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SURJECTIVITY OF CONSTANT COEFFICIENT PARTIAL DIFFERENTIAL OPERATORS ON $\mathcal{A}(\mathbb{R}^4)$ AND WHITNEY'S C_4-CONE

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Abstract. Constant coefficient partial differential operators on the space of all real analytic functions in four variables are considered. The variety of their symbol is decomposed using methods of algorithmic algebraic geometry. This decomposition is needed for the application of a geometric characterization, given recently by the present authors, of those operators whose symbol satisfies Hörmander's Phragmén-Lindelöf condition, which, by earlier work of Hörmander, is equivalent to the surjectivity of the differential operator on the space of real analytic functions.

1. Historical background. Let $P(D)$ be a constant coefficient partial differential operator in $\mathcal{A}(\mathbb{R}^n)$, the space of all real analytic functions on $\mathbb{R}^n$ with values in $\mathbb{C}$. It was shown in 1971 by DeGiorgi and Cattabriga [11] that $P(D)$ is onto provided $n = 2$. They also conjectured that the analogous result need not hold for $n \geq 3$. In fact, they conjectured that the heat equation would be a counterexample. Their claim was proved in 1973 by Piccinini [20]. In the same year, Hörmander [13] provided a characterization of all surjective constant coefficient partial differential operators on an arbitrary convex set in terms of a Phragmén-Lindelöf condition. A vast amount of research followed, of which we can mention only some: Andreotti and Nacinovich [1] and Boiti and Nacinovich [3] considered related problems for systems and Zampieri [24], Braun, Meise, and Vogt [8], and Braun [5] considered the analogous problem in spaces of ultradifferentiable functions.

It turned out that some more questions lead to characterizations in terms of conditions of Phragmén-Lindelöf type. This includes investigations of phenomena of

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Hartogs type by Kaneko [14], the characterization of the constant coefficient partial differential operators that admit a continuous linear right inverse on $C^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ a convex domain, by Meise, Taylor, and Vogt [17, 18], and existence problems for extension operators investigated by Meise and Franken [12].

2. Hörmander's Phragmén-Lindelöf condition. Let $P(D)$ be a constant coefficient partial differential operator and let V be the variety of the principal part of the symbol P. Hörmander has shown in [13] that $P(D)\colon \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective if and only if the following Phragmén-Linelöf condition holds for the plurisubharmonic functions on V; here plurisubharmonicity in a regular point is defined using a chart, while in a singular point we require only that ϕ is locally bounded and satisfies $\limsup_{\zeta\to z} \phi(\zeta) = \phi(z)$.

The precise result is:

Theorem (Hörmander [13]). *$P(D)\colon \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective if and only if there is $A > 0$ such that each plurisubharmonic function $\phi\colon V \to [-\infty, \infty[$ which satisfies (α) and (β) also satisfies (γ), where*

(α) $\phi(z) \le |z|$ *for all* $z \in V$,
(β) $\phi(z) \le 0$ *for all* $z \in V \cap \mathbb{R}^n$,
(γ) $\phi(z) \le A|\operatorname{Im} z|$ *for all* $z \in V$.

3. Introduction to the present paper. In general, it is not easy to relate this (or any other) Phragmén-Lindelöf condition to geometric properties of V. Until recently, $n = 3$ was the only case for which a characterization was known. This result, due to Zampieri, will be stated as Theorem 7 below. We will then recall our recent characterization for the case $n = 4$ from [7]. Roughly speaking, in $n = 4$ the investigation consists of two steps: First, the variety V has to be decomposed. Second the local Phragmén-Lindelöf principle (see Section 5) has to be investigated in one point of each of the sets that were constructed in the first step. There are several examples in [7] for the second step. Thus, in the present paper, our main emphasis will be on the first step. It turns that in order to decompose V in the way needed for the first step, some algorithmic commutative algebra (i.e., calculations with Gröbner bases) is needed. We will describe this in detail in Sections 18 and 19. The operator that will serve as an example is

$$(1) \qquad P(D) = \frac{\partial^5}{\partial y\,\partial x^2\,\partial w^2} - \frac{\partial^5}{\partial y^3\,\partial w^2} - \frac{\partial^5}{\partial y\,\partial z^3\,\partial w} + \frac{\partial^5}{\partial z^5}.$$

We have chosen it because we think its variety has an interesting geometry, while still being tractable. We show in Example 22 that $P(D)\colon \mathcal{A}(\mathbb{R}^4) \to \mathcal{A}(\mathbb{R}^4)$ is not surjective. Note that, by Hörmander [13], surjectivity of $P(D)$ depends on the principal part alone. So it suffices to consider homogeneous polynomials.

4. General setting. We denote by $B(x,r) := \{y \in \mathbb{C}^n \mid |x - y| < r\}$ the euclidean ball of radius r. For an open set $\Omega \subset \mathbb{R}^n$ the space $\mathcal{A}(\Omega)$ consists of all real analytic function $f\colon \Omega \to \mathbb{C}$.

If $P = \sum_{|\alpha|\le m} a_\alpha z^\alpha$ is a polynomial of degree m, then $P_m := \sum_{|\alpha|=m} a_\alpha z^\alpha$ denotes its principal part and

$$P(D) := \sum_{|\alpha|\le m} i^{-|\alpha|} a_\alpha \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

is the corresponding partial differential operator. The variety of its principal part is denoted by $V = V(P_m) = \{z \in \mathbb{C}^n \mid P_m(z) = 0\}$.

The set of regular points of a variety V is denoted by V_{reg}, the set of its singular points by V_{sing}.

Instead of the Phragmén-Lindelöf condition given in Section 2 we use the following one:

5. The local Phragmén-Lindelöf condition. For $\xi \in \mathbb{R}^n$ and $r_0 > 0$ let V be an analytic variety in $B(\xi, r_0)$ which contains ξ. We say that V satisfies the *local Phragmén-Lindelöf condition* $\text{PL}_{\text{loc}}(\xi)$ if there exist positive numbers A and $r_0 \geq r_1 \geq r_2$ such that each $u \in \text{PSH}(V \cap B(\xi, r_1))$ satisfying

(α) $u(z) \leq 1, \ z \in V \cap B(\xi, r_1)$

and

(β) $u(z) \leq 0, \ z \in V \cap \mathbb{R}^n \cap B(\xi, r_1)$

also satisfies

(γ) $u(z) \leq A|\text{Im}\, z|, \ z \in V \cap B(\xi, r_2)$.

6. **Theorem** (Hörmander [13]). *The operator $P(D)\colon \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ as in Section 4 is surjective if and only if $V(P_m)$ satisfies $\text{PL}_{\text{loc}}(\xi)$ at each $\xi \in V(P_m) \cap \mathbb{R}^n \setminus \{0\}$.*

The advantage of the local condition $\text{PL}_{\text{loc}}(\xi)$ is that near $\xi \neq 0$, a homogeneous variety V looks like the product of a variety W, of dimension $\dim V - 1$, with a complex line. This fact will be exploited in Proposition 15.

In the case $n = 2$, the variety V in Theorem 6 is just a finite set of straight lines through the origin. Hence the initial result of DeGiorgi and Cattabriga [11], referred to in Section 1, is an immediate consequence of Theorem 6. The case $n = 3$ was solved by Zampieri:

7. **Theorem** (Zampieri [23], see also Braun [4]). *The operator $P(D)\colon \mathcal{A}(\mathbb{R}^3) \to \mathcal{A}(\mathbb{R}^3)$ is surjective if and only if for each $\xi \in V \cap \mathbb{R}^3 \setminus \{0\}$ the following holds:*

Each irreducible component $[W]_\xi$ of the germ $[V]_\xi$ is regular at ξ and the dimension at ξ of $W \cap \mathbb{R}^3$ as a real analytic manifold is 2.

It is easy to see that the wave operator with one silent parameter,

$$P(D) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \colon \mathcal{A}(\mathbb{R}^4) \to \mathcal{A}(\mathbb{R}^4),$$

is surjective, although $V(P) = V(P_2)$ is singular and irreducible at $(0, 0, 0, 1)$. This shows that the condition of Theorem 7 is not necessary if $n \geq 4$.

8. Overview over the case $n = 4$. For $n = 4$, it is shown in [7] that the geometric characterization of the surjective operators $P(D)$ on $\mathcal{A}(\mathbb{R}^4)$ involves the solution of two different problems:

(I) characterize when a germ $[V]_\xi$ satisfies $\text{PL}_{\text{loc}}(\xi)$,

(II) identify a finite subset S of $V \cap \mathbb{R}^4 \setminus \{0\}$ such that it suffices to check $\text{PL}_{\text{loc}}(\xi)$ at all points in S.

Both steps are carried out in [7]. In fact, in [7] detailed instructions are given how to do part (I) in finitely many steps. In the present paper, we explain how (II) can be accomplished using Gröbner basis algorithms. We will start, however, by recalling step (I) from [7]. To do so, some notions have to be introduced first.

9. **Real simple curves.** A map $\gamma\colon]0,\alpha[\to \mathbb{C}^n$, $\alpha > 0$, is called a *real simple curve* if it admits a convergent Puiseux series expansion $\gamma(t) = \sum_{j=q}^{\infty} \xi_j t^{j/q}$ with $q \in \mathbb{N}$ and $|\xi_q| = 1$. The vector ξ_q is called *tangent vector* to γ at the origin. We are only concerned with properties of γ near $t = 0$. Hence it is no restriction to assume that α is chosen so small that γ is injective. The image of γ is called its *trace*, it is denoted by $\operatorname{tr}\gamma = \gamma(]0,\alpha[)$. A simple curve γ with $\operatorname{tr}\gamma \subset \mathbb{R}^n$ is called a *real simple curve*.

10. **Conoids.** Let γ be a real simple curve in $\mathbb{C}^n$, let $d \geq 1$, let M be a subset of $\mathbb{C}^n$, and let $R > 0$ be sufficiently small. Define

$$\Gamma(\gamma, d, M, R) := \bigcup_{0<t<R} \left(\gamma(t) + t^d M\right).$$

The set $\Gamma(\gamma, d, M, R)$ is called a *conoid* of opening exponent d, core γ, and profile M, truncated at R, provided that it does not contain the origin.

If the opening exponent is 1 and $\operatorname{tr}\gamma$ is a line segment, then $\Gamma(\gamma, d, M, R)$ is a cone. The idea behind the concept of a conoid is the following: When one wants to investigate directional properties of V, then one looks at V inside a sufficiently small cone. However, sometimes this investigation is not fine enough. Then one can look at V inside a suitably chosen conoid.

11. **The limit varieties $T_{\gamma,d}V$.** First, recall the definition of the classical tangent cone T_xV to a variety $V \subset \mathbb{C}^n$ at some point x. It consists of all $\zeta \in \mathbb{C}^n$ such that there are sequences $(z_n)_{n\in\mathbb{N}}$ in V and $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ with $\lim_{n\to\infty} z_n = x$ and $\lim_{n\to\infty} r_n(x - z_n) = \zeta$.

To define an extension of this notion, fix $d \geq 1$ and a real simple curve $\gamma\colon]0,\alpha[\to \mathbb{R}^n$ and set for $t \in]0,\alpha[$

$$V_{\gamma,t,d} := \left\{ w \in \mathbb{C}^n \mid \gamma(t) + wt^d \in V \right\} = \frac{1}{t^d}(V - \gamma(t)).$$

Then the *limit variety* $T_{\gamma,d}V$ is defined to be the set of all $\zeta \in \mathbb{C}^n$ such that there are sequences $(t_j)_{j\in\mathbb{N}}$ in $]0,\alpha[$ and $(w_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^n$ satisfying

$$\lim_{j\to\infty} t_j = 0, \quad \lim_{j\to\infty} w_j = \zeta, \quad \text{and} \quad w_j \in V_{\gamma,t_j,d}.$$

It is shown in [6] that $T_{\gamma,d}V$ is an algebraic subvariety of $\mathbb{C}^n$. In fact, in [6] convergence is proved in a stronger sense: There we show that the current of integration over $V_{\gamma,d,t}$ converges to a current $T_{\gamma,d}[V]$ as t tends to zero. Furthermore, we show the existence of finitely many algebraic subvarieties $W_1, \ldots, W_\mu$ of $\mathbb{C}^n$ such that $T_{\gamma,d}[V] = \sum_{j=1}^{\mu} n_j [W_j]$, where $n_j \in \mathbb{N}$ and $[W_j]$ denotes the current of integration over W_j (i.e., we show that $T_{\gamma,d}[V]$ is a holomorphic chain with nonnegative multiplicities).

For $d > 1$ the limit variety $T_{\gamma,d}V$ is invariant under translations in the direction of ξ_0, the tangent to γ. Furthermore, $T_{\gamma,1}V = T_0V - \xi_0$ (see [6], Proposition 4.1 for both results).

12. (γ, d)-hyperbolicity. Let W be an analytic variety of pure dimension $k \geq 1$ in a neighborhood of $\zeta \in \mathbb{R}^n$. A projection $\pi: \mathbb{C}^n \to \mathbb{C}^n$ is called *noncharacteristic* for W at ζ if the rank of π is k, image and kernel of π are spanned by real vectors, and $T_\zeta W \cap \ker \pi = \{0\}$.

Let V be an analytic variety of pure dimension k in $\mathbb{C}^n$, let γ be a real simple curve, let $d \geq 1$, and fix $\zeta \in T_{\gamma,d}V \cap \mathbb{R}^n$. We say that V is *(γ, d)-hyperbolic* at ζ with respect to the projection π if π is noncharacteristic for $T_{\gamma,d}V$ at ζ and there is a conoid $\Gamma(\gamma, d, U, R)$ where U is a neighborhood of ζ such that $z \in V \cap \Gamma(\gamma, d, U, R)$ is real whenever $\pi(z)$ is real. We say that V is (γ, d)-hyperbolic if V is (γ, d)-hyperbolic with respect to some projection.

13. Hyperbolicity in conoids. Let V be an analytic surface in $\mathbb{C}^3$ which contains the origin. We say that V is *hyperbolic in conoids* if for each real simple curve γ and each $d \geq 1$ the following two conditions hold:

(a) $T_{\gamma,d}V$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\zeta)$ for each $\zeta \in T_{\gamma,d}V \cap \mathbb{R}^3$,
(b) V is (γ, d)-hyperbolic at each real point of $T_{\gamma,d}V$ which is regular.

Note that by the last remark of Section 11 the variety $T_{\gamma,d}V$ is simpler than V: for $d > 1$, the limit variety $T_{\gamma,d}V$ is the product of a curve and a line and $T_{\gamma,1}V$ is the translated tangent cone T_0V, which is a cone over a curve. In particular, there are geometric criteria for condition (a): In the case $d > 1$, the limit variety $T_{\gamma,d}V$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\zeta)$ for each $\zeta \in T_{\gamma,d}V \cap \mathbb{R}^3$ if and only if the criterion of Theorem 7 holds at each $\zeta \in T_{\gamma,d}V \cap \mathbb{R}^3$. In the case $d = 1$, the limit variety $T_{\gamma,1}V$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\zeta)$ for each $\zeta \in T_{\gamma,1}V \cap \mathbb{R}^3$ if and only if T_0V satisfies the criterion of Theorem 7 and T_0V contains no irreducible component which is elliptic (i.e., intersects with $\mathbb{R}^3$ only in $\{0\}$). For details and proofs see [7], Lemma 3.17.

14. Theorem ([7], 5.3). *Let V be an analytic surface in $\mathbb{C}^3$ which contains the origin. Then V satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$ if and only if V is hyperbolic in conoids.*

It is shown in [7], Theorem 5.3, that it suffices to check conditions (a) and (b) of Section 13 only at a finite set of pairs (γ, d) and that this set can be determined from the geometry of V. The details of that construction are given in [7], 5.1. The essence is the following: For $\xi \in S^2$ let B_ξ denote the branch locus of the restriction of an arbitrary projection along ξ to V, i.e.,

$$B_\xi = \{z \in V \mid \langle \nabla f(z), \xi \rangle = 0\},$$

where f is a square-free function whose variety is V. Then basically only those pairs (γ, d) have to be considered such that the origin is singular in $T_{\gamma,d}V$ and such that for almost all $\xi \in S^2$ there are at least two branches S_1 and S_2 of $B_\xi \cap \mathbb{R}^3$ such that the Puiseux series expansions of S_1 and S_2 coincide with the Puiseux series expansion of γ up to, but not including, the term with exponent d, and such that the coefficient of t^d of at least one of the curves S_1 or S_2 differs from the corresponding coefficient for γ.

In particular, the verification of hyperbolicity in conoids can be reduced to the investigation of a finite number of conditions.

So far, we have accomplished step (I) of our general scheme 8. The connection between step (I) and step (II) is established by the following reduction argument:

15. **Proposition** ([7], 6.1). *Let $P \in \mathbb{C}[z_1, \ldots, z_{n+1}]$ be homogeneous and let $\xi \in V(P) \cap \mathbb{R}^{n+1}, \xi \neq 0$ be given. Choose $\xi_1, \ldots, \xi_n \in \mathbb{R}^{n+1}$ such that $\{\xi_1, \ldots, \xi_n, \xi\}$ is a basis of $\mathbb{R}^{n+1}$ and define*

$$Q(z') := P\Big(\sum_{j=1}^{n} z_j \xi_j + \xi\Big), \quad z' := (z_1, \ldots, z_n) \in \mathbb{C}^n.$$

Then $V(P)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ if and only if $V(Q) \subset \mathbb{C}^n$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$.

Unfortunately, Proposition 15 is insufficient from a practical point of view, since it requires the investigation of $\mathrm{PL}_{\mathrm{loc}}(\xi)$ for an infinite number of points. However, concrete calculations show that the points in $V(P) \cap \mathbb{R}^n$ fall into only finitely many classes of points which are all "alike". To derive a general result from this observation, the definition of one of the more exotic Whitney cones has to be recalled.

16. **The Whitney cone $C_4(V, p)$.** Let V be an analytic variety in $\mathbb{C}^n$, let $p \in V$. A vector v belongs to the *Whitney cone* $C_4(V, p)$ if there are sequences $(x_j)_{j \in \mathbb{N}}$ in V_{reg} and $(v_j)_{j \in \mathbb{N}} \in \mathbb{C}^n$ such that each v_j is tangential to V at x_j and such that $\lim_{j \to \infty} x_j = p$ and $\lim_{j \to \infty} v_j = v$ (see Whitney [22], Definition 7.1H). Furthermore, we define

$$C_4(V) = \{(z, v) \in \mathbb{C}^{2n} \mid z \in V,\ v \in C_4(v, z)\}.$$

Note that $C_4(V, p)$ is algebraic and that it contains the tangent cone T_pV, which is $C_3(V, p)$ in Whitney's notation. In particular, $\dim C_4(V, p) \geq \dim V$. Contrary to the behavior of T_pV it may happen that $\dim C_4(V, p) > \dim V$. However, it was shown by Stutz [21], Proposition 3.6, that the set of p with $\dim C_4(V, p) > \dim V$ is an analytic subset of V_{sing} of dimension not exceeding $\dim V - 2$. We need the following version of this result for the algebraic case:

17. **Proposition.** *If V is a homogeneous algebraic hypersurface of $\mathbb{C}^n$ and $p \in V$, then*

(a) *$C_4(V)$ is a homogeneous algebraic subvariety of $\mathbb{C}^{2n}$,*
(b) *$S := \{p \in V \mid \dim C_4(V, p) = n\}$ is an algebraic subset of V_{sing} of dimension not exceeding $n - 3$.*

Proof. By Chirka [9], Proposition 9.2, the set $C_4(V)$ is analytic. Since V is homogeneous, it is easy to see that $C_4(V, \lambda w) = C_4(V, w)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Since $C_4(V, p)$ is a cone, the definition shows that $C_4(V)$ is homogeneous. By Chow's theorem ([9], Remark 7.1) we infer that $C_4(V)$ is algebraic. This proves part (a) of the claim.

For the proof of part (b), note first that by the result of Stutz mentioned above, S is an analytic subset of V_{sing} of dimension not exceeding $n - 3$. Let now $f_1, \ldots, f_k \in \mathbb{C}[z_0, \ldots, z_n]$ be generators for the ideal of all polynomials vanishing on $C_4(V)$. Note that $z \in S$ if and only if $\{z\} \times \mathbb{C}^n \subset C_4(V)$. Thus

$$S = \{z \in \mathbb{C}^n \mid f_j(z, v) = 0 \text{ for all } j \leq k,\ v \in \mathbb{C}^n\}.$$

Hence S is the intersection of a family of algebraic sets, thus algebraic. □

18. **Calculation of S.** Assume $V = V(P)$ for a square-free homogeneous polynomial P. The proof of Proposition 17 provides a hint how to determine the set S explicitly. Since S will play a role later on, we describe this algorithm now:

By the proof of Chirka [9], Proposition 9.2, $C_4(V)$ is the euclidean closure of

$$A := \bigcup_{w \in V_{\mathrm{reg}}} \{w\} \times T_w V.$$

Since $C_4(V)$ is Zariski closed by Proposition 17, it contains the Zariski closure of A. On the other hand, the Zariski topology is coarser than the euclidean topology. Hence $C_4(V)$ coincides with the Zariski closure of A. Elimination theory provides an algorithm to determine the Zariski closure of the projection of an algebraic set. In our case, we write $A = \pi(B)$ for $\pi\colon \mathbb{C}^{3n} \to \mathbb{C}^{2n}$, $(z, v, \tau) \mapsto (z, v)$, and

$$B = \left\{(z, v, \tau) \in \mathbb{C}^{3n} \mid P(z) = 0,\ \langle \nabla P(z), v\rangle = 0,\ \langle \nabla P(z), \tau\rangle = 1\right\};$$

note that the purpose of the last equation in the definition of B is to exclude points z in V_{sing}. Let $I \subset \mathbb{C}[z, v, \tau]$ denote the ideal generated by $g_1 := P$, $g_2 := \langle \nabla P(z), v\rangle$, and $g_3 := \langle \nabla P(z), \tau\rangle - 1$, and let $I_{2n} := I \cap \mathbb{C}[z, v]$ be the corresponding elimination ideal. Then

$$C_4(V) = \left\{(z, v) \in \mathbb{C}^{2n} \mid g(z, v) = 0 \text{ for all } g \in I_{2n}\right\}$$

by Cox, Little, and O'Shea [10], Theorem 4.4.3 To determine the elimination ideal I_{2n}, let G be a Gröbner basis for I with respect to an elimination order where the τ-variables are considered larger than all others. (See [10], Exercise 3.1.6, or Becker and Weispfenning [2], Remark before 6.14, for the definition of an elimination order.) By [10], Exercise 3.1.5, or [2], Proposition 6.15, the elimination ideal I_{2n} is generated by

$$G_{2n} := G \cap \mathbb{C}[z, v].$$

Denote the elements of G_{2n} by $b_1, \dots, b_t$. Then

$$C_4(V) = \{(z, v) \in \mathbb{C}^{2n} \mid b_j(z, v) = 0 \text{ for } j = 1, \dots, t\}.$$

To determine S, it is necessary to find those $z \in V$ for which the elements of G_{2n} provide no restrictions on v, i.e., if b_j is decomposed as $b_j(z, v) = \sum_{\alpha \in \mathbb{N}_0^n} b_{j,\alpha}(z)\, v^\alpha$, then

$$S = \{z \in \mathbb{C}^n \mid b_{j,\alpha}(z) = 0 \text{ for each } \alpha\}.$$

This is a description in terms of finitely many polynomials since all but a finite number of the $b_{j,\alpha}$ vanish identically.

19. A partition of V. To go on with step (II) of our general plan (see Section 8), we decompose an arbitrary homogeneous hypersurface $V \subset \mathbb{C}^4$ into four parts:

(a) V_3 is the set of regular points of $V \setminus \{0\}$,
(b) $V_2 := \{z \in V_{\mathrm{sing}} \mid z \neq 0,\ V_{\mathrm{sing}} \text{ regular in } z,\ \dim C_4(V, z) = 3\}$,
(c) $V_1 := V_{\mathrm{sing}} \setminus (V_2 \cup \{0\})$,
(d) $V_0 := \{0\}$.

It is shown in [7], Lemma 6.6, that for $0 \le d \le 3$ the set V_d is either empty or of dimension d and that V_d and $V_d \cap \mathbb{R}^4$ consist of only a finite number of connected components.

Remark. In Example 22 we will present a case where $V_1 \not\subset \overline{V}_2$ although $V_1 \cap \overline{V}_2 \neq \emptyset$. This shows that $(V_d)_{0 \le d \le 3}$ is not a stratification of V.

The usefulness of the partition $V(P) \cap \mathbb{R}^4 = \bigcup_{d=0}^{3} V_d \cap \mathbb{R}^4$ stems from the observation [7], Proposition 6.7, that $\mathrm{PL}_{\mathrm{loc}}(\xi)$ does not change its boolean value if ξ varies in a connected component of $V_d \cap \mathbb{R}^4$ for fixed $d \in \{0, 1, 2, 3\}$. This leads to the following result:

20. **Theorem** ([7], Theorem 6.9). *Let $P \in \mathbb{C}[Z_1, \dots, Z_4]$ be a polynomial and let P_m be its principal part. Let $Z_{d,1}, \dots, Z_{d,N_d}$ be the connected components of $V_d \cap \mathbb{R}^4$, $d = 1, 2, 3$. Then the following are equivalent:*

(a) *$P(D)\colon \mathcal{A}(\mathbb{R}^4) \to \mathcal{A}(\mathbb{R}^4)$ is surjective.*
(b) *$V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ for each $\xi \in V(P_m) \cap \mathbb{R}^4 \setminus \{0\}$.*
(c) *For each $d \in \{1, 2, 3\}$ and each $j \in \{1, \dots, N_d\}$ there is $\xi \in Z_{d,j}$ such that $V(P_m)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$.*
(d) *For each $\xi \in V(P_m) \cap \mathbb{R}^4 \setminus \{0\}$ all reductions of P_m at ξ are hyperbolic in conoids at the origin.*
(e) *For each $d \in \{1, 2, 3\}$ and each $j \in \{1, \dots, N_d\}$ there is $\xi \in Z_{d,j}$ such that a reduction of P_m at ξ is hyperbolic in conoids at the origin.*

21. *Remark.* (a) Our algorithm calls for a decomposition of $V_d \cap \mathbb{R}^4$, $d = 0, 1, 2, 3$, into connected components. If this is too difficult, it obviously suffices to give, for each d, a finite sequence $z_{d,1}, \dots, z_{d,k_d}$ in $V_d \cap \mathbb{R}^4$ such that each point in $V_d \cap \mathbb{R}^4$ can be connected by an arc to at least one $z_{d,j}$.

(b) The most difficult part in the construction of the sets V_d, $d = 1, 2, 3$, is to determine the set $S = \{z \in V \mid \dim C_4(V, z) = 4\}$. This part of the construction is described in Section 18. Then the points $z_{d,j}$ of part (a) have to be found. To do that, it is useful to write V and V_{sing} as analytic covers. Finally, for all d and j it has to be checked whether V satisfies $\mathrm{PL}_{\mathrm{loc}}(z_{d,j})$. This process is described in [7], Theorem 5.3.

(c) We give a short sketch of the proof of Theorem 20 because it shows why the result is limited to dimension 4 or less. It can be seen from the definition that the set of points $\xi \in V \cap \mathbb{R}^n$ such that V satisfies $\mathrm{PL}_{\mathrm{loc}}(\xi)$ is open in $V \cap \mathbb{R}^n$ for all n. To see that $\mathrm{PL}_{\mathrm{loc}}(\xi)$ is either true or false for all points in a connected component of $V_d \cap \mathbb{R}^4$ simultaneously, it has to be shown that $\mathrm{PL}_{\mathrm{loc}}(\xi)$ fails in a whole neighborhood in $V_d \cap \mathbb{R}^n$ of ξ if it fails at ξ. This is easy in the case $d = 3$ since at a regular point ξ the property $\mathrm{PL}_{\mathrm{loc}}(\xi)$ holds if and only if the real dimension of $V \cap \mathbb{R}^n$ at ξ coincides with the complex dimension of V at ξ. In more abstract terms, this argument uses the fact that since all regular points in a suitable neighborhood share the same complex structure the only important aspect is how well this complex structure overlaps with the real analytic structure of $V \cap \mathbb{R}^n$. Of course, in singular points, the complex structure may vary even if one stays inside V_{sing}. However, Proposition 4.2 of Stutz [21], which is one of the main ingredients in the proof of Theorem 20, states roughly that the complex structure of V_{sing} does not change as long as one stays within a connected component of V_2. In general, for higher codimension no similar result is known to the authors. Fortunately, in our case, i.e., for $n = 4$, the remaining set V_1 consists of finitely many lines. By homogeneity, it suffices to consider only two antipodal points on $L \cap \mathbb{R}^4$ for each of these lines.

Note also that attempts to generalize Theorem 14 suffer from a similar problem.

22. Main example, part I: Construction of a partition. Let P denote the polynomial associated with the partial differential operator defined by (1), i.e.,

$$P(x,y,z,w) = x^2yw^2 - y^3w^2 - yz^3w + z^5.$$

Then $P(D)\colon \mathcal{A}(\mathbb{R}^4) \to \mathcal{A}(\mathbb{R}^4)$ is not surjective.

To follow the scheme presented in Remark 21(b), we begin with the singular set of $V := V(P)$. It is easy to see that

$$V_{\text{sing}} = \big(\{(0,0,0)\} \times \mathbb{C}\big) \cup \big(\mathbb{C}^2 \times \{(0,0)\}\big).$$

Hence $V_0 = \{0\}$ and $V_3 = V \setminus \big(\{(0,0,0)\} \times \mathbb{C}\big) \cup \big(\mathbb{C}^2 \times \{(0,0)\}\big)$. To determine V_1 and V_2 it is necessary to calculate all points z with $\dim C_4(V,z) = 4$. To do so by the method of Section 18, define the following three polynomials in $(x,y,z,w,\xi,\eta,\zeta,\omega,\kappa,\lambda,\mu,\nu) \in \mathbb{C}^{12}$:

$$\begin{aligned}
g_1 &= P = x^2yw^2 - y^3w^2 - yz^3w + z^5,\\
g_2 &= \big\langle \nabla P(x,y,z,w), (\xi,\eta,\zeta,\omega)\big\rangle\\
&= 2\xi yxw^2 + \eta(w^2x^2 - 3y^2w^2 - z^3w)\\
&\qquad + \zeta(-3yz^2w + 5z^4) + \omega(2ywx^2 - 2y^3w - yz^3)\\
g_3 &= \big\langle \nabla P(x,y,z,w), (\kappa,\lambda,\mu,\nu)\big\rangle - 1,\\
&= 2\kappa yxw^2 + \lambda(w^2x^2 - 3y^2w^2 - z^3w)\\
&\qquad + \mu(-3yz^2w + 5z^4) + \nu(2ywx^2 - 2y^3w - yz^3) - 1.
\end{aligned}$$

We will call the variables κ, λ, μ, and ν auxiliary. Following the algorithm described in Section 18 we have to find a Gröbner basis for the ideal I generated by g_1, g_2, and g_3 with respect to an elimination order for which the auxiliary variables are considered larger than the others. We have used the computer algebra system MapleV to calculate such a Gröbner basis. The termorder that was actually applied was a block order in the sense of Becker and Weispfenning [2], Example 5.8(iv), where a graded reverse lexicographic order (see Cox, Little, and O'Shea [10], Definition 2.2.6) was used in each block and the variables in the blocks were ordered as $\omega < \zeta < \eta < \xi < w < z < y < x$ and $\kappa < \lambda < \mu < \nu$. The resulting (reduced) Gröbner basis, which we denote by G, contains 32 elements. It will not be reproduced here; G and the program for the calculation of the points p with $C_4(V,p) = \mathbb{C}^4$ can be found on our homepage.

The Gröbner basis G enables us to determine the elimination ideal I_{2n}. To do this, we have to remove all elements from G that depend on the auxiliary variables. The following five elements remain: $h_1 := g_1$, $h_2 := g_2$, and

$$\begin{aligned}
h_3 := \big(&(20x^3zyw - 20y^3xzw)\zeta + (2y^2xz^3 - 4y^4xw + 4x^3y^2w - 2y^3xzw)\omega\big)\xi\\
&+ \big((30y^4zw + 10x^4zw - 40x^2zy^2w + 10y^2z^4 - 10x^2z^4)\zeta\\
&+ (-x^2yz^3 + 2y^4zw - 8x^2y^3w + 2x^4yw - y^3z^3 + 6y^5w)\omega\big)\eta\\
&+ (-50y^5w - 20y^3z^3 + 100x^2y^3w - 50x^4yw + 20x^2yz^3)\zeta^2\\
&+ (-11x^2z^2y^2 + 3x^2y^3w - 3y^5w + 20x^4zy + 20zy^5 - 40zy^3x^2 + 11z^2y^4)\omega\zeta\\
&+ (4x^4y^2 + zy^5 - zy^3x^2 - 8y^4x^2 + 4y^6)\omega^2,
\end{aligned}$$

$$
\begin{aligned}
h_4 := {} & (2yxz^4 + 4x^3zyw - 4y^3xzw)\xi + (-y^4w^2 - y^2z^4 - x^2z^4 - 8x^2zy^2w + x^2y^2w^2 \\
& \quad + 2x^4zw + z^2yx^2w - wy^2z^3 - z^2y^3w + 6y^4zw)\eta \\
& + (2y^2z^4 + 2y^4zw - 10y^5w + 20x^2y^3w \\
& \quad + y^3z^3 - x^2yz^3 - 10x^4yw - 2x^2zy^2w)\zeta \\
& + (-z^2y^4 - 8zy^3x^2 - y^5w + 4zy^5 - y^3z^3 + 4x^4zy + x^2z^2y^2 + x^2y^3w)\omega, \\
h_5 := {} & 2yxwz^2\xi + (z^2wx^2 - y^3w^2 - 3z^2y^2w - yz^3w + yw^2x^2)\eta \\
& + (5y^3zw + 2yz^4 - 5x^2wyz)\zeta + (-wy^4 + 2yz^2x^2 - z^3y^2 + x^2y^2w - 2z^2y^3)\omega.
\end{aligned}
$$

A point $p = (x, y, z, w)$ satisfies $\dim C_4(V, p) = 4$ if and only if $h_i(p, \cdot) \equiv 0$ for $i = 1, \ldots, 5$, i.e., at all points where the coefficients of all monomials in ξ, η, ζ, and ω vanish. (These coefficients are polynomials in x, y, z, and w.) It is easy to read off these 21 equations from $h_1, \ldots, h_5$. They are again solved with the help of a computer algebra system. The set of solutions consists of four lines:

$$
\begin{aligned}
L_1 &= \mathbb{C} \times \{(0,0,0)\}, & L_3 &= \{(x, x, 0, 0) \mid x \in \mathbb{C}\}, \\
L_2 &= \{(0,0,0)\} \times \mathbb{C}, & L_4 &= \{(x, -x, 0, 0) \mid x \in \mathbb{C}\}.
\end{aligned}
$$

This implies at once

$$
V_1 = \bigcup_{j=1}^{4} L_j \setminus \{0\} \quad \text{and} \quad V_2 = \left(\mathbb{C}^2 \times \{(0,0)\}\right) \setminus \bigcup_{j=1}^{4} L_j.
$$

Since we are proving that $P(D)$ is not onto, we will skip the construction of all $z_{d,j}$ and give only one point ξ where $\mathrm{PL}_{\mathrm{loc}}(\xi)$ fails. There are several choices for ξ. The general idea is that points in $V_1 \cap \mathbb{R}^4$ carry the most information. Therefore, we use $\xi = (1, 0, 0, 0)$ in [7], Example 7.13, where the present result is announced. On the other hand, the points in $V_1 \cap \mathbb{R}^4$ are the most difficult ones to investigate; so here we use a point in $V_2 \cap \mathbb{R}^4$, namely $\xi = (0, 1, 0, 0)$.

23. Main example, part II: Investigation of $\mathrm{PL}_{\mathrm{loc}}(0, 1, 0, 0)$**.** The first thing to do is to calculate a reduction of P at $\xi = (0, 1, 0, 0)$. To do so, use the basis $(\xi_1, \xi_2, \xi_3, \xi)$ with $\xi_1 = (1, 0, 0, 0)$, $\xi_2 = (0, 0, 1, 0)$, and $\xi_3 = (0, 0, 0, 1)$. We set $a = (a_1, a_2, a_3) \in \mathbb{C}^3$ and define

$$
Q(a) = P\Big(\sum_{j=1}^{3} a_j \xi_j + \xi\Big) = a_1^2 a_3^2 - a_3^2 - a_2^3 a_3 + a_2^5.
$$

We have to investigate whether $W := V(Q)$ satisfies $\mathrm{PL}_{\mathrm{loc}}(0)$. The first step in this investigation is to determine the tangent cone. Since the lowest order term of Q is a_3^2, it follows that $T_0 W = \mathbb{C}^2 \times \{0\}$. Hence it is immediate that $\mathrm{PL}_{\mathrm{loc}}(\xi)$ holds for each $\xi \in T_0 W \cap \mathbb{R}^3$. For $v \in \mathbb{R}^3 \setminus \{0\}$ denote by γ_v the real simple curve $\gamma_v(t) = tv$, $0 < t < 1$. The condition of Theorem 14 now requires the investigation of $(\gamma_v, 1)$-hyperbolicity for all v in $T_0 W \cap S^2$. However, in [7], Remark 5.4, we describe a method to single out a finite system $(S_j)_{1 \le j \le J}$ of subsets of $T_0 W \setminus \{0\}$ such that it suffices to check only one v in each S_j. Doing so, it turns out that an interesting direction is $v = (0, -1, 0)$.

We claim that W is not $(\gamma_v, 1)$-hyperbolic. To see this, note first that by [7], 3.19, one can pick an arbitrary noncharacteristic projection in the Definition 12 of (γ, d)-hyperbolicity. We set $\pi(a_1, a_2, a_3) = (a_1, a_2, 0)$. To look at the inverse images of points of the form $(0, -t, 0)$, we have to solve the equation $-a_3^2-(-t)^3a_3+(-t)^5 = 0$. The Puiseux series expansion of the solution is

$$a_3 = it^{5/2} + \text{higher order terms}.$$

Thus W is not $(\gamma_v, 1)$-hyperbolic at 0 with respect to π, and thus, because of [7], 3.19, it is not $(\gamma_v, 1)$-hyperbolic at 0 at all. We have finally proved our claim that $P(D)\colon \mathcal{A}(\mathbb{R}^4) \to \mathcal{A}(\mathbb{R}^4)$ is not surjective.

24. **Connection to properties of elementary solutions.** Hörmander's investigations are based on Fourier analysis. Thus his work and all work that is based on it are limited to convex domains. Starting from Kawai [15] in 1972, criteria for the surjectivity of $P(D)$ on $\mathcal{A}(\Omega)$ for arbitrary domains Ω have been given in terms of elementary solutions. This has finally led to a characterization by Langenbruch [16] in terms of conditions on the real analytic singular support in the real analytic category of hyperfunction elementary solutions.

This has the following interesting consequence: Take a partial differential operator $P(D)$ in four variables whose variety V satiesfies the geometric conditions of [7] (see also Theorem 14). Then Langenbruch's result implies the existence of hyperfunction elementary solutions with large lacunas in the real analytic singular support. Except for very simple cases, no direct proof of existence for these elementary solutions is known. It is an open problem, posed by Langenbruch in [16], to construct these hyperfunction elementary solutions with large lacunas in the real analytic singular support for operators $P(D)$ which are surjective on $\mathcal{A}(\mathbb{R}^n)$ but not locally hyperbolic.

25. **Connection to the existence of a continuous linear right inverse.** Meise, Taylor, and Vogt [19], Corollary 3.14, have shown that a homogeneous partial differential operator $P(D)\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ admits a continuous linear right inverse if and only if P has no elliptic factor and $P(D)\colon \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective. Thus, the results of [7] and of the present work also yield a characterization of the homogeneous constant coefficient partial differential operators on $C^\infty(\mathbb{R}^n)$ that admit a continuous linear right inverse.

It should be noted that Meise, Taylor and Vogt have shown in [17] that the existence of a continuous linear right inverse for $P(D)\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ implies the existence of distributional elementary solutions with large holes in the support. Except in simple cases, there is no explicit construction for these elementary solutions.

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