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ON BANACH SPACES INVARIANT UNDER DIFFERENTIATION

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Summary

If E is a Fréchet space which is continuously embedded in $\mathcal{D}'(\Omega)$ and invariant under differentiations then E is already contained in $C^\infty(\Omega)$. If additionally E is a Banach space then E cannot contain $\mathcal{D}(\Omega)$. We investigate these and several related properties.

In analysis it is a widely accepted fact that there is no Banach space E such that

$$\mathcal{D}(\Omega) \subset E \subset \mathcal{D}'_\sigma(\Omega) \quad \text{with continuous inclusions} \qquad (*)$$

and such that

$$\text{all partial differentiations act continuously in } E. \qquad (**)$$

Here and in the following $\Omega \subset \mathbb{R}^n$ is open, $\mathcal{D}(\Omega)$ is the space of test functions and $\mathcal{D}'_\sigma(\Omega)$ denotes the space of distributions on Ω endowed with the weak topology.

Surprisingly, a proof of this folklore fact seems to be missing. The aim of the present paper is to provide a short proof and to discuss some related questions. We will show in particular that a Banach space E satisfying $(**)$ and the right hand inclusion of $(*)$ is already contained in a canonical weighted space of entire functions.

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Let $H(\mathbb{C}^n)$ denote the space of entire functions and

$$H_A := \left\{ f \in H(\mathbb{C}^n) \mid \int |f(z)|^2 e^{-2A|z|} dz < \infty \right\}, \quad \text{for } A > 0.$$

The theorem below is the main result of this paper. We consider here a situation slightly more general than (*) and (**) which can easily be dualized (see Theorem 6).

Theorem 1. *Let E be a Fréchet space. Let $\varkappa\colon E \longrightarrow \mathcal{D}'(\Omega)$ and $d_j\colon E \longrightarrow E$, for $j = 1, \ldots, n$, be linear mappings such that*

$$\varkappa\colon E \longrightarrow \mathcal{D}'_\sigma(\Omega) \quad \textit{is continuous} \tag{1}$$

and

$$\varkappa \circ d_j = \partial_j \circ \varkappa \quad \textit{for } j = 1, \ldots, n, \tag{2}$$

for the distributional derivatives $\partial_j = \partial/\partial x_j$.

a) Then $\varkappa(E)$ is contained in $C^\infty(\Omega)$, and

$$\varkappa\colon E \longrightarrow C^\infty(\Omega) \quad \textit{is continuous.}$$

b) Let E be a Banach space and let Ω be connected. Then there is $A > 0$ such that any $h \in \varkappa(E)$ can be extended uniquely to a function $\widetilde{h} \in H_A$, and the resulting mapping

$$\widetilde{\varkappa}\colon E \longrightarrow H_A \quad \textit{is continuous.}$$

Proof. a) i) Passing to $\hat{\varkappa}\colon \hat{E} := E/\ker(\varkappa) \longrightarrow \mathcal{D}'_\sigma(\Omega)$ and identifying $\hat{E}$ with a subspace of $\mathcal{D}'(\Omega)$ (via $\hat{\varkappa}$) we may assume that E is contained in $\mathcal{D}'(\Omega)$,

$$\varkappa = \mathrm{id}\colon E \longrightarrow \mathcal{D}'_\sigma(\Omega) \quad \text{is continuous,} \tag{3}$$

and

$$d_j = \partial_j\colon E \longrightarrow E \quad \text{is continuous.} \tag{4}$$

Indeed, d_j is well-defined on $\hat{E}$ because of (2), and d_j is continuous by the closed graph theorem (and (1)).

ii) For a compact $K \subset \Omega$ let $\mathcal{D}(K) := \{f \in C^\infty(\mathbb{R}^n) \mid \operatorname{supp} f \subset K\}$. The mapping

$$T\colon E \times \mathcal{D}(K) \longrightarrow \mathbb{C}, \quad T(h, \varphi) := \langle h, \varphi \rangle$$

is separately continuous by (3). Since E and $\mathcal{D}(K)$ are Fréchet spaces, T is continuous and thus there are a continuous seminorm $\|\cdot\|$ on E and $k \in \mathbb{N}$ such that

$$|\langle h, \varphi \rangle| \leqslant C_1 \|h\| \, \|\varphi\|_k \quad \text{for any } h \in E \text{ and any } \varphi \in \mathcal{D}(K) \tag{5}$$

where $\|\varphi\|_k := \sup\limits_{\substack{|\alpha|\leqslant k \\ x\in K}} |\varphi^{(\alpha)}(x)|$. For a distribution f with compact support the Fourier Laplace transform is denoted by

$$\hat{f}(z) := \langle f, \exp(-i\langle \cdot, z\rangle)\rangle, \quad \text{for } z \in \mathbb{C}^n.$$

By Leibniz' rule, (5) implies

$$|z^\alpha (h\varphi)^\wedge(z)| \leqslant C_2 \sup_{\beta\leqslant\alpha} \|\partial^\beta h\| \, \|\varphi\|_{|\alpha|+k} \, (1+|z|)^k \, e^{c|\operatorname{Im} z|}$$

for all $z \in \mathbb{C}^n$, $\alpha \in \mathbb{N}_0^n$. Thus, $\varphi h \in \mathcal{D}(\mathbb{R}^n)$ for any $\varphi \in \mathcal{D}(K)$ by the theorem of Paley-Wiener, and therefore $h \in C^\infty(\Omega)$. The inclusion $\mathrm{id}\colon E \longrightarrow C^\infty(\Omega)$ is continuous by the closed graph theorem and (3).

b) Let $\|\cdot\|$ be the norm on E. By (4) there is $C \geqslant 1$ such that for $j = 1, \ldots, n$

$$\|\partial_j h\| \leqslant C\|h\| \quad \text{for all } h \in E.$$

From a) we know that for any compact $K \subset \Omega$ there is $C_K \geqslant 1$ such that

$$\|h\|_K := \sup_{x\in K} |h(x)| \leqslant C_K \|h\| \quad \text{for all } h \in E.$$

This implies that for all $h \in E$

$$\|h^{(\alpha)}\|_K \leqslant C_K \|h^{(\alpha)}\| \leqslant C_K C^{|\alpha|} \|h\| \quad \text{for all } \alpha \in \mathbb{N}_0^n. \tag{6}$$

By Taylor's theorem, h is a real analytic function on Ω with everywhere convergent Taylor series. Since Ω is connected, we get for fixed $x_0 \in \Omega$

$$h(x) = \sum_\alpha h^{(\alpha)}(x_0)(x-x_0)^\alpha/\alpha! \quad \text{for any } x \in \Omega.$$

This formula also defines an entire extension $\widetilde{h}$ of h and

$$|\widetilde{h}(z)| \leqslant C_3 \, e^{C|z|} \quad \text{for any } z \in \mathbb{C}^n$$

by (6). This implies that $\widetilde{h} \in H_{2C}$ and $\mathrm{id}\colon E \longrightarrow H_{2C}$ is continuous by the closed graph theorem, property (3), and the identity theorem. □

Corollary 2. *a) The space $C^\infty(\Omega)$ is the maximal Fréchet space satisfying* $(*)$ *and* $(**)$.

b) There exists no Banach space E satisfying $(*)$ *and* $(**)$.

Proof. a) directly follows from Theorem 1a).

b) If E is a Fréchet space satisfying $(*)$ and $(**)$, then Theorem 1a) implies that E induces the standard Fréchet topology on $\mathcal{D}(K)$ for any compact $K \subset \Omega$. This shows b). □

Example 3. Let $W\colon \mathbb{C}^n \longrightarrow \mathbb{R}$ satisfy

$$\sup_{|\xi|\leqslant 1} W(\xi + z) \leqslant C + W(z) \quad \text{on } \mathbb{C}^n. \tag{7}$$

Then

$$H(W) := \left\{ f \in H(\mathbb{C}^n) \;\middle|\; \int |f(z)|^2 e^{-2W(z)}\, dz < \infty \right\}$$

satisfies (1) and (2) for $\varkappa(f) := f\big|_\Omega$.

Proof. (1) is trivial. Since

$$|\partial_j f(0)| \leqslant C_1 \Big(\int_{|z|\leqslant 1} |f(z)|^2\, dz \Big)^{1/2} \quad \text{for } f \in H(\mathbb{C}^n)$$

we get for $f \in H(W)$ by (7)

$$\int |\partial_j f(z)|^2 e^{-2W(z)}\, dz \leqslant C_1^2 \iint_{|\xi|\leqslant 1} |f(z+\xi)|^2 d\xi\, e^{-2W(z)}\, dz$$

$$\leqslant C_2 \int_{|\xi|\leqslant 1} \int |f(z+\xi)|^2 e^{-2W(z+\xi)} dz\, d\xi = C_3 \int |f(z)|^2 e^{-2W(z)} dz.$$

□

A similar series of examples can also be given using L_p-norms for $p \in [1, \infty]$ instead of the L_2-norm.

By Theorem 1a) and Example 3, H_A is the "largest" Banach space satisfying (1) and (2).

If Ω is not connected, we can choose a space from Example 3 (or a subspace closed under differentiation) on each connected component of Ω. In particular, the space of locally constant bounded functions provides an example of a Banach space satisfying (1) and (2).

In Theorem 1a), the differentiations ∂_j in (2) can be substituted by a more general set of partial differential operators:

Proposition 4. *Let $P_1(\partial), \dots, P_d(\partial)$ be partial differential operators with constant coefficients. Let E be a Fréchet space satisfying* (1) *and let*

$$P_j(\partial)\varkappa(E) \subset \varkappa(E) \quad \textit{for } j = 1, \dots, d.$$

If the algebra generated by $\{P_1(\partial), \dots, P_d(\partial)\}$ contains a non-constant hypoelliptic operator $P(\partial)$, then $\varkappa(E) \subset C^\infty(\Omega)$.

Proof. We modify the proof of Theorem 1a): We may suppose that (3) holds. By (5) we then get for $\varphi \in \mathcal{D}(K)$, $h \in E$ fixed, and all $j \in \mathbb{N}$

$$\big| \big((P^j(\partial)h)\varphi \big)^\wedge(x) \big| \leqslant C \|P^j(\partial)h\| \, (1 + |x|)^k = C_j (1 + |x|)^k \quad \text{for } x \in \mathbb{R}^n.$$

Since $P(\partial)$ is hypoelliptic we can use Hörmander [1, Theorem 11.1.8] and conclude that

$$h \in \bigcap_{j\in\mathbb{N}} B^{\mathrm{loc}}_{\infty, g\widetilde{P}^j}(\Omega) \subset C^\infty(\Omega),$$

where $g(x) := 1/(1+|x|)^k$. The inclusion follows from Sobolev's lemma (Hörmander [1, Theorem 10.1.25]) since

$$\widetilde{P}(x) := \left(\sum |P^{(\alpha)}(x)|^2\right)^{1/2} \geqslant C(1+|x|)^c$$

for some $c > 0$ again by the hypoellipticity of $P \neq \text{const}$ (use Hörmander [1, Theorem 11.1.3IIb]). □

Conversely, if $P(\partial)$ is a non-constant partial differential operator such that for any Fréchet space E satisfying (1) and $P(\partial)\varkappa(E) \subset \varkappa(E)$ we have $\varkappa(E) \subset C^\infty(\Omega)$, then $P(\partial)$ is hypoelliptic. Indeed, we set

$$E := \{h \in C(\Omega) \mid P(\partial)h = 0\}$$

with the topology induced by $C(\Omega)$. Then E is closed in $C(\Omega)$ and invariant under $P(\partial)$, hence $E \subset C^\infty(\Omega)$ by assumption. Thus $P(\partial)$ is hypoelliptic.

One could guess that a Banach space E with (**) could be found satisfying (*) in a suitably weaker form. We will show now that this is impossible in a rather general setting. In fact, let E be a space of hyperfunctions on Ω. Then by the definition of hyperfunctions (see e.g. Hörmander [1, Chapter 9])

$$E(K) := \{h \in E \mid \operatorname{supp} h \subset K\} \subset A(K)' \quad \text{for any compact } K \subset \Omega,$$

where $A(K)$ denotes the space of holomorphic germs near K. Condition (8) below thus is a very weak substitute for (*).

Theorem 5. *Let E be a Banach space of hyperfunctions on Ω such that for some compact $K \subset \Omega$*

$$\{0\} \neq E(K) \quad \text{and} \quad \mathrm{id}\colon E(K) \longrightarrow A(K)'_\sigma \quad \text{is continuous.} \tag{8}$$

Then $P(\partial)E \not\subset E$ for any non-constant partial differential operator $P(\partial)$.

Proof. $E(K)$ is closed in E by (8). Let $P(\partial)E \subset E$. Then $P(\partial)E(K) \subset E(K)$ and

$$P(\partial)\colon E(K) \longrightarrow E(K) \quad \text{is continuous} \tag{9}$$

by (8) and the closed graph theorem. Since

$$T\colon E(K) \times H(\mathbb{C}^n) \longrightarrow \mathbb{C}, \quad (h, f) \longmapsto \langle h, f\rangle,$$

is separately continuous by (8) and hence continuous, there is $k \in \mathbb{N}$ such that

$$|\langle h, f\rangle| \leqslant C\|h\| \sup_{|z|\leqslant k} |f(z)| \quad \text{for any } h \in E(K) \text{ and any } f \in H(\mathbb{C}^n).$$

Using also (9) we get for any $h \in E(K)$

$$|P^j(iz)\hat{h}(z)| \leqslant C\|P^j(\partial)h\| \, e^{k|z|} \leqslant CC_1^j\|h\|e^{k|z|} \quad \text{for any } j \in \mathbb{N} \text{ and any } z \in \mathbb{C}^n.$$

This implies that

$$\left(\frac{|P(iz)|}{(2C_1)}\right)^j \longrightarrow 0 \quad \text{for } j \to \infty \text{ if } \hat{h}(z) \neq 0,$$

and therefore

$$|P(iz)| \leqslant 2C_1 \quad \text{if } \hat{h}(z) \neq 0. \tag{10}$$

On the other hand there are $C_2 > 0$ and an open cone $\Gamma \subset \mathbb{C}^n$ such that

$$|P(iz)| \geqslant C_2|z|^{\deg P} \quad \text{if } z \in \Gamma \text{ and } |z| \geqslant 1/C_2. \tag{11}$$

By assumption, there is $0 \neq h \in E(K)$. Since then $\{z \in \Gamma \mid \hat{h}(z) \neq 0\}$ is unbounded, P is constant by (10) and (11). □

Finally, Theorem 1b) admits the following dual formulation:

Theorem 6. *Let E be a Banach space and let Ω be connected. Let*

$$\varkappa\colon \mathcal{D}(\Omega) \longrightarrow E \quad \textit{and} \quad d_j\colon E \longrightarrow E \quad \textit{be linear and continuous} \tag{12}$$

and let

$$\varkappa \circ \partial_j = d_j \circ \varkappa \quad \textit{for } j = 1, \ldots, n. \tag{13}$$

Then there is $A > 0$ such that $\varkappa$ can be uniquely extended to a continuous linear mapping $\tilde{\varkappa}\colon (H_A)' \longrightarrow E$.

Proof. a) The assumptions of Theorem 1b) are satisfied for $\{E', {}^t\varkappa, {}^t d_j\}$. Thus there is $A > 0$ such that ${}^t\varkappa\colon E' \longrightarrow H_A$ is continuous, that is,

$$\|{}^t\varkappa(e')\| \leqslant C\|e'\|_{E'} \quad \text{for any } e' \in E'$$

where $\|\cdot\|$ is the norm in H_A. For the norm $\|\cdot\|^*$ in $(H_A)'$ we thus get

$$\|\varkappa(f)\|_E = \sup_{\|e'\| \leqslant 1} |\langle f, {}^t\varkappa(e')\rangle| \leqslant C\|f\|^* \quad \text{for any } f \in \mathcal{D}(\Omega)$$

and

$$\varkappa\colon (\mathcal{D}(\Omega), \|\cdot\|^*) \longrightarrow E \quad \text{is continuous.}$$

b) $\mathcal{D}(\Omega)$ is dense in $(H_A)'$. Indeed, any $f \in H_A$ which vanishes on $\mathcal{D}(\Omega)$, is zero on Ω, thus $f = 0$ on $\mathbb{C}^n$ by the identity theorem. □

Examples for the situation in Theorem 6 are provided by the duals of the spaces from Example 3.

In contrast to the situation of Theorem 1b), the continuity of d_j is needed to prove Theorem 6. In fact, Let $E := L_2(\mathbb{R})$, $\varkappa :=$ id and choose a linear complement F of $\mathcal{D}(\mathbb{R})$ in E. If we now set $d_1(f) := f'$ for $f \in \mathcal{D}(\mathbb{R})$ and $d_1(f) := 0$ for $f \in F$, then (12) and (13) are satisfied (except for the continuity of d_1), but

$$\mathrm{id}\colon \big(\mathcal{D}(\mathbb{R}), \|\cdot\|^*\big) \longrightarrow E = L_2(\mathbb{R}) \quad \text{is not continuous}$$

for the norm $\|\cdot\|^*$ in $(H_A)'$ since

$$\mathrm{id}\colon \big(\mathcal{D}(\mathbb{R}), \tau\big) \longrightarrow E \quad \text{is not continuous}$$

for the topology τ induced by $C^\infty(\mathbb{R})'_b$.

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