}')$. By the closedness of S, Q is relatively closed in $[x_{k-1}, x_k]$. We claim that it is also relatively open in $[x_{k-1}, x_k]$. Select any point $z_0 \in Q$ together with an associated point z_0'. Observe first that, by the assumption, there are open balls B_{z_0} and $B_{z_0'}$ centered at z_0 and z_0', respectively, such that $B_{z_0} \cap (x_{k-1} x_k)_{x_{k+1}} \cap (z_0 z_0')_{x_{k-1}} \subseteq S$ and $B_{z_0'} \cap (x_{k+1} x_{k+2})_{x_{k-1}} \cap (z_0 z_0')_{x_{k-1}} \subseteq S$. Suppose, to reach a contradiction, that Q is not relatively open at z_0. By the observation just made, there exists a point $w_0 \in (z_0, z_0')$ such that for every open ball B_{w_0} at w_0 we have $B_{w_0} \cap (x_{k-1} x_k)_{x_{k+1}} \cap (z_0 z_0')_{x_{k-1}} \not\subseteq S$. Let $w \in (z_0, w_0]$ be the point with this property lying closest to z_0. By assumption, S is $\mathcal{V}$-convex at $w \in \mathrm{bdry} S$, so that there exists an open ball B_w at w such that the intersection of any straight line parallel to a vector in $\mathcal{V}$ with $S \cap B_w$ is connected or empty. It follows from the choice of w that there is a point $u \in B_w \cap (w, z_0)$ with an open ball B_u at u such that $B_u \cap (z_0 z_0')_{x_{k-1}} \subseteq S$. The $\mathcal{V}$-convexity of S at w and $[z_0, z_0'] \subseteq S$ imply the existence of an open ball B_w' at w for which $B_w' \cap (z_0 z_0')_{x_{k-1}} \subseteq S$, a contradiction. Hence, S is simultaneously relatively open and closed in $[x_{k-1}, x_k]$, and, by virtue of $x_k \in Q$, also nonempty, so that it must be $Q = [x_{k-1}, x_k]$. The rest of the argument consisting in an appropriate modification of $\mathcal{P}$ to reach a contradiction proceeds as in case of S open and connected. Finally, the case of $p = x_l$ for some $1 \leq l \leq n$, with none of segments $[x_{l-1}, x_l], [x_l, x_{l+1}]$ parallel to $\mathfrak{l}$ is analysed as above.

The proof is complete. □

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