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ON SEPARATELY CONTINUOUS MAPPINGS

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Abstract. A criterion for the equicontinuity of certain separately equicontinuous sets of mappings is obtained, and a few consequences of it are presented.

In this note we use a Baire category argument to prove that certain separately equicontinuous sets of mappings are equicontinuous. As a consequence, we deduce the equicontinuity of certain sets of bilinear mappings and we show that Banach's closed graph theorem for group homomorphisms remains valid for $\mathbb{Z}$-bilinear mappings.

Let X, Y and Z be three topological spaces. Let us remember that: (a) a set $\mathfrak{X}$ of mappings from $X \times Y$ into Z is *separately equicontinuous* if, for each $x \in X$ and for each $y \in Y$, the sets

$$\{u \in Y \mapsto f(x,u) \in Z;\ f \in \mathfrak{X}\} \quad \text{and} \quad \{v \in X \mapsto f(v,y) \in Z;\ f \in \mathfrak{X}\}$$

are equicontinuous; (b) X is *first countable* [4] if each point of X has a countable fundamental system of neighborhoods.

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Let us also remember that a topological group is an *SIN-group* [6] if its identity element has a fundamental system of neighborhoods consisting of conjugation-invariant sets.

Theorem 1. Let X be a first countable topological space, and let F and G be two topological groups such that F is a Baire space and G is an SIN-group. If $\mathfrak{X}$ is a separately equicontinuous set of mappings from $X \times F$ into G such that the mapping $y \in F \mapsto f(x,y) \in G$ is a group homomorphism for each $x \in X$ and for each $f \in \mathfrak{X}$, then $\mathfrak{X}$ is equicontinuous.

Proof. Let e_F (resp. e_G) be the identity element of F (resp. G), and let $(x_0, y_0) \in X \times F$ be arbitrary. Let $(U_n)_{n \in \mathbb{N}}$ be a countable fundamental system of neighborhoods of x_0 in X and let W be an arbitrary neighborhood of e_G in G. Take a conjugation-invariant, symmetric and closed neighborhood W_1 of e_G in G such that $(W_1)^4 \subset W$. We have

$$f(x, yy_0)\big(f(x_0, y_0)\big)^{-1} = f(x,y)\big(f(x_0,y)\big)^{-1} f(x_0,y) f(x,y_0)\big(f(x_0,y_0)\big)^{-1}$$

for all $(x,y) \in X \times F$ and for all $f \in \mathfrak{X}$.

For each $n \in \mathbb{N}$, put

$$F_n = \{y \in F;\ f(x,y)\big(f(x_0,y)\big)^{-1} \in W_1 \text{ for all } x \in U_n \text{ and for all } f \in \mathfrak{X}\}$$

$$= \bigcap_{(x,f) \in U_n \times \mathfrak{X}} \{y \in F;\ f(x,y)\big(f(x_0,y)\big)^{-1} \in W_1\}.$$

F_n is symmetric because W_1 is conjugation-invariant and symmetric. Moreover, F_n is closed. In fact, for each $x \in X$ and for each $f \in \mathfrak{X}$, the mapping $g_{x,f}\colon y \in F \mapsto f(x,y)\big(f(x_0,y)\big)^{-1} \in G$ is continuous. Since

$$F_n = \bigcap_{(x,f) \in U_n \times \mathfrak{X}} g_{x,f}^{-1}(W_1)$$

and since W_1 is closed in G, then F_n is closed in F.

We claim that $F = \bigcup_{n\in\mathbb{N}} F_n$. Indeed, let $y \in F$ be given. By the equicontinuity of the set

$$\{x \in X \mapsto f(x,y) \in G;\ f \in \mathfrak{X}\}$$

at x_0, there exists a neighborhood U of x_0 in X such that the relations $x \in U$, $f \in \mathfrak{X}$ imply $f(x,y)\big(f(x_0,y)\big)^{-1} \in W_1$. If m is chosen in such a way that $U_m \subset U$ we conclude that $y \in F_m$, proving our claim. Since F is a Baire space, there exists an integer ℓ such that $\mathrm{int}(F_\ell) \neq \phi$, which implies that $V = F_\ell F_\ell$ is a neighborhood of e_F in F. If $x \in U_\ell$, $y = y_1 y_2 \in V$ $(y_1, y_2 \in F_\ell)$ and $f \in \mathfrak{X}$, then

$$f(x,y)\big(f(x_0,y)\big)^{-1} = f(x,y_1)f(x,y_2)\big(f(x_0,y_2)\big)^{-1}\big(f(x_0,y_1)\big)^{-1} \in W_1 W_1$$

because W_1 is conjugation-invariant. Finally, by the equicontinuity of the set

$$\{x \in X \mapsto f(x,y_0) \in G;\ f \in \mathfrak{X}\}$$

at x_0 and the equicontinuity of the set of group homomorphisms

$$\{y \in F \mapsto f(x_0,y) \in G;\ f \in \mathfrak{X}\}$$

at e_F, there exist a neighborhood U^* of x_0 in X and a neighborhood V^* of e_F in F such that $U^* \subset U_\ell$ and $V^* \subset V$, and such that $f(x,y_0)\big(f(x_0,y_0)\big)^{-1} \in W_1$ and $f(x_0,y) \in W_1$ for all $x \in U^*$, $y \in V^*$ and $f \in \mathfrak{X}$. Consequently,

$$f(x,yy_0)\big(f(x_0,y_0)\big)^{-1} \in (W_1)^4 \subset W$$

for all $x \in U^*$, $y \in V^*$ and $f \in \mathfrak{X}$. This proves that $\mathfrak{X}$ is equicontinuous at (x_0,y_0), thereby concluding the proof of the theorem.

Remark 2. Theorem 1 is already known in the particular case where X is metrizable,

F and G are topological vector spaces and F is a Baire space; see Exercise 11, TVS III.42 of [3].

We have the following important consequence of Theorem 1, which extends Theorem 11.15 of [9] (and hence its consequences mentioned on p.487 of [9]), as well as Corollary 4, p.47 of [1].

Corollary 3. Let E, F and G be three commutative topological groups such that E is semimetrizable and F is a Baire space. Then every separately equicontinuous set of $\mathbb{Z}$-bilinear mappings from $E \times F$ into G is equicontinuous.

In [8] we have investigated equicontinuity and sequential equicontinuity for sets of separately continuous bilinear mappings between topological modules. A result in this direction, which also follows from Proposition 2.4 and Corollary 3.3 of [7], reads as follows:

Corollary 4. Let A be a commutative topological ring with a non-zero identity element such that there exists a countable subset C of the multiplicative group of all invertible elements of A such that $0 \in \overline{C}$. Let E, F and G be unitary topological A-modules such that E and F are complete and metrizable. If $\mathfrak{X}$ is a set of separately continuous A-bilinear mappings from $E \times F$ into G such that $\mathfrak{X}(x,y) = \{f(x,y); f \in \mathfrak{X}\}$ is bounded in G for each $(x,y) \in E \times F$, then $\mathfrak{X}$ is equicontinuous.

Proof. By Proposition 2.4 and Theorem 3.1 of [7], $\mathfrak{X}$ is separately equicontinuous. Therefore $\mathfrak{X}$ is equicontinuous by Corollary 3.

We close our note with a closed graph theorem for $\mathbb{Z}$-bilinear mappings.

Corollary 5. Let E, F and G be three commutative topological groups which are complete, metrizable and separable. If f is a $\mathbb{Z}$-bilinear mapping from $E \times F$ into G

whose graph is closed, then f is continuous.

Proof. Let $x \in E$. Arguing as in the proof of the theorem proved in [5], we conclude that the graph of the group homomorphism $u \in F \mapsto f(x,u) \in G$ is closed. By Theorem 10 of [2], this mapping is continuous. Similarly, the group homomorphism $v \in E \mapsto f(v,y) \in G$ is continuous for each $y \in F$. Therefore f is separately continuous. By Corollary 3, f is continuous, as was to be shown.

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