ON BANACH SPACES IN Variant
UNDER DIFFERENTIATION

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Summary

If $E$ is a Fréchet space which is continuously embedded in $\mathcal{D}'(\Omega)$ and invariant under differentiations then $E$ is already contained in $C^\infty(\Omega)$. If additionally $E$ is a Banach space then $E$ cannot contain $\mathcal{D}(\Omega)$. We investigate these and several related properties.

In analysis it is a widely accepted fact that there is no Banach space $E$ such that

$$\mathcal{D}(\Omega) \subset E \subset \mathcal{D}_e'(\Omega)$$

with continuous inclusions \((*)\)

and such that

all partial differentiations act continuously in $E$. \((**\))

Here and in the following $\Omega \subset \mathbb{R}^n$ is open, $\mathcal{D}(\Omega)$ is the space of test functions and $\mathcal{D}_e'(\Omega)$ denotes the space of distributions on $\Omega$ endowed with the weak topology.

Surprisingly, a proof of this folklore fact seems to be missing. The aim of the present paper is to provide a short proof and to discuss some related questions. We will show in particular that a Banach space $E$ satisfying \((**)\) and the right hand inclusion of \((*)\) is already contained in a canonical weighted space of entire functions.

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Let $H(C^n)$ denote the space of entire functions and

$$H_A := \left\{ f \in H(C^n) \mid \int |f(z)|^2 e^{-2A|z|} dz < \infty \right\}, \quad \text{for } A > 0.$$ 

The theorem below is the main result of this paper. We consider here a situation slightly more general than (**) and (**) which can easily be dualized (see Theorem 6).

**Theorem 1.** Let $E$ be a Fréchet space. Let $\kappa: E \to \mathcal{D}'(\Omega)$ and $d_j: E \to E$, for $j = 1, \ldots, n$, be linear mappings such that

$$\kappa: E \to \mathcal{D}'(\Omega) \text{ is continuous} \quad \text{(1)}$$

and

$$\kappa \circ d_j = \partial_j \circ \kappa \text{ for } j = 1, \ldots, n, \quad \text{(2)}$$

for the distributional derivatives $\partial_j = \partial / \partial x_j$.

a) Then $\kappa(E)$ is contained in $C^\infty(\Omega)$, and

$$\kappa: E \to C^\infty(\Omega) \text{ is continuous.}$$

b) Let $E$ be a Banach space and let $\Omega$ be connected. Then there is $A > 0$ such that any $h \in \kappa(E)$ can be extended uniquely to a function $\tilde{h} \in H_A$, and the resulting mapping

$$\tilde{\kappa}: E \to H_A \text{ is continuous.}$$

**Proof.** a) i) Passing to $\hat{E} := E / \ker(\kappa) \to \mathcal{D}'(\Omega)$ and identifying $\hat{E}$ with a subspace of $\mathcal{D}'(\Omega)$ (via $\hat{\kappa}$) we may assume that $E$ is contained in $\mathcal{D}'(\Omega)$,

$$\kappa = \text{id}: E \to \mathcal{D}'(\Omega) \text{ is continuous,} \quad \text{(3)}$$

and

$$d_j = \partial_j: E \to E \text{ is continuous.} \quad \text{(4)}$$

Indeed, $d_j$ is well-defined on $\hat{E}$ because of (2), and $d_j$ is continuous by the closed graph theorem (and (1)).

ii) For a compact $K \subset \Omega$ let $\mathcal{D}(K) := \{ f \in C^\infty(\mathbb{R}^n) \mid \text{supp } f \subset K \}$. The mapping

$$T: E \times \mathcal{D}(K) \to \mathbb{C}, \quad T(h, \varphi) := \langle h, \varphi \rangle$$

is separately continuous by (3). Since $E$ and $\mathcal{D}(K)$ are Fréchet spaces, $T$ is continuous and thus there are a continuous seminorm $\| \cdot \|$ on $E$ and $k \in \mathbb{N}$ such that

$$|\langle h, \varphi \rangle| \leq C_k \| h \| \| \varphi \|_k \text{ for any } h \in E \text{ and any } \varphi \in \mathcal{D}(K)$$

(5)
where $\|\varphi\|_k := \sup_{z \in K} |\varphi^{(k)}(z)|$. For a distribution $f$ with compact support the Fourier Laplace transform is denoted by

$$\hat{f}(z) := \langle f, \exp(-i\zeta \cdot z) \rangle, \quad z \in \mathbb{C}^n.$$  

By Leibniz' rule, (5) implies

$$|z^\alpha (h\varphi)^\wedge(z)| \leq C_2 \sup_{\beta \leq \alpha} \|\partial^\beta \varphi\| \|\varphi\|_{\|o+\|} (1 + |z|)^{\beta} e^{c|\text{Im} z|}$$

for all $z \in \mathbb{C}^n$, $\alpha \in \mathbb{N}_0^n$. Thus, $\varphi h \in \mathcal{D}(\mathbb{R}^n)$ for any $\varphi \in \mathcal{D}(K)$ by the theorem of Paley-Wiener, and therefore $h \in C^\infty(\Omega)$. The inclusion $\text{id} : E \to C^\infty(\Omega)$ is continuous by the closed graph theorem and (3).

b) Let $\| \cdot \|$ be the norm on $E$. By (4) there is $C \geq 1$ such that for $j = 1, \ldots, n$

$$\|\partial_j h\| \leq C\|h\| \quad \text{for all } h \in E.$$  

From a) we know that for any compact $K \subset \Omega$ there is $C_K \geq 1$ such that

$$\|h\|_K := \sup_{z \in K} |h(z)| \leq C_K \|h\| \quad \text{for all } h \in E.$$  

This implies that for all $h \in E$

$$\|h^{(\alpha)}\|_K \leq C_K \|h^{(\alpha)}\| \leq C_K C^{\|\alpha\|}_K \|h\| \quad \text{for all } \alpha \in \mathbb{N}_0^n. \quad (6)$$

By Taylor's theorem, $h$ is a real analytic function on $\Omega$ with everywhere convergent Taylor series. Since $\Omega$ is connected, we get for fixed $x_0 \in \Omega$

$$h(x) = \sum_{\alpha} h^{(\alpha)}(x_0)(x - x_0)^{\alpha}/\alpha! \quad \text{for any } x \in \Omega.$$  

This formula also defines an entire extension $\tilde{h}$ of $h$ and

$$|\tilde{h}(z)| \leq C_3 e^{C|z|} \quad \text{for any } z \in \mathbb{C}^n$$

by (6). This implies that $\tilde{h} \in H_{2C}$ and $\text{id} : E \to H_{2C}$ is continuous by the closed graph theorem, property (3), and the identity theorem.

**Corollary 2.** a) The space $C^\infty(\Omega)$ is the maximal Fréchet space satisfying $(\ast)$ and $(\ast\ast)$.

b) There exists no Banach space $E$ satisfying $(\ast)$ and $(\ast\ast)$.

**Proof.** a) directly follows from Theorem 1a).

b) If $E$ is a Fréchet space satisfying $(\ast)$ and $(\ast\ast)$, then Theorem 1a) implies that $E$ induces the standard Fréchet topology on $\mathcal{D}(K)$ for any compact $K \subset \Omega$. This shows b).
Example 3. Let $W: \mathbb{C}^n \rightarrow \mathbb{R}$ satisfy
\[
\sup_{|z| \leq 1} W(\xi + z) \leq C + W(z) \quad \text{on } \mathbb{C}^n.
\] (7)

Then
\[
H(W) := \left\{ f \in H(\mathbb{C}^n) \mid \int |f(z)|^2 e^{-2W(z)} \, dz < \infty \right\}
\]
satisfies (1) and (2) for $\mathcal{A}(f) := f|_\Omega$.

Proof. (1) is trivial. Since
\[
|\partial_j f(0)| \leq C_1 \left( \int_{|z| \leq 1} |f(z)|^2 \, dz \right)^{1/2} \quad \text{for } f \in H(\mathbb{C}^n)
\]
we get for $f \in H(W)$ by (7)
\[
\left\| \partial_j f(z) \right\|^2 e^{-2W(x)} \, dz \leq C_2 \int \int_{|\xi| \leq 1} |f(z + \xi)|^2 e^{-2W(z)} \, d\xi \, dz
\]
\[
\leq C_3 \int \int_{|\xi| \leq 1} |f(z + \xi)|^2 e^{-2W(z + \xi)} \, d\xi \, dz = C_3 \int |f(z)|^2 e^{-2W(z)} \, dz.
\]

\[\square\]

A similar series of examples can also be given using $L_p$-norms for $p \in [1, \infty]$ instead of the $L_2$-norm.

By Theorem 1a) and Example 3, $H_A$ is the "largest" Banach space satisfying (1) and (2).

If $\Omega$ is not connected, we can choose a space from Example 3 (or a subspace closed under differentiation) on each connected component of $\Omega$. In particular, the space of locally constant bounded functions provides an example of a Banach space satisfying (1) and (2).

In Theorem 1a), the differentiations $\partial_j$ in (2) can be substituted by a more general set of partial differential operators:

Proposition 4. Let $P_1(\partial), \ldots, P_d(\partial)$ be partial differential operators with constant coefficients. Let $E$ be a Fréchet space satisfying (1) and let
\[
P_j(\partial)\mathcal{A}(E) \subset \mathcal{A}(E) \quad \text{for } j = 1, \ldots, d.
\]

If the algebra generated by $\{P_1(\partial), \ldots, P_d(\partial)\}$ contains a non-constant hypoelliptic operator $P(\partial)$, then $\mathcal{A}(E) \subset C^\infty(\Omega)$.

Proof. We modify the proof of Theorem 1a): We may suppose that (3) holds. By (5) we then get for $\varphi \in D(K), h \in E$ fixed, and all $j \in \mathbb{N}$
\[
\left| ((P^j(\partial)h)\varphi)^{\wedge}(x) \right| \leq C\|P^j(\partial)h\|_1 (1 + |x|)^k = C_j(1 + |x|)^k \quad \text{for } x \in \mathbb{R}^n.
\]
Since $P(\partial)$ is hypoelliptic we can use Hörmander [1, Theorem 11.1.8] and conclude that

$$h \in \bigcap_{j \in \mathbb{N}} B_{C^0, \mathbb{B}}^{\max}(\Omega) \subset C^\infty(\Omega),$$

where $g(x) := 1/(1 + |x|^4)$. The inclusion follows from Sobolev's lemma (Hörmander [1, Theorem 10.1.25]) since

$$\bar{P}(x) := \left( \sum |P^{(a)}(x)|^2 \right)^{1/2} \geq C(1 + |x|)^c$$

for some $c > 0$ again by the hypoellipticity of $P \neq \text{const}$ (use Hörmander [1, Theorem 11.1.31b]).

Conversely, if $P(\partial)$ is a non-constant partial differential operator such that for any Fréchet space $E$ satisfying (1) and $P(\partial) \mathcal{H}(E) \subset \mathcal{H}(E)$ we have $\mathcal{H}(E) \subset C^\infty(\Omega)$, then $P(\partial)$ is hypoelliptic. Indeed, we set

$$E := \{ h \in C(\Omega) \mid P(\partial)h = 0 \}$$

with the topology induced by $C(\Omega)$. Then $E$ is closed in $C(\Omega)$ and invariant under $P(\partial)$, hence $E \subset C^\infty(\Omega)$ by assumption. Thus $P(\partial)$ is hypoelliptic.

One could guess that a Banach space $E$ with $(\ast)$ could be found satisfying $(\ast)$ in a suitably weaker form. We will show now that this is impossible in a rather general setting. In fact, let $E$ be a space of hyperfunctions on $\Omega$. Then by the definition of hyperfunctions (see e.g. Hörmander [1, Chapter 9])

$$E(K) := \{ h \in E \mid \text{supp } h \subset K \} \subset A(K)'$$

for any compact $K \subset \Omega$, where $A(K)$ denotes the space of holomorphic germs near $K$. Condition (8) below thus is a very weak substitute for $(\ast)$.

**Theorem 5.** Let $E$ be a Banach space of hyperfunctions on $\Omega$ such that for some compact $K \subset \Omega$

$$\{0\} \neq E(K) \quad \text{and} \quad \text{id}: E(K) \rightarrow A(K)'_\sigma \quad \text{is continuous.} \quad (8)$$

Then $P(\partial)E \not\subset E$ for any non-constant partial differential operator $P(\partial)$.

**Proof.** $E(K)$ is closed in $E$ by (8). Let $P(\partial)E \subset E$. Then $P(\partial)E(K) \subset E(K)$ and

$$P(\partial): E(K) \rightarrow E(K) \quad \text{is continuous} \quad (9)$$

by (8) and the closed graph theorem. Since

$$T: E(K) \times H(\mathbb{C}^n) \rightarrow \mathbb{C}, \quad (h, f) \mapsto \langle h, f \rangle,$$

is separately continuous by (8) and hence continuous, there is $k \in \mathbb{N}$ such that

$$|\langle h, f \rangle| \leq C \|h\| \sup_{|x| \leq k} |f(x)|$$

for any $h \in E(K)$ and any $f \in H(\mathbb{C}^n)$.
Using also (9) we get for any \( h \in E(K) \)
\[
|P^j(iz)h(z)| \leq C\|P^j(\delta)h\|e^{k|z|} \leq CC_j\|\delta\|e^{k|z|} \quad \text{for any } j \in \mathbb{N} \text{ and any } z \in \mathbb{C}^n.
\]
This implies that
\[
\left( \frac{|P(iz)|}{2C_1} \right)^j \to 0 \quad \text{for } j \to \infty \text{ if } \hat{h}(z) \neq 0,
\]
and therefore
\[
|P(iz)| \leq 2C_1 \quad \text{if } \hat{h}(z) \neq 0.
\] (10)
On the other hand there are \( C_2 > 0 \) and an open cone \( \Gamma \subset \mathbb{C}^n \) such that
\[
|P(iz)| \geq C_2|z|^\alpha \quad \text{if } z \in \Gamma \text{ and } |z| \geq 1/C_2.
\] (11)
By assumption, there is \( 0 \neq h \in E(K) \). Since then \( \{z \in \Gamma \mid \hat{h}(z) \neq 0\} \) is unbounded, \( P \) is constant by (10) and (11).

Finally, Theorem 1b) admits the following dual formulation:

**Theorem 6.** Let \( E \) be a Banach space and let \( \Omega \) be connected. Let
\[
\kappa : \mathcal{D}(\Omega) \to E  \quad \text{and} \quad d_j : E \to E \quad \text{be linear and continuous}
\] (12)
and let
\[
\kappa \circ \partial_j = d_j \circ \kappa \quad \text{for } j = 1, \ldots, n.
\] (13)
Then there is \( A > 0 \) such that \( \kappa \) can be uniquely extended to a continuous linear mapping \( \kappa : (H_A)' \to E \).

**Proof.** a) The assumptions of Theorem 1b) are satisfied for \( \{E', i\kappa, i'd_j\} \). Thus there is \( A > 0 \) such that \( i\kappa : E' \to H_A \) is continuous, that is,
\[
\|i\kappa(e')\| \leq C\|e'\|_{E'} \quad \text{for any } e' \in E'
\]
where \( \|\cdot\| \) is the norm in \( H_A \). For the norm \( \|\cdot\|^* \) in \( (H_A)' \) we thus get
\[
\|\kappa(f)\|_{E} = \sup_{|e'| \leq 1} |\langle f, i\kappa(e') \rangle| \leq C\|f\|^* \quad \text{for any } f \in \mathcal{D}(\Omega)
\]
and
\[
\kappa : (\mathcal{D}(\Omega), \|\cdot\|^*) \to E \quad \text{is continuous}.
\]
b) \( \mathcal{D}(\Omega) \) is dense in \( (H_A)' \). Indeed, any \( f \in H_A \) which vanishes on \( \mathcal{D}(\Omega) \), is zero on \( \Omega \), thus \( f = 0 \) on \( \mathbb{C}^n \) by the identity theorem. \( \square \)
Examples for the situation in Theorem 6 are provided by the duals of the spaces from Example 3.

In contrast to the situation of Theorem 1b), the continuity of $d_j$ is needed to prove Theorem 6. In fact, let $E := L_2(\mathbb{R})$, $\times := \text{id}$ and choose a linear complement $F$ of $\mathcal{D}(\mathbb{R})$ in $E$. If we now set $d_0(f) := f'$ for $f \in \mathcal{D}(\mathbb{R})$ and $d_1(f) := 0$ for $f \in F$, then (12) and (13) are satisfied (except for the continuity of $d_1$), but

$$\text{id}: (\mathcal{D}(\mathbb{R}), \| \cdot \|^*) \rightarrow E = L_2(\mathbb{R})$$

is not continuous for the norm $\| \cdot \|^*$ in $(H_A)'$ since

$$\text{id}: (\mathcal{D}(\mathbb{R}), \tau) \rightarrow E$$

is not continuous for the topology $\tau$ induced by $C^\infty(\mathbb{R})'$.

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