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Duality for Monomorphisms and Almost Open Operators Between Locally Convex Spaces.

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Dedicated to the memory of Pascal Laubin.

Abstract

A continuous and linear operator $T : E \to F$ between Banach spaces E and F is a bounded below, resp. surjective (and open), if and only if its transposed operator $T' : F'_b \to E'_b$ is surjective (and open), resp. bounded below. This survey paper presents a complete analysis of the possible extensions of these two results when E and F are both Fréchet spaces or both complete (DF)-spaces.

1 Introduction, Notation and Preliminaries

Bounded below and almost open continuous linear operators between normed spaces, their relation with the topological divisors of zero in the normed algebra of all operators, and the approximate point spectrum have been extensively studied. We refer to the books of Berberian [3] and Harte [14] and to the article of Abramovich, Aliprantis and Polyrakis [1]. We recall the necessary definitions in the frame of locally convex spaces. These classes of operators in the general frame were investigated by the authors in [7]. See also Arizmendi and Harte [2]. A continuous linear operator T between locally convex spaces E and F is called *bounded below* (or a *monomorphism*) if it is injective and the image of every open subset of E is open in $T(E) \subset F$. An operator T is called *almost open* if for every 0-neighborhood U in E there is a 0-neighborhood V in F such that $V \subset \overline{T(U)}$, the closure taken in F. Every almost open operator has dense range. The operator is called *open* if it maps open subsets of E into open subsets of F. Every open operator is surjective. The open mapping theorem of Banach-Schauder assures that a continuous linear operator between Fréchet spaces is surjective if and only if it is open, or equivalently in

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this case, if it is almost open. A continuous linear operator between locally convex spaces which is open onto its image is called a *homomorphism* (e.g. [17, § 32]).

The following classical result, which can be seen in [3, Theorems 57.18 and 57.16] and [14, Theorems 5.5.3 and 5.5.2], is very useful when dealing with operators between Banach spaces.

Theorem 1.1 *Let $T : X \to Y$ be a linear continuous operator between Banach spaces X and Y with transposed operator $T' : Y' \to X'$. Then*

(A) T is bounded below if and only if T' is open,

(B) T is open if and only if T' is bounded below.

The possibility of extending Theorem 1.1 to arbitrary locally convex spaces has been considered several times in the literature, and it is a quite delicate question even for Fréchet or complete (DF)-spaces E and F. We avoided this duality problem in our research in [7] with direct proofs. There are many scattered results which include for example the classical Dieudonné, Köthe, Schwartz homomorphism theorem for Fréchet spaces [17, § 32.3(4)]. The case of Montel (gDF)-spaces (also called (DCF)-spaces) was investigated by Hollstein [15]. Further information can be seen in Köthe's book [17, § 32 and § 33]. The relation to short exact sequences of Fréchet spaces and the lifting of bounded sets is investigated by Meise and Vogt [19, Chapter 26]. See also Dierolf [11]. Very interesting results about the strong transposed operator of a homomorphism were given by Dierolf and Zarnadze [12]. Very recently Wengenroth has utilized the derived functors and homological algebra to investigate in his Habilitationsschrift [25, Chapter 7] when the transposed operator of a homomorphism in the category of locally convex spaces is again a homomorphism. See also Palamodov [20]. Our purpose is to collect some of these results, adding a few new remarks and counterexamples 2.7, 3.3, 3.4, and 3.7. We hope that the present article would be useful as a future reference.

We use the standard notation for operator theory, Banach spaces and locally convex spaces that can be found in [3, 17, 19, 21, 22]. Unless explicitly mentioned, E and F denote Hausdorff locally convex spaces (l.c.s.). The family of all closed absolutely convex neighbourhoods of the origin in E is denoted by $\mathcal{U}_0(E)$ and the set of all closed absolutely convex bounded subsets of a locally convex space E is denoted by $\mathcal{B}(E)$. The set of all continuous and linear operators between E and F is denoted by $L(E, F)$. E'_b stands for the topological dual of E endowed with the topology of the uniform convergence on the bounded subsets of E, and for any operator $T \in L(E, F)$ we represent the transposed operator of T by $T' : F'_b \to E'_b$. If E is a Fréchet space, we denote by E'_{ind} the dual E' of E endowed with the bornological topology associated with E. Grothendieck proved that $E'_{\text{ind}} = (E', \beta(E', E''))$, see [17, § 29.4.2]. A Fréchet space is called *distinguished* if $E'_b = E'_{\text{ind}}$ holds topologically. We refer to [4, 9, 17, 19, 21] for details about distinguished spaces.

The following important result will be used several times.

Theorem 1.2 *(Grothendieck's Homomorphism Theorem) [17, § 32.4(3)].*

(1) The operator $T \in L(E, F)$ is a homomorphism if and only if the following two conditions are satisfied:

(i) $T'(F')$ is $\sigma(E', E)$-closed in E', and

(ii) for every $M_1 \subset E'$-equicontinuous set such that $M_1 \subset T'(F')$ there exists $M_2 \subset F'$-equicontinuous satisfying $M_1 \subset T'(M_2)$.

(2) T is almost open if and only if condition (1)(ii) is verified [17, § 34.1(4)].

A l.c.s. E is a *Pták* space if and only if every $T \in L(E, F)$ which is almost open is open, see [17, § 34.3]. Every Fréchet space is a Pták space. The strong dual of a reflexive Fréchet space is a Pták space. Every quotient of a Pták space is complete [17, § 34.3(3)]. A *Montel* space E is a quasibarrelled space in which each bounded set is relatively compact. A *(DFM)*-space is a (DF)-space which is also Montel.

The class of *quasinormable* spaces was introduced and studied by Grothendieck. It contains Banach spaces and nuclear spaces, and it is also stable under the formation of quotients. Every (DF)-space in the sense of Grothendieck is quasinormable. A l.c.s. is called quasinormable if for every $U \in \mathcal{U}_0(E)$ there is $V \in \mathcal{U}_0(E)$ such that for every $\epsilon > 0$ there is $B \in \mathcal{B}(E)$ with $V \subseteq B + \epsilon U$. Quasinormable Fréchet spaces and their relevance for the lifting of bounded sets can be seen in [19, Chapter 26] and [21, Section 8.3].

2 Extending Theorem 1.1.(A)

This section studies the relation between T being a monomorphism and T' being surjective and open, thus finding extensions of the result (A). Our first result is well known and generalizes [14, Theorem 5.5.3] for l.c.s.

Remark 2.1 Let E and F be l.c.s.

(a) *If $T \in L(E, F)$ is a monomorphism, then $T' : F'_b \to E'_b$ is surjective.* This is easy, fix $u \in E'_b$ and define $v : T(E) \to \mathbb{K}$ by $v := u \circ T^{-1}$. By the Hahn-Banach theorem there exists $w \in F'$ such that $w|_{T(E)} = v$. This implies $T'(w) = u$.

(b) *If $T \in L(E, F)$ satisfies that $T' : F'_b \to E'_b$ is surjective, then T is injective,* as follows from [17, § 32.1(5)].

We analyze the necessity in theorem 1.1.(A). We start with Fréchet spaces E and F.

Proposition 2.2 *Let $T \in L(E, F)$ be a monomorphism between Fréchet spaces. Suppose that one of the following conditions are satisfied:*

(1) Every bounded subset of $F/T(E)$ is contained in the image of a bounded subset of F by the quotient map $q : F \to F/T(E)$,

(2) $F/T(E)$ is a Montel space,

(3) E is distinguished,

then $T' : F'_b \to E'_b$ is a surjective homomorphism.

The assumption (3) holds, in particular, if E is quasinormable or if E is reflexive (this last case follows from [17, § 32.4(7)]).

Proof.

(1) This follows from a deep recent result due to Meise and Vogt [19, Lemma 26.11].

(2) If $F/T(E)$ is Montel, then every bounded subset of $F/T(E)$ is relatively compact. A classical consequence of the Banach-Dieudonné theorem [17, § 21.10(1)] shows that every compact subset of $F/T(E)$ is contained in the image of a compact subset of F by $q : F \to F/T(E)$ (see e.g. [19, Corollary 26.22]). The conclusion now follows from case (1).

(3) The operator $T' : F'_b \to E'_b$ is surjective and continuous by 2.1. Therefore $T' : F'_{\text{ind}} \to E'_{\text{ind}}$ is a surjective continuous operator between (LB)-spaces. By the open mapping theorem, $T' : F'_{\text{ind}} \to E'_{\text{ind}}$ is open. Since E is distinguished. Then $E'_{\text{ind}} = E'_b$. This implies that $T' : F'_b \to E'_b$ is open. Thus $T' : F'_b \to E'_b$ is a surjective homomorphism.

□

Example 2.3 The implication in proposition 2.2 does not hold in general for arbitrary Fréchet spaces. Let E be a non-distinguished Fréchet space (e.g. [17, § 31.7] or [19, Corollary 27.18 and Example 27.19]). Let F be a countable product of Banach spaces such that E is isomorphic to a closed subspace of F. We denote by $T : E \to F$ the continuous inclusion. The operator T is a monomorphism. Suppose that $T' : F'_b \to E'_b$ is a surjective homomorphism. Since F'_b is bornological, its quotient E'_b is also bornological. A contradiction, since E is not distinguished.

Remark 2.4 If $T \in L(E, F)$ is a monomorphism between Fréchet spaces such that $T' : F'_b \to E'_b$ is open and F is distinguished, then E is distinguished too.

It is easy to see that the necessity in Theorem 1.1.(A) does hold for (DF)-spaces.

Proposition 2.5 *If $T \in L(E, F)$ is a monomorphism between (DF)-spaces, then $T' : F'_b \to E'_b$ is surjective and open.*

Proof. $T' : F'_b \to E'_b$ is a continuous linear surjection between Fréchet spaces by 2.1. The conclusion follows from the Banach-Schauder open mapping theorem.

□

We investigate now the converse implication in theorem 1.1.(A). This one always holds for Fréchet spaces as a consequence of Banach-Dieudonné theorem.

Proposition 2.6 *[17, § 33.3(2)] Let E and F be Fréchet spaces and let $T \in L(E, F)$ satisfy that $T' : F'_b \to E'_b$ is surjective and open, then T is a monomorphism.*

The sufficiency in theorem 1.1.(A) is false in general for (DF)-spaces.

Example 2.7 *A continuous linear operator $T : E \to F$ between complete (LB)-spaces which is not a monomorphism but whose transposed operator $T' : F'_b \to E'_b$ is surjective and open*: The following example due to Grothendieck can be seen in [21, 8.6.13]. There is a countable direct sum of reflexive Banach spaces $(F, t) = \oplus_{k=1}^{\infty} F_k$ with a closed subspace E such that the (LB)-space $(E, s) := \mathrm{ind}_n((\oplus_{k=1}^{n} F_k) \bigcap E)$ has a topology strictly finer than the restriction $t|E$ of t to E but such that $(E, s)' = (E, t|E)'$. We denote by $T : (E, s) \hookrightarrow (F, t)$ the inclusion, which is not a monomorphism. On the other hand $T' : (F, t)'_b \to (E, s)'_b$ is surjective by the Hahn-Banach theorem. Since $(F, t)'_b$ and $(E, s)'_b$ are Fréchet spaces, T' is open.

Proposition 2.8 *Let E be a (DFM)-space and F a quasibarrelled (DF)-space. If $T' : F'_b \to E'_b$ is surjective and open, then $T \in L(E, F)$ is a monomorphism.*

Proof. This follows from Grothendieck's homomorphism theorem 1.2.(1)(i), because $T' : F'_b \to E'_b$ lifts bounded subsets of E'_b, as every bounded subset of the Fréchet space E'_b is relatively compact.

□

The best criterion to ensure that a continuous linear operator between (DF)-spaces is a monomorphism is the so-called Baernstein's lemma. It was successfully applied by Bierstedt, Meise and Summers [5] to show that if a weighted inductive limit of Banach spaces of holomorphic functions on an open subset of $\mathbb{C}^n$ satisfies that the linking maps are compact, then the inductive limit topology can be described by weighted sup-seminorms; see e.g. [21, Theorem 11.9.12].

Proposition 2.9 *(Baernstein lemma) [21, Theorem 8.3.55]. Let F be a (DF)-space. Let E be a l.c.s. such that every closed bounded set is compact. Suppose that $T \in L(E, F)$ satisfies that $T^{-1}(B)$ is bounded in E for every bounded subset B of F. Then T is a monomorphism.*

Corollary 2.10 *[21, Proposition 8.6.8(v)]. Let $(F, t) := \mathrm{ind}_n(F_n, t_n)$ be an (LB)-space such that every bounded subset in F is bounded in a step. Let E be a subspace of F. If $(E, t|E)$ is semi-Montel, then $(E, t|E) = \mathrm{ind}_n(E \cap F_n, t_n|(E \cap F_n))$ holds topologically, i.e. E is a limit subspace of (F, t).*

Corollary 2.11 *If E and F are both Fréchet-Montel spaces or both (DFM)-spaces, then $T : E \to F$ is a monomorphism if and only if $T' : F'_b \to E'_b$ is surjective and open.*

In the case of (LB)-spaces (or even (LF)-spaces) E and F, the problems considered in this section are related to the acyclicity of inductive limits and the condition (M) of Retakh. We refer the reader to [23] or [21, Section 8.6 and Remark 8.9.20]. We briefly explain the connection: Let $(F, t) := \mathrm{ind}_n(F_n, t_n)$ an (LF)-space, let E be a subspace of F such that $E \cap F_n$ is closed in (F_n, t_n) for each n. We define $(E, s) := \mathrm{ind}_n(E \cap F_n, t_n|(E \cap F_n))$. Clearly, the injection $T : (E, s) \to (F, t)$ is continuous and injective. The operator T is a weak homomorphism if and only if $T' : (F, t)' \to (E, s)'$ is surjective and T is a monomorphism if and only if $s = t|E$ on E. Characterizations of these two properties in terms of the properties of $\mathrm{ind}_n(F_n/E_n)$ are due to Palamodov, Retakh and Vogt. See [23]. Compare also with Proposition 2.9. The characterizations are useful to study the surjectivity of convolution operators between spaces of (ultra)-distributions. See [6, 13, 23].

3 Extending Theorem 1.1.(B)

This section establishes the relationship between T being almost open and T' being a monomorphism and it extends result (B) mentioned in the introduction. We have a general characterization to determine when T' is a monomorphism.

Theorem 3.1 *[17, § 32.5(2)] Let $T \in L(E, F)$ with E, F l.c.s. Then $T' : F'_b \to E'_b$ is a monomorphism if and only if for every $B \in \mathcal{B}(F)$ there is some $C \in \mathcal{B}(E)$ such that $B \subseteq \overline{T(C)}$, with the closure taken in F.*

An operator $T \in L(E, F)$ is said to *lift bounded sets with closure* if for every $B \in \mathcal{B}(F)$ there is $C \in \mathcal{B}(E)$ such that $B \subseteq \overline{T(C)}$, the closure taken in F. If T lifts bounded sets with closure, then T has dense range. The operator T is said to *lift bounded sets* if every bounded set B in F is contained in the image $T(C)$ of a bounded set C in E by T. Clearly if T lifts bounded sets, then it is surjective. It was proved by Bonet and Dierolf [8] that a surjection between Fréchet spaces which lifts bounded sets with closure, actually lifts bounded sets (see [19, Lemma 26.7]).

The following rather trivial examples show that more conditions are required to ensure that an operator $T \in L(E, F)$ such that $T' : F'_b \to E'_b$ is a monomorphism is (almost) open or surjective.

Example 3.2 (a) Let E be a proper dense subspace of the Banach space F and let $T : E \to F$ be the canonical inclusion. The transpose operator $T' : F'_b \to E'_b$ is an isomorphism and T is continuous, linear and open onto its image but not surjective.

(b) Let (E, t) be an infinite dimensional Banach space, $F := (E, \sigma(E, E'))$, and $T : E \to F$ the identity. Then $T' : F'_b \to E'_b$ is an isomorphism, and T is surjective but not almost open. In fact t is strictly finer than $\sigma(E, E')$ and they are both topologies of the same dual pair, hence the closures of convex sets for the two topologies coincide.

Proposition 3.3 *Let F be a quasibarrelled l.c.s. and let $T \in L(E, F)$ satisfy that $T' : F'_b \to E'_b$ is a monomorphism. Then T is almost open.*

Proof. By theorem 3.1, T has dense range. We apply theorem 1.2.(2) to complete the proof. To check condition (1)(ii) in the statement of 1.2, we fix an E-equicontinuous subset $M_1 \subset T'(F')$. Consequently M_1 is bounded in E'_b. Since T' is a monomorphism, $M_2 := (T')^{-1}(M_1)$ is bounded in F'_b, hence F-equicontinuous, as F is quasibarrelled. Since $T'(M_2) = M_1$ the proof is complete. □

The assumption on F is necessary, even if E and F are complete (DF)-spaces.

Example 3.4 (1) Let (E, t) be the Hilbert space $l_2(I)$ with the index set I of uncountable cardinal. We denote by t' the topology on E of the uniform convergence on the separable bounded subsets of E', and we denote F the space (E, t'). Clearly t and t' are topologies of the dual pair (E, E'), hence they have the same bounded sets. Since E is reflexive, $(E, \sigma(E, E'))$ is quasicomplete, and we can apply [17, § 18.4(4)] to conclude that F is also quasicomplete. Since the space F is (DF) (see e.g. [17, p. 401 remark after § 29.4(6)]), F is a complete (DF)-space by [17, § 29.5(3)]. Let $T : E \to F$ denote the identity. Its transposed operator $T' : F'_b \to E'_b$ is an isomorphism, and T is continuous, linear and surjective but not almost open.

(2) Let G be a non-distinguished Fréchet space. See e.g. [19, Proposition 27.18]. We set $E := G'_{\text{ind}}$, $F := G'_b$ and $T : E \to F$ the identity, which is continuous but not open. Since E and F have the same bounded sets, $T' : F'_b \to E'_b$ is a monomorphism. In this case T is not almost open. (Observe that F is not quasibarrelled.) To see this we denote by $(U_n)_{n=1}^{\infty}$ a basis of 0-neighborhoods in G. Since G is not distinguished, there are $\alpha_n > 0$, $n \in \mathbb{N}$ such that the 0-neighborhood $U := \Gamma(\cup_{n=1}^{\infty} \alpha_n U_n^\circ)$ in G'_{ind} is not a 0-neighborhood in G'_b. Since E is (DF), we apply [21, Proposition 8.2.27] to conclude that the closure $\overline{U}$ of U in G'_b is contained in $2U$. If T were almost open, then $\overline{T(U)} = \overline{U}$ would be a neighborhood in G'_b. This is a contradiction since U is not a 0-neighborhood in G'_b.

Corollary 3.5 *Let E be a Fréchet space or the dual of a reflexive Fréchet space, let F be a quasibarrelled space. If $T \in L(E, F)$ satisfies that $T' : F'_b \to E'_b$ is a monomorphism, then T is surjective and open.*

Proof. The space E is a Pták space by [17, § 34.3(5) and the comments before it]. By proposition 3.3, T is almost open, hence nearly open in the sense of Pták in [17, § 34]. We apply [17, § 34.2(2)] to conclude that T is open into its image. This implies that $T : E \to T(E)$ is a surjective homomorphism. By [17, § 34.3(3)], $T(E)$

is also complete. Since it is a dense subspace of F, we conclude that $T(E) = F$, and $T : E \to F$ is a surjective homomorphism.

□

The main application of results like the corollary above is to conclude the surjectivity of an operator between Fréchet spaces from properties of the transpose operator. A very useful "surjectivity criterion" is due to Meise and Vogt [19, 26.1]. It has been successfully applied e.g. in [6, 2.2]

Proposition 3.6 *[19, 26.1]. Let $T \in L(E,F)$ be an operator between Fréchet spaces. The operator T is surjective if and only if for every bounded subset B in E'_b, then $(T')^{-1}(B)$ is bounded in F'_b.*

Example 3.7 *There are bornological (DF)-spaces E and F and $T \in L(E,F)$ such that $T' : F'_b \to E'_b$ is a monomorphism but T is not open. Our example shows that an almost open operator between bornological (DF)-spaces need not be open*

Let F be a complete (LB)-space which contains a dense barrelled subspace $\tilde{E}$ which is not bornological. Such examples were constructed by Valdivia [24]. We denote by E the bornological space associated with $\tilde{E}$ and by $T : E \to F$ the continuous inclusion. The operator T lifts bounded sets with closure. Indeed, T has dense range and $\tilde{E}$ and F are (DF)-spaces. We can apply [21, Corollaries 8.3.17 and 8.3.25] to conclude that every bounded subset of F is contained in the closure of a bounded subset of $\tilde{E}$. Since E and $\tilde{E}$ have the same bounded sets, it follows that T lifts bounded sets with closure, hence $T' : F'_b \to E'_b$ is a monomorphism. By proposition 3.3, T is almost open. As $\tilde{E}$ is not bornological, we conclude that T is not open onto its image.

We now discuss the converse, and suppose that $T : E \to F$ is a continuous, almost open, linear operator, between l.c.s. We investigate whether $T' : F'_b \to E'_b$ is a monomorphism or, equivalently by Theorem 3.1, whether T lifts bounded sets with closure.

We first suppose that E and F are (DF)-spaces. Our next result if T is supposed to be open is [12, 2.8]. Recall from 3.7 that almost open operators between (DF)-spaces need not be open onto their image.

Proposition 3.8 *Let $T : E \to F$ be continuous, almost open and linear operator. If E is a (DF)-space, then $T' : F'_b \to E'_b$ is a monomorphism.*

Proof. We apply Theorem 3.1. Let $(C_n)_{n=1}^{\infty}$ be a fundamental sequence of bounded sets in E. Assume that there is $C \in \mathcal{B}(F)$ such that for every $n \in \mathbb{N}$ there is $c_n \in C$ with $c_n \notin n\overline{T(C_n)}$, the closure taken in F. By the Hahn-Banach theorem, for every $n \in \mathbb{N}$ there exists $u_n \in F'$ with $u_n(c_n) = n$ and $|u_n(T(x))| \leq 1$ for every $x \in C_n$. The set $\{u_n \circ T | n \in \mathbb{N}\} \subseteq E'$ is $\beta(E', E)$-bounded. Indeed, fix $m \in \mathbb{N}$, as $C_m \subseteq C_n$ for every $n \geq m$ then $|u_n \circ T(z)| \leq 1$ for every $z \in C_m$. Now $u_n \circ T \in C_m^{\circ}$ for every $n \geq m$. Using that $u_1 \circ T, u_2 \circ T, \ldots, u_{m-1} \circ T \in E'$ the conclusion follows.

Since E is (DF), $U := \bigcap_{n=1}^{\infty} \{x \in E \,|\, |u_n \circ T(x)| \leq 1\}$ is a 0-neighborhood in E. As T is almost open, $\overline{T(U)}$ is a 0-neighborhood in F, and we can find $\lambda > 0$ such that $C \subseteq \lambda \overline{T(U)}$. For $n \in \mathbb{N}$ and $x \in U$ we have that $|u_n(Tx)| \leq 1$. Consequently $|u_n(c)| \leq \lambda$ for every $c \in C$ and every $n \in \mathbb{N}$. A contradiction.

□

The assumptions of Proposition 3.8 imply that F is a (DF)-space too.

We suppose now that E is a Fréchet space and that $T : E \to F$ is almost open. Since E is a Pták space, T is open. This implies that $T(E)$ is a Fréchet space. As $T(E)$ is dense in F, we conclude that $T(E) = F$, hence T is surjective and F is a Fréchet space. Consequently the problem we have to consider is the following: *Let $T : E \to F$ be a continuous, linear, surjective operator between Fréchet spaces, when does T lift bounded sets with closure?* This question has been thoroughly studied. We mention [19, Chapter 26] as a reference and we collect in the next proposition some of the known results.

Proposition 3.9 *Let $T : E \to F$ be a continuous, linear, surjective operator between Fréchet spaces. We have:*

(1) There are a Fréchet-Montel space E, a Banach space F and $T : E \to F$ continuous, linear and surjective such that $T' : F'_b \to E'_b$ is not a monomorphism.

(2) $T' : F'_b \to E'_b$ is a monomorphism if and only if T lifts bounded sets.

(3) If F is Montel or ker T *is quasinormable, then $T' : F'_b \to E'_b$ is a monomorphism.*

(4) If E is quasinormable and F is Banach, then $T' : F'_b \to E'_b$ is a monomorphism.

(5) If E is quasinormable and $T' : F'_b \to E'_b$ is a monomorphism, then ker T *is quasinormable.*

Proof.

(1) There are well known examples due to Grothendieck. See [17, § 33.6].

(2) This is a result of Bonet and Dierolf [8]. See also [19, Lemma 26.7].

(3) If F is Montel, this follows from the Banach-Dieudonné theorem, cf. [19, Corollary 26.22, Proposition 26.23]. If ker T is quasinormable, the result, originally due to Palamodov and De Wilde, follows from [19, Lemma 26.13].

(4) It is a result of Miñarro [18]. In [7] we extended the result for a quasinormable l.c.s. E and a normed space F.

(5) This is a result of Cholodovskij [10]. See also [19, Proposition 26.18].

□

Extensions of some of these results can be seen in [25, Chapter 7].

Corollary 3.10 *Let E and F be both Fréchet Montel spaces or both (DFM)-spaces, and let $T : E \to F$ be continuous and linear. The operator T is surjective and open if and only if $T' : F'_b \to E'_b$ is a monomorphism.*

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