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Differential Operators on Conic Manifolds: Maximal Regularity and Parabolic Equations

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Dedicated to the memory of Pascal Laubin

ABSTRACT. We study an elliptic differential operator A on a manifold with conic points. Assuming A to be defined on the smooth functions supported away from the singularities, we first address the question of possible closed extensions of A to L_p Sobolev spaces and then explain how additional ellipticity conditions ensure maximal regularity for the operator A. Investigating the Lipschitz continuity of the maps $f(u) = |u|^\alpha$, $\alpha \geq 1$, and $f(u) = u^\alpha$, $\alpha \in \mathbb{N}$, and using a result of Clément and Li, we finally show unique solvability of a quasilinear equation of the form $(\partial_t - a(u)\Delta)u = f(u)$ in suitable spaces.

1. Introduction

Parabolic equations and associated initial value problems or boundary value problems are common models appearing in science and engineering. A well-known example is the mixed initial-boundary value problem for the heat equation

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = g(t,x) & \text{on }]0,T[\times\Omega, \\ u(0,x) = u_0(x) & \text{on } \Omega, \\ u(t,x)|_{\partial\Omega} = u_1(x) & \text{for } t \in]0,T[, \end{cases} \tag{1.1}$$

where Ω is a domain (or manifold) with smooth boundary $\partial\Omega$.

A typical approach to solve (1.1) consists in rewriting it as an abstract evolution equation

$$\begin{cases} \dot{u}(t) + Au(t) = g(t) & \text{on }]0,T[, \\ u(0) = u_0 \end{cases} \tag{1.2}$$

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with an unbounded operator A on a Banach space E_0, whose domain $E_1 = \mathcal{D}(A)$ is densely and continuously embedded into E_0 and incorporates the choice of the boundary condition. The investigation of existence, uniqueness, and regularity of the solution to the problem (1.2), both in the C^1 and L_q setting, has attracted the attention of many authors, see, e.g., Amann [**1**], Arendt et al. [**3**], Da Prato and Grisvard [**8**], Lunardi [**16**], and Prüss [**21**].

(1.1) can be viewed as a special case of the following semilinear problem

$$\begin{cases} \partial_t u(t,x) + Au(t,x) = f(t,u(t,x)) + g(t,x) & \text{on }]0,T[\times \Omega, \\ u(0,x) = u_0(x) & \text{on } \Omega, \\ u(t,x)|_{\partial\Omega} = u_1(x) & \text{for } t \in]0,T[. \end{cases} \tag{1.3}$$

For $A = -\Delta$, the problem (1.3) is a so-called reaction-diffusion system which models phenomena in physics, chemistry and biology (see, e.g., [**5**]). A problem in superconductivity for example is described by the so-called (non-stationary) Ginzburg-Landau equation, where Ω is a cube in $\mathbb{R}^3$, $A = -\Delta$ is the Laplacian, $g \equiv 0$, and the nonlinearity is $f(t,u) = u - u^3$.

More generally, the operator A might also depend on u. In [**6**], Clément and Li developed a method for solving the quasilinear abstract problem

$$\begin{cases} \dot{u}(t) + A(u)u(t) = f(t,u(t)) + g(t) & \text{on }]0,T[\\ u(0) = u_0 \end{cases} \tag{1.4}$$

in the L_q setting, which relies on the properties of the linear problem (1.2) associated with $A(u_0)$. The main requirement is that $A(u_0)$ be of "maximal L_q regularity", which, in short, means that, for every choice of $g \in L_q([0,T], E_0)$ and u_0 in the real interpolation space $(E_1, E_0)_{\frac{1}{q},q}$, (1.2) admits a unique solution $u \in L_q([0,T], E_1) \cap W_q^1([0,T], E_0)$. This together with appropriate Lipschitz continuity of A and f gives existence and uniqueness of the solution to (1.4).

In this paper we consider the case where A is a differential operator on a manifold with conic singularities which we denote by B. Formally, B is a compact Hausdorff space which is a smooth manifold outside a finite number of singular points, while, near each of these points, it has the structure of a cone whose cross-section is a smooth closed manifold.

In order to describe the class of operators we treat in this paper, we blow up B at the singularities so that we obtain a manifold $\mathbb{B}$ with boundary $X = \partial\mathbb{B}$. When we speak of a differential operator A on the conic manifold B or a *cone differential operator*, we shall mean a differential operator on $\operatorname{int}\mathbb{B}$, the interior of $\mathbb{B}$, which has a Fuchs type degeneracy near the boundary, i.e., with respect to a splitting of coordinates $(t,x) \in [0,1[\times X$ near the boundary, it is of the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \qquad a_j \in C^\infty([0,1[, \operatorname{Diff}^{\mu-j}(X)) \tag{1.5}$$

(note that from now on we shall use t no longer as the time variable; instead it corresponds to the distance from the boundary in this neighborhood).

While one should keep the intuitive picture of the conic manifold B in mind, it is important and a great simplification that all the analysis takes place on $\mathbb{B}$. In fact, the only way the singularity of the underlying manifold then enters into the considerations is through the particular form of the operators we study. The Fuchs type degeneracy encodes that the singularities are conic; other types of singularities can be modelled by corresponding degeneracies, see, e.g., Schulze [**24**], Melrose [**18**], Mazzeo [**17**].

The choice of Fuchs type operators is motivated by two observations. First, consider $\mathbb{R}^{n+1}$ as the cone over S^n with vertex at the origin. The blow-up and the use of variables (t, x) described above correspond to the choice of polar coordinates. A simple computation shows that every differential operator with smooth coefficients on $\mathbb{R}^{n+1}$ then takes the form (1.5). Note, however, that the class of Fuchs type operators is considerably larger. It includes operators with discontinuous coefficients at 0 (only the radial limits have to exist). The second observation is that the Laplace-Beltrami operator with respect to a Riemannian metric with a conic degeneracy also has this form, cf. Example 2.1.

As the main result of this paper we shall show in Section 5 that Clément and Li's method yields L_q solvability of certain problems of type (1.4). Our argument relies on the results we obtained in [**7**] on the existence and boundedness of imaginary powers of cone differential operators which – according to a theorem of Dore and Venni [**10**] – implies the maximal L_q-regularity. As a specific example we can treat the case where $A(u) = -a(t^c u)\Delta$ is the Laplace-Beltrami operator for a conic manifold of dimension greater than four, multiplied by a positive C^∞-function a depending on $t^c u$. Here t is a smooth function on $\mathbb{B}$, which is strictly positive and extends the above coordinate t; c is a positive constant. The nonlinearity at the right hand side can be taken to be a linear combination of functions of the form $f(u) = |u|^\alpha$, $\alpha \geq 1$, or $f(u) = u^\alpha$, $\alpha \in \mathbb{N}$.

The analysis on conic manifolds shows many interesting features. One basic problem concerns the domain of the operators. On a closed manifold, an elliptic differential operator defined on all smooth functions has a single closed extension in L_p; its domain is the corresponding Sobolev space, which depends only on the order of the operator. For cone differential operators the situation is quite different. They naturally act on scales of weighted L_p-Sobolev spaces which coincide with the usual ones in the interior and are characterized by a weight function of the type t^γ, $\gamma \in \mathbb{R}$, close to the boundary. For an elliptic operator, defined a priori on $C^\infty_{\mathrm{comp}}(\operatorname{int}\mathbb{B})$, there are, in general, many different closed extensions, parametrized by the subspaces of a finite-dimensional space of singular functions. They depend on the form of A near $t = 0$, as we shall see in Section 3. If one tries to employ maximal regularity techniques, the choice of the domain therefore is of crucial importance.

It is our intention to make the paper readable also for non-specialists in singular calculus. We shall highlight the specific difficulties of the subject and study many examples.

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2. Differential operators on conic manifolds

In this section we recall some basic notions on cone differential operators and weighted Sobolev spaces. We summarize how ellipticity of such operators is described in terms of the symbolic structure, and how it is connected to the Fredholm property of the associated mapping between the Sobolev spaces.

2.1. Operators of Fuchs type. Let $\mathbb{B}$ be a compact manifold with boundary $X = \partial\mathbb{B}$. Fix once and for all a splitting of coordinates $(t, x) \in [0, 1[\times X$ near the boundary X of $\mathbb{B}$. A *cone differential operator* or *Fuchs type operator* on $\mathbb{B}$ is a differential operator – or also a system of differential operators – with smooth coefficients on $\operatorname{int}\mathbb{B}$ which near the boundary has the form

$$A = t^{-\mu} \sum_{j=0}^{\mu} a_j(t)(-t\partial_t)^j, \qquad a_j \in C^\infty([0,1[, \operatorname{Diff}^{\mu-j}(X)), \tag{2.1}$$

where μ is the order of A. Let us stress the three main features of a Fuchs type operator: the singular factor $t^{-\mu}$ determined by the order of A, the smoothness of the coefficients a_j up to $t = 0$, and the totally characteristic derivatives in t-direction. Without difficulty, one could also treat a singular factor $t^{-\nu}$ for some real $\nu > 0$.

Besides the usual homogeneous principal symbol

$$\sigma_\psi^\mu(A) \in C^\infty((T^*\operatorname{int}\mathbb{B}) \setminus 0) \tag{2.2}$$

taking values in bundle homomorphisms, we associate with a cone differential operator A two further symbolic levels: The *rescaled symbol*

$$\tilde{\sigma}_\psi^\mu(A) \in C^\infty((T^*X \times \mathbb{R}) \setminus 0) \tag{2.3}$$

is defined, in local terms, by

$$\tilde{\sigma}_\psi^\mu(A)(x, \xi, \tau) = \sum_{j=0}^{\mu} \sigma_\psi^\mu(a_j)(0, x, \xi)(-i\tau)^j.$$

The *conormal symbol*

$$\sigma_M^\mu(A) \in \mathcal{A}(\mathbb{C}, \operatorname{Diff}^\mu(X)) \subset \mathcal{A}(\mathbb{C}, \mathcal{L}(H_p^s(X), H_p^{s-\mu}(X))) \tag{2.4}$$

is an entire function taking values in (systems of) differential operators on the boundary X. It is given by

$$\sigma_M^\mu(A)(z) = \sum_{j=0}^{\mu} a_j(0) z^j.$$

Ellipticity of A shall be described in terms of the invertibility of the symbols (2.2), (2.3), and (2.4). As the case of systems of operators does not present additional analytical difficulties, we shall not stress this point in the text, below.

EXAMPLE 2.1. *Let $g(t)$ be a family of smooth metrics on X that depends smoothly on a parameter $t \in [0,1[$. Equip $\operatorname{int}\mathbb{B}$ with a metric that coincides with $dt^2 + t^2 g(t)$ near $t = 0$. Near the boundary, the associated Laplace-Beltrami operator Δ is given by*

$$t^{-2}\left\{(t\partial_t)^2 + (n - 1 + t(\log G)'(t))t\partial_t + \Delta_X(t)\right\},$$

where $G = \det(g_{ij})^{1/2}$ and $\Delta_X(t)$ is the Laplacian on X with respect to the metric $g(t)$. Thus Δ is a second order Fuchs type operator on $\mathbb{B}$ with rescaled symbol

$$\tilde{\sigma}_\psi^2(\Delta)(x, \tau, \xi) = \tau^2 + |\xi|^2,$$

where $|\xi|$ refers to the metric $g(0)$. Its conormal symbol is

$$\sigma_M^2(\Delta)(z) = z^2 - (n-1)z + \Delta_X(0).$$

2.2. Weighted cone Sobolev spaces. The intention to find a class of spaces on which Fuchs type operators are naturally continuous leads to the definition of the following scale of weighted Sobolev spaces on the interior of $\mathbb{B}$:

DEFINITION 2.2. *Let $s \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$, and $1 < p < \infty$. Then $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ denotes the space of all distributions $u \in H^s_{p,\mathrm{loc}}(\operatorname{int}\mathbb{B})$ such that*

$$t^{\frac{n+1}{2}-\gamma}(t\partial_t)^k\partial_x^\alpha(\omega u)(t,x) \in L_p([0,1[\times X, \tfrac{dt}{t}dx) \qquad \forall\, k+|\alpha| \le s$$

for some cut-off function ω (the particular choice of ω is irrelevant).

Recall that a cut-off function is a function $\omega \in C^\infty_{\mathrm{comp}}([0,1[)$ such that $\omega \equiv 1$ near $t = 0$. The index s indicates the smoothness of functions, while the weight index γ measures the flatness or rate of vanishing near the boundary. We shall extend this definition to arbitrary $s \in \mathbb{R}$ in the sequel. A Fuchs type operator A as in (2.1) clearly induces continuous mappings

$$A : \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$$

for any γ, s, and p. Apart from a certain normalization, the particular choice of the weight factor $t^{\frac{n+1}{2}}$ and the measure $\frac{dt}{t}dx$ ensures that essential properties of A, like Fredholm property and invertibility, are independent of $s \in \mathbb{R}$ and $1 < p < \infty$, cf. Theorem 2.4.

EXAMPLE 2.3. *If we identify $\mathbb{R}^{1+n}$ with $\mathbb{R}_+ \times S^n$ via polar coordinates (i.e. we regard 0 as a conic singularity), a function u belongs to $L_p(\mathbb{R}^{1+n})$ if and only if $t^{\frac{n+1}{p}}u(t,x) \in L_p(\mathbb{R}_+ \times S^n, \frac{dt}{t}dx)$. This suggests to regard*

$$L_p(B) := \mathcal{H}_p^{0,\gamma_p}(\mathbb{B}), \qquad \gamma_p = (n+1)\left(\tfrac{1}{2} - \tfrac{1}{p}\right), \tag{2.5}$$

as the natural L_p-spaces on the conic manifold B.

Writing the standard Sobolev spaces $H^s_p(\mathbb{R}^{1+n})$ in polar coordinates leads to more complicated spaces (so-called subspaces with asymptotics), cf. [**9**], Appendix A, [**24**], Theorem 1.1.22. We shall illustrate this later on, see (4.4) in Example 4.3.

There are various ways of extending the definition of cone Sobolev spaces to real smoothness parameters s, for example by interpolation and duality. For later purposes we want to sketch a definition based on the use of local coordinates. To this end let

$$H_p^{s,\gamma}(\mathbb{R}^{1+n}_{(t,x)}) = \left\{u \in \mathcal{D}'(\mathbb{R}^{1+n}) \mid e^{-\gamma\langle t\rangle}u(t,x) \in H^s_p(\mathbb{R}^{1+n})\right\} \tag{2.6}$$

with the canonically induced norm. As usual, $\langle t\rangle = (1+t^2)^{1/2}$. Moreover let $S_\gamma : \mathcal{D}'(\mathbb{R}^{1+n}) \to \mathcal{D}'(\mathbb{R}^{1+n})$ be defined by

$$S_\gamma : C^\infty_{\mathrm{comp}}(\mathbb{R}^{1+n}) \to C^\infty_{\mathrm{comp}}(\mathbb{R}^{1+n}), \quad u(t,x) \mapsto e^{(\frac{n+1}{2}-\gamma)t}u(e^{-t},x). \tag{2.7}$$

Let $\kappa_j : U_j \to \mathbb{R}^n$, $j = 1,\dots,N$, and $\chi_j : V_j \to \mathbb{R}^{1+n}$, $j = 1,\dots,M$, provide coverings by coordinate charts of X and $\mathbb{B}$, respectively, and $\{\varphi_j\}$, $\{\psi_j\}$ be corresponding subordinate partitions of unity. Then $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is the space of all distributions such that

(2.8)
$$\begin{aligned}\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} &= \sum_{j=1}^{N} \|S_0(1\times\kappa_j)_*(\omega\varphi_j u)\|_{H_p^{s,\gamma}(\mathbb{R}^{1+n})} + \sum_{j=1}^{M} \|\chi_{j*}((1-\omega)\varphi_j u)\|_{H_p^s(\mathbb{R}^{1+n})}\\ &= \sum_{j=1}^{N} \|S_\gamma(1\times\kappa_j)_*(\omega\varphi_j u)\|_{H_p^s(\mathbb{R}^{1+n})} + \sum_{j=1}^{M} \|\chi_{j*}((1-\omega)\varphi_j u)\|_{H_p^s(\mathbb{R}^{1+n})}\end{aligned}$$

is defined and finite. Here, $\omega \in \mathcal{C}^\infty_{\mathrm{comp}}([0,1[)$ is a cut-off function and $*$ refers to the push-forward of distributions. Up to equivalence of norms, this construction is independent of the choice of κ_j and χ_j.

2.3. Ellipticity of cone differential operators. Each Fuchs type operator A of the form (2.1) induces continuous actions $A: \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$ for any $\gamma, s \in \mathbb{R}$ and $1 < p < \infty$. We next address the question when it is a Fredholm operator.

A cone differential operator A is called *elliptic* with respect to the weight $\gamma \in \mathbb{R}$ if the following conditions are satisfied:

(1) Both the homogeneous principal symbol $\sigma_\psi^\mu(A)$ and the rescaled symbol $\tilde{\sigma}_\psi^\mu(A)$ are invertible,

(2) the conormal symbol is invertible on the line $\mathrm{Re}\, z = \frac{n+1}{2} - \gamma$, i.e.

$$\sigma_M^\mu(A)(z): H_p^s(X) \xrightarrow{\cong} H_p^{s-\mu}(X) \qquad \forall\, \mathrm{Re}\, z = \tfrac{n+1}{2} - \gamma.$$

Due to the spectral invariance of pseudodifferential operators on closed manifolds, condition (2) is independent of the choice of s and p.

Under condition (1) the conormal symbol $\sigma_M^\mu(A)(z) \in \mathcal{A}(\mathbb{C}, L^\mu(X))$ is meromorphically invertible with only finitely many poles in each vertical strip $|\mathrm{Re}\, z| < k$, $k \in \mathbb{N}$, cf. [**24**], Theorem 2.4.20. Condition (2) is imposed in order to ensure that none of these poles lies on the line $\mathrm{Re}\, z = \frac{n+1}{2} - \gamma$.

Together, (1) and (2) allow the construction of a parametrix, cf. [**24**], [**18**].

The following theorem was shown in [**23**]:

THEOREM 2.4. *Let A be a cone differential operator. Then the operator $A: \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{s-\mu,\gamma-\mu}(\mathbb{B})$ is Fredholm if and only if A is elliptic with respect to the weight γ. The Fredholm property as well as the index are independent of s and p.*

EXAMPLE 2.5. *Let Δ be the Laplacian on $\mathbb{B}$ as described in Example* 2.1. *If $0 = \lambda_0 > \lambda_1 > \ldots$ are the eigenvalues of $\Delta_X(0)$, then $\sigma_M^2(\Delta)(z) = z^2 - (n-1)z + \Delta_X(0)$ is not bijective if and only if*

$$z \in \left\{ \tfrac{n-1}{2} \pm \left(\tfrac{(n-1)^2}{4} - \lambda_j\right)^{1/2} \mid j \in \mathbb{N}_0 \right\}.$$

Accordingly, Δ is elliptic with respect to all γ not belonging to this set. For later purpose let us point out that in any case $\sigma_M^2(\Delta)$ is invertible in the strip $0 < \mathrm{Re}\, z < n-1$.

3. Closed extensions of cone differential operators

We consider A as an unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$,

(3.1) $$A: \mathcal{C}^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B}) \subset \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \longrightarrow \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$

and shall investigate its closed extensions. The material in Proposition 3.1 through Corollary 3.4 goes back to Lesch's work [**14**] for the case $p=2$ and we omit proofs. We shall assume that A is elliptic in the interior, i.e. satisfies the ellipticity condition (1) of Section 2.3.

In contrast to elliptic pseudodifferential operators on closed manifolds, a cone differential operator A has in general infinitely many closed extensions. There are two natural extensions - the minimal and maximal extension $A_{\min}=A^{\gamma,p}_{\min}$ and $A_{\max}=A^{\gamma,p}_{\max}$. The minimal extension is the closure of the operator A in (3.1), hence

$$\mathcal{D}(A_{\min})=\bigl\{u\in\mathcal{H}^{0,\gamma}_p(\mathbb{B})\mid\exists\,(u_n)\subset C^\infty_{\mathrm{comp}}(\operatorname{int}\mathbb{B}):\\ u_n\longrightarrow u\text{ and }Au_n\longrightarrow v=:Au\text{ in }\mathcal{H}^{0,\gamma}_p(\mathbb{B})\bigr\};$$

the maximal extension is given by the action of A on the domain

$$\mathcal{D}(A_{\max})=\bigl\{u\in\mathcal{H}^{0,\gamma}_p(\mathbb{B})\mid Au\in\mathcal{H}^{0,\gamma}_p(\mathbb{B})\bigr\}.$$

These two special cases are the key to understanding the general situation.

PROPOSITION 3.1. *The domain of the closure of A is given by*

$$\mathcal{D}(A_{\min})=\Bigl\{u\in\bigcap_{\varepsilon>0}\mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B})\mid Au\in\mathcal{H}^{0,\gamma}_p(\mathbb{B})\Bigr\}.$$

In particular,

$$\mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B})\hookrightarrow\mathcal{D}(A_{\min})\hookrightarrow\mathcal{H}^{\mu,\gamma+\mu-\varepsilon}_p(\mathbb{B})\qquad\forall\,\varepsilon>0.$$

If A additionally satisfies condition (2) *of Section* 2.3 *with respect to the weight $\gamma+\mu$, then (topologically)*

$$\mathcal{D}(A_{\min})=\mathcal{H}^{\mu,\gamma+\mu}_p(\mathbb{B}).$$

If the coefficients a_j in (2.1) are independent of t for t close to 0, this result follows from a simpler version of the above mentioned parametrix construction. The general case can be treated by means of perturbation theory, since for

$$A^{(0)}=\omega\,t^{-\mu}\sum_{j=0}^{\mu}a_j(0)(-t\partial_t)^j\,\omega_0+(1-\omega)\,A\,(1-\omega_1)$$

the cut-off functions $\omega,\omega_j\in C^\infty_{\mathrm{comp}}([0,1[)$ can be chosen in such a way that $A-A^{(0)}$ is $A^{(0)}$-bounded with $A^{(0)}$-bound less than 1, hence $\mathcal{D}(A_{\min})=\mathcal{D}(A^{(0)}_{\min})$, cf. [**12**], Theorem 1.1 on page 190.

Assuming merely the interior ellipticity of A, one obtains:

PROPOSITION 3.2. *$A_{\min}$ is a Fredholm operator.*

Let us now turn to the description of the maximal extension of A. As mentioned in Section 2.3, the conormal symbol of A is meromorphically invertible. Let $p_1,\ldots,p_N$ denote the finitely many poles in the strip $\frac{n+1}{2}-\gamma-\mu<\operatorname{Re}z<\frac{n+1}{2}-\gamma$. Near each p_j we write

$$\sigma^\mu_M(A)(z)^{-1}\equiv\sum_{k=0}^{m_j}R_{jk}(z-p_j)^{-k-1}\tag{3.2}$$

modulo a function holomorphic near p_j. It can be shown that the Laurent coefficients R_{jk} belong to $L^{-\infty}(X)$ and have finite dimensional range. Define

$$G_A = G_A^{\gamma,p} = \operatorname{diag}(G_1, \dots, G_N) : \bigoplus_{j=1}^{N} \mathcal{C}^\infty(X)^{m_j+1} \longrightarrow \bigoplus_{j=1}^{N} \mathcal{C}^\infty(X)^{m_j+1}$$

by the left upper triangular matrices

$$G_j = (g_{ik}^j)_{0\le i,k\le m_j} \quad \text{with} \quad g_{ik}^j = \begin{cases} R_{j,i+k} & \text{if } i+k \le m_j \\ 0 & \text{else} \end{cases}.$$

G_A is a finite rank operator.

PROPOSITION 3.3. *There exists a finite dimensional vector space $\mathcal{E}_A = \mathcal{E}_A^\gamma \subset \mathcal{H}_p^{\infty,\gamma}(\mathbb{B})$ with*

$$\mathcal{D}(A_{\max}) = \mathcal{D}(A_{\min}) \oplus \mathcal{E}_A, \qquad \dim \mathcal{E}_A = \operatorname{rank} G_A,$$

as a topologically direct sum. The space $\mathcal{E}_A$ does not depend on $1 < p < \infty$.

COROLLARY 3.4. *Let $A : \mathcal{C}_{\text{comp}}^\infty(\operatorname{int}\mathbb{B}) \subset \mathcal{H}_p^{0,\gamma}(\mathbb{B}) \to \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ be given. Then:*

a) *Any closed extension of A is given by the action of A on a domain $\mathcal{D}(A_{\min}) \oplus V$ with a subspace V of $\mathcal{E}_A$.*
b) *A has a unique closed extension $A_{\min}^{\gamma,p} = A_{\max}^{\gamma,p}$ if and only if the conormal symbol $\sigma_M^\mu(A)(z)$ is invertible for all z with $\frac{n+1}{2} - \gamma - \mu < \operatorname{Re} z < \frac{n+1}{2} - \gamma$.*

EXAMPLE 3.5. *If Δ is the Laplacian on $\mathbb{B}$ as introduced in Example 2.1, we saw in Example 2.5 that the conormal symbol $\sigma_M^2(\Delta)(z)$ is invertible for all z with $0 < \operatorname{Re} z < n-1$ but not for $z = 0$ and $z = n-1$. Hence*

$$\Delta : \mathcal{C}_{\text{comp}}^\infty(\operatorname{int}\mathbb{B}) \subset \mathcal{H}_q^{0,\gamma_p}(\mathbb{B}) \longrightarrow \mathcal{H}_q^{0,\gamma_p}(\mathbb{B}), \qquad 1 < p, q < \infty,$$

cf. (2.5), has a unique closed extension if and only if $\frac{n+1}{2} - \gamma_p - 2 \ge 0$ and $\frac{n+1}{2} - \gamma_p \le n-1$. These conditions are satisfied if and only if

$$2\max(p, p') \le n+1, \tag{3.3}$$

where, as usual, p' denotes the number dual to p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

To describe the space $\mathcal{E}_A$ from Proposition 3.3, let us assume for simplicity that the coefficients a_j in (2.1) are independent of t for t close to 0. If the inverted conormal symbol of A is as in (3.2), then

$$\mathcal{E}_A = \left\{ \omega \sum_{j=1}^{N} \sum_{l=0}^{m_j} \zeta_{jl}(u) t^{-p_j} \log^l t \mid u \in \mathcal{C}_{\text{comp}}^\infty(]0,1[\times X) \right\} \tag{3.4}$$

with an arbitrary (fixed) cut-off function $\omega \in \mathcal{C}_{\text{comp}}^\infty([0,1[)$ and the linear finite rank mappings $\zeta_{jl} : \mathcal{C}_{\text{comp}}^\infty(]0,1[\times X) \to \mathcal{C}^\infty(X)$ being defined by

$$\zeta_{jl}(u) = \frac{(-1)^l}{l!} \sum_{k=l}^{m_j} \frac{1}{(k-l)!} R_{jk} \frac{\partial^{k-l}(\mathcal{M}u)}{\partial z^{k-l}}(p_j + \mu);$$

here, $\mathcal{M}u = \mathcal{M}_{t\to z}u \in \mathcal{A}(\mathbb{C}, \mathcal{C}^\infty(X))$ denotes the Mellin transform of u. In case the coefficients depend on t, one can show that $\mathcal{E}_A \subset V$, where V is a finite-dimensional space of singular functions which is similar to the right hand side in (3.4).

EXAMPLE 3.6. *Let us reconsider the Laplacian introduced in Example 2.1 specializing to* $\dim \mathbb{B} = 2$ *with* $X = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ *and metric* $dt^2 + t^2 d\vartheta^2$ *on* $[0,1[\times S^1$, *where* $d\vartheta^2$ *is the standard metric on* S^1. *Then the conormal symbol* $\sigma_M^2(\Delta)(z) = z^2 + \partial_\vartheta^2$ *has the non-bijectivity points* $p_j = j \in \mathbb{Z}$. *Passing to Fourier series, outside* $\mathbb{Z}$ *the inverse is given by*

$$(z^2+\partial_\vartheta^2)^{-1}: \quad \sum_k c_k e^{ik\vartheta} \mapsto \sum_k \tfrac{c_k}{z^2-k^2} e^{ik\vartheta}.$$

For fixed j, *only the terms coming from* $k = \pm j$ *are not holomorphic near* j. *This shows that*

$$(z^2+\partial_\vartheta^2)^{-1} \equiv R_{01}\, z^{-2}$$

near $z = 0$ *modulo holomorphic functions, while, near* $z = j \neq 0$,

$$(z^2+\partial_\vartheta^2)^{-1} \equiv R_{j0}\,(z-j)^{-1},$$

where the Laurent coefficients are given by

$$R_{01} f(\vartheta) = \hat{f}_0, \qquad R_{j0} f(\vartheta) = \tfrac{1}{2j}\,\hat{f}_{-j}\, e^{-ij\vartheta} + \tfrac{1}{2j}\,\hat{f}_j\, e^{ij\vartheta}$$

($\hat{f}_k$ *denoting the* k*-th Fourier coefficient of* f). *Thus we obtain*

$$\dim \mathcal{D}(\Delta_{\max}^{\gamma,p})/\mathcal{D}(\Delta_{\min}^{\gamma,p}) = \begin{cases} 2 & \text{if } \gamma \in \mathbb{Z} \\ 4 & \text{else} \end{cases}$$

In the particular case $\gamma_p = 1 - \frac{2}{p}$, *cf.* (2.5), *the domain of the maximal extension is given by*

$$\mathcal{D}(\Delta_{\max}^{0,q}) = \mathcal{D}(\Delta_{\min}^{0,q}) \oplus \omega\,\mathrm{span}(1, \log t), \qquad p = 2,$$

for any $1 < q < \infty$, *and for* $p \neq 2$ *by*

$$\mathcal{D}(\Delta_{\max}^{\gamma_p,q}) = \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B}) \oplus \omega \begin{cases} \mathrm{span}(1, \log t, e^{i\vartheta}t, e^{-i\vartheta}t) & p > 2 \\ \mathrm{span}(1, \log t, e^{i\vartheta}t^{-1}, e^{-i\vartheta}t^{-1}) & p < 2 \end{cases}$$

4. Bounded imaginary powers

The boundedness of purely imaginary powers A^{iy}, $y \in \mathbb{R}$, of an operator $A : \mathcal{D}(A) \subset Y \to Y$ is closely related to the unique solvability of the parabolic equation

$$\begin{cases} \dot{u} + Au = f & \text{on }]0,T[\\ u(0) = u_0. \end{cases} \tag{4.1}$$

In **[10]** Dore and Venni proved the following theorem:

THEOREM 4.1. *Let* $A : \mathcal{D}(A) \subset Y \to Y$ *be a closed densely defined and positive operator in a* UMD*-space* Y. *If the imaginary powers of* A *exist and satisfy the estimate*

$$\|A^{iy}\|_{\mathcal{L}(Y)} \le c\, e^{\theta|y|} \qquad \forall\, y \in \mathbb{R} \tag{4.2}$$

for some $0 < \theta < \frac{\pi}{2}$, *then the initial value problem* (4.1) *with* $u_0 = 0$ *has, for any* $f \in L_r([0,T], Y)$, $1 < r < \infty$, *a unique solution*

$$u \in W_r^1([0,T], Y) \,\cap\, L_r([0,T], \mathcal{D}(A)).$$

Moreover, u, $\dot{u}$ *and* Au *depend continuously on* f.

Positivity of a linear operator here means that the resolvent set $\varrho(A)$ contains all non-negative reals, and $\|(A+\lambda)^{-1}\|_{\mathcal{L}(Y)} = O(\lambda^{-1})$ for $\lambda \geq 0$. In applications, the assumption on Y to be a UMD-space is not very restrictive. For example, $L_p(\Omega, d\mu)$, $1 < p < \infty$, is a UMD-space for any σ-finite measure space (Ω, μ), cf. [**1**, Theorem 4.5.2]. This is then also true for the cone Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$, since $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is a weighted L_p-space on $\mathbb{B}$ and, by the existence of order reductions, $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is isomorphic to $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ for any $s \in \mathbb{R}$.

The key assumption of Theorem 4.1 is the existence of the imaginary powers together with the estimate (4.2). In [**7**] we gave criteria, when this assumption holds true for the minimal or maximal extension of a cone differential operator A. To describe these criteria let us recall the notion of the *model cone operator* $\widehat{A}$ associated with A. The idea is to freeze the coefficients of A at $t = 0$ so that we obtain an operator that lives on the infinite cone over X. On this cone we have a natural choice of Sobolev spaces together with a weight function at the origin. For practical reasons, we work on the cylinder $X^\wedge := \mathbb{R}_+ \times X$ with (t, x)-coordinates. Then

$$\widehat{A} = t^{-\mu} \sum_{j=0}^{\mu} a_j(0)(t\partial_t)^j \, : \, C^\infty_{\text{comp}}(X^\wedge) \subset \mathcal{K}_p^{0,\gamma}(X^\wedge) \longrightarrow \mathcal{K}_p^{0,\gamma}(X^\wedge), \tag{4.3}$$

for A as in (2.1) and the scale of Sobolev spaces is defined as follows:

DEFINITION 4.2. *$\mathcal{K}_p^{s,\gamma}(X^\wedge)$ consists of all distributions $u \in H^s_{p\,\text{loc}}(X^\wedge)$ such that for some cut-off function $\omega \in C^\infty_{\text{comp}}([0,1[)$*

i) *$\omega u \in \mathcal{H}_p^{s,\gamma}(\mathbb{B})$,*
ii) *if $\kappa : U \to \mathbb{R}^n$ is a coordinate chart of X and we set $\chi(t,x) = (t, t\kappa(x))$, then $\chi_*[(1-\omega)\phi u] \in H_p^s(\mathbb{R}^{1+n})$ for any $\phi \in C^\infty_{\text{comp}}(U)$.*

For $p = 2$, the spaces $\mathcal{K}_p^{s,\gamma}(X^\wedge)$ were introduced by Schulze, see [**24**].

If A satisfies ellipticity condition (1) of Section 2.3, the extensions of $\widehat{A}$ can be described quite similar to the extensions of A. In particular,

$$\mathcal{D}(\widehat{A}_{\max}) = \mathcal{D}(\widehat{A}_{\min}) \oplus \mathcal{E}_A$$

with $\mathcal{E}_A$ from (3.4) (now considered as a function space on $X^\wedge$). In case A also satisfies condition (2) of Section 2.3 with respect to the weight $\gamma + \mu$,

$$\mathcal{D}(\widehat{A}_{\min}) = \mathcal{K}_p^{\mu,\gamma+\mu}(X^\wedge).$$

EXAMPLE 4.3. *Let Δ be the Laplacian on $\mathbb{B}$ from Example 3.6. Then $\widehat{\Delta}$ is given by $t^{-2}((t\partial_t)^2 + \partial_\vartheta^2)$ acting on $\mathcal{K}_p^{0,\gamma}(S^{1\wedge})$. In the special case $\gamma = \gamma_p$ we have $\mathcal{K}_p^{0,\gamma_p}(S^{1\wedge}) = L_p(\mathbb{R}^2)$ via polar coordinates, cf. Example 2.3. Hence the associated model cone operator is just the standard Laplacian, i.e.*

$$\widehat{\Delta} = \widehat{\Delta}^p = \Delta_{\mathbb{R}^2} \, : \, C^\infty_{\text{comp}}(\mathbb{R}^2 \setminus \{0\}) \subset L_p(\mathbb{R}^2) \longrightarrow L_p(\mathbb{R}^2).$$

As in Example 3.6 we get

$$\mathcal{D}(\widehat{\Delta}^p_{\max}) = \mathcal{D}(\widehat{\Delta}^p_{\min}) \oplus \omega \begin{cases} \operatorname{span}(1, \log t, e^{i\vartheta}t, e^{-i\vartheta}t) & p > 2 \\ \operatorname{span}(1, \log t) & p = 2 \\ \operatorname{span}(1, \log t, e^{i\vartheta}t^{-1}, e^{-i\vartheta}t^{-1}) & p < 2 \end{cases}$$

with $\mathcal{D}(\widehat{\Delta}^p_{\min}) = \mathcal{K}_p^{2,\gamma_p+2}(S^{1\wedge})$ *in case* $p \neq 2$. *A particular closed extension of* $\widehat{\Delta}^p$ *is given by the action of* $\Delta_{\mathbb{R}^2}$ *on* $H^2_p(\mathbb{R}^2)$. *We then obtain*

$$H^2_p(\mathbb{R}^2) = \mathcal{D}(\widehat{\Delta}^p_{\min}) \oplus \omega \begin{cases} \operatorname{span}(1, e^{i\vartheta}t, e^{-i\vartheta}t) & p > 2 \\ \operatorname{span}(1) & p \leq 2 \end{cases} \tag{4.4}$$

In fact, we know that $H^2_p(\mathbb{R}^2) \subset \mathcal{D}(\widehat{\Delta}^p_{\max})$. *Moreover, the Sobolev embedding theorem ensures that* $H^2_p(\mathbb{R}^2)$ *is a subspace of* $C(\mathbb{R}^2)$. *Thus, in case* $p > 2$, *the function* $\log t$ *cannot belong to* $H^2_p(\mathbb{R}^2)$. *For the same reason, we can exclude* $\operatorname{span}(\log t, e^{i\vartheta}t^{-1}, e^{-i\vartheta}t^{-1})$ *in case* $p \leq 2$. *Finally, in cartesian coordinates,* $e^{\pm i\vartheta}t = x \pm iy$ *is a smooth function on* $\mathbb{R}^2$ *and thus* $\omega e^{\pm i\vartheta}t$ *belongs to* $H^2_p(\mathbb{R}^2)$.

THEOREM 4.4. *Let the cone differential operator* A *satisfy*

(E) *both* $\sigma^\mu_\psi(A)$ *and* $\tilde{\sigma}^\mu_\psi(A)$ *have no spectrum in* Λ_θ,

where $\Lambda_\theta = \{z \in \mathbb{C} \mid |\arg z| \geq \theta\} \cup \{0\}$ *is a closed sector and* $0 \leq \theta < \pi$.

a) *If* A *satisfies condition* (2) *of Section* 2.3 *with respect to* $\gamma + \mu$ *and*

(E$_{\min}$) $\widehat{A}_{\min}$ *has no spectrum in* $\Lambda_\theta \setminus \{0\}$

then there exists a $\varrho \geq 0$ *such that*

$$\|(A_{\min} + \varrho)^{it}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} \leq c\, e^{\theta|t|} \qquad \forall\, t \in \mathbb{R}.$$

b) *If* A *satisfies condition* (2) *of Section* 2.3 *and*

(E$_{\max}$) $\widehat{A}_{\max}$ *has no spectrum in* $\Lambda_\theta \setminus \{0\}$

then there exists a $\varrho \geq 0$ *such that*

$$\|(A_{\max} + \varrho)^{it}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} \leq c\, e^{\theta|t|} \qquad \forall\, t \in \mathbb{R}.$$

Note that Theorem 4.1 then also holds true for $A_{\min}$ and $A_{\max}$, respectively, since equation (4.1) is equivalent to $\dot{v} + (A + \varrho)v = g$ with $g(t) = e^{\varrho t}f(t)$ and $v(t) = e^{\varrho t}u(t)$.

EXAMPLE 4.5. *Let* $\Delta = \Delta^{\gamma_p,q}$ *be as in Example* 3.5. *If* $2\max(p,p') - 1 < n$, *cf.* (3.3), *then* $-\Delta$ *satisfies both conditions* (E) *and* (E$_{\min}$) *for any* $0 < \theta < \frac{\pi}{2}$. *In fact, condition* (E) *is clearly fulfilled. It is more difficult to check* (E$_{\min}$). *Details can be found in Theorem* 7.1 *of* [**7**]. *Correspondingly, the heat equation* $\partial_t u - \Delta u = f$, $u(0) = 0$, *has a unique solution* $u \in W^1_r([0,T], \mathcal{H}^{0,\gamma_p}_q(\mathbb{B})) \cap L_r([0,T], \mathcal{H}^{2,\gamma_p+2}_q(\mathbb{B}))$ *for any* $f \in L_r([0,T], \mathcal{H}^{0,\gamma_p}_q(\mathbb{B}))$, $1 < q < \infty$.

The idea of proving Theorem 4.4 is to consider $\lambda - A$ as an element of a suitable parameter-dependent pseudodifferential calculus on $\mathbb{B}$ (i.e., the cone algebra as introduced by Schulze [**24**]). This technique was also used by Gil [**11**].

Conditions (E) and (E$_{\min}$) then assure that $\lambda - A$ is an elliptic element in this calculus, and therefore we find a parametrix $R(\lambda)$ which coincides with the resolvent $(\lambda - A)^{-1}$ for large $\lambda \in \Lambda_\theta$. This yields, cf. Proposition 4.7 of [**7**], that $\|(\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} = \|R(\lambda)\|_{\mathcal{L}(\mathcal{H}^{0,\gamma}_p(\mathbb{B}))} = O(|\lambda|^{-1})$ for $|\lambda| \to +\infty$ and thus complex powers of A can be defined by the Dunford integral

$$A^z = \frac{1}{2\pi i}\int_{\mathcal{C}} \lambda^z (\lambda - A)^{-1} d\lambda, \qquad \operatorname{Re} z < 0, \tag{4.5}$$

where C is an appropriate path that coincides with $\partial\Lambda_\theta$ away from 0. Assuming that $(\lambda - A)^{-1}$ exists in the whole sector Λ_θ (which can be achieved replacing A by $A+c$), the use of the microlocal structure of $(\lambda - A)^{-1} = R(\lambda)$ allows to show that $\|A^z\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_p e^{\theta|\mathrm{Im}\, z|}$ for $|\mathrm{Re}\, z|$ sufficiently small. This estimate then extends to the purely imaginary powers. The result for the maximal extension follows from the one for the closure by passing to the adjoint.

The proof of Theorem 4.4 relies only on the structure of the parametrix to $\lambda - A$. Thus, corresponding results are true for others than the minimal or maximal extension of A, as soon as one finds criteria that ensure the existence of such a parametrix.

The following example shows that, for the two-dimensional Laplacian, it is neither the minimal nor the maximal extension which is most interesting. Instead we show that there is an intermediate extension generating a holomorphic semigroup.

Let $A = -\Delta$ be the Laplacian as in Example 3.6. Denote by A_p the extension with domain

$$\mathcal{D}(A_p) = \mathcal{D}(\Delta_{\min}^{\gamma_p,p}) \oplus \omega \begin{cases} \mathrm{span}(1, te^{i\vartheta}, te^{-i\vartheta}), & p > 2 \\ \mathrm{span}(1), & p \leq 2 \end{cases} \tag{4.6}$$

cf. (4.4). For $p \leq 2$, the functions $\omega e^{i\vartheta}t$ and $\omega e^{-i\vartheta}t$ both are elements of $\mathcal{D}(\Delta_{\min}^{\gamma_p,p})$. In fact, this follows from Proposition 3.1, since $\omega e^{\pm i\vartheta}t \in \mathcal{H}_p^{\infty,\gamma_p+2-\varepsilon}(\mathbb{B})$ for every $\varepsilon > 0$, and $\Delta(e^{\pm i\vartheta}t) = 0$ implies that $\Delta(\omega e^{\pm i\vartheta}t) \in \mathcal{H}_p^{\infty,\gamma_p}(\mathbb{B})$. For $p < 2$, we even have $\omega\, t\, u \in \mathcal{H}_p^{\infty,\gamma_p+2}(\mathbb{B})$ for every function u which is smooth up to the boundary of $\mathbb{B}$; this is no longer true in case $p = 2$.

Let $0 < \theta < \frac{\pi}{2}$ be arbitrary. We shall show that the resolvent $(\lambda - A_p)^{-1}$ exists for all but finitely many $\lambda \in \Lambda_\theta$ and satisfies

$$\|(\lambda - A_p)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))} = O(|\lambda|^{-1}) \text{ for } |\lambda| \to \infty.$$

COROLLARY 4.6. $-\Delta$ *fulfills neither condition* $(\mathrm{E}_{\min})$ *of Theorem 4.4 nor* $(\mathrm{E}_{\max})$. *Even more is true: both* $\lambda + \Delta_{\min}^{\gamma_p,p}$ *and* $\lambda + \Delta_{\max}^{\gamma_p,p}$ *are non-invertible for all* $\lambda \in \mathbb{C}$.

PROOF. Due to the compact embedding $\mathcal{D}(\Delta_{\max}^{\gamma_p,p}) \hookrightarrow \mathcal{H}_p^{0,\gamma_p}(\mathbb{B})$, the spectrum of $\Delta_{\max}^{\gamma_p,p}$ is either all of $\mathbb{C}$ or a discrete set. In the second case we thus would find a point λ, which belongs simultaneously to the resolvent sets of A_p and $\Delta_{\max}^{\gamma_p,p}$. However, this cannot be true, since $\mathcal{D}(A_p)$ is a proper subspace of the domain of the maximal extension. The argument for the minimal operator is analogous, since its domain is a proper subspace of $\mathcal{D}(A_p)$. □

To obtain the statement on the resolvent of A_p, let $0 < \varrho < 1$ be fixed and let $\Delta_{2\mathbb{B}}$ denote the Laplacian on $2\mathbb{B}$ (which is the double of $\mathbb{B}$ or any smooth closed manifold containing $\mathbb{B}$ as a submanifold) with respect to a metric that coincides with the given metric on $\mathbb{B} \setminus ([0, \frac{\varrho}{2}] \times X)$. Then define $R_p(\lambda) : \mathcal{H}_p^{0,\gamma_p}(\mathbb{B}) \to \mathcal{D}(A_p)$ by

$$R_p(\lambda) = \omega(\lambda - \widehat{A}_p)^{-1}\omega_0 + (1 - \omega)(\lambda - \Delta_{2\mathbb{B}}^p)^{-1}(1 - \omega_1) \tag{4.7}$$

where $\omega, \omega_j \in C_0^\infty([0,1[)$ are cut-off functions satisfying $\omega \equiv \omega_j \equiv 1$ on $[0, \varrho]$ and $\omega\omega_1 = \omega_1$, $\omega_0\omega = \omega$. Moreover, let $\Delta_{2\mathbb{B}}^p$ be the unbounded operator in $L_p(2\mathbb{B})$ acting like $\Delta_{2\mathbb{B}}$ on the domain $H^2_p(2\mathbb{B})$. Note that (4.7) makes sense for sufficiently large $\lambda \in \Lambda_\theta$ and

$$\omega(\lambda - \widehat{A}_p)^{-1}\omega_0 = \omega\, \mathrm{op}_y\Big(\frac{1}{\lambda - |\eta|^2}\Big)\omega_0 \qquad \forall\, \lambda \notin [0, \infty[.$$

On the right-hand side, op denotes the usual pseudodifferential action, y and η are the variables and covariables for $\mathbb{R}^2$, if we identify $L_p(\mathbb{R}^2)$ with $\mathcal{K}^{0,\gamma_p}_p(S^{1\wedge})$ and $\mathcal{D}(\widehat{A}_p)$ with $H^2_p(\mathbb{R}^2)$ via polar coordinates, cf. (4.4).

The next lemma states that choosing other cut-off functions in (4.7) changes $R_p(\lambda)$ only modulo "good" remainders.

LEMMA 4.7. *Let $\sigma, \sigma_0, \sigma_1 \in C^\infty_0([0,1[)$ be arbitrary cut-off functions with $\sigma\sigma_1 = \sigma_1$, $\sigma_0\sigma = \sigma$ and $\sigma \equiv \sigma_j \equiv 1$ on $[0, \varrho]$. Then*

$$R_p(\lambda) = \sigma(\lambda - \widehat{A}_p)^{-1}\sigma_0 + (1-\sigma)(\lambda - \Delta^p_{2\mathbb{B}})^{-1}(1-\sigma_1) + G(\lambda)$$

with a remainder $G(\lambda)$

$$G(\lambda) \in \mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{L}(\mathcal{H}^{0,\gamma_p}_p(\mathbb{B}), \mathcal{D}(A_p)))$$

where $\tilde{\Lambda}_\theta = \{z \in \Lambda_\theta \mid |z| \geq c\}$, $0 < \theta < \frac{\pi}{2}$, with a sufficiently large constant $c > 0$.

PROOF. Let us write $C^\infty(\mathbb{B})$ for the space of functions which are smooth up to the boundary of $\mathbb{B}$. Since the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}^{0,0}_2(\mathbb{B})}$ induces an identification of the dual space $\mathcal{H}^{0,\gamma_p}_p(\mathbb{B})'$ with $\mathcal{H}^{0,\gamma_{p'}}_{p'}(\mathbb{B})$ and since $C^\infty(\mathbb{B}) \subset \mathcal{H}^{0,\gamma_{p'}}_{p'}(\mathbb{B})$, the result follows if we can show that $G(\lambda)$ has an integral kernel $k(\lambda) \in \mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{D}(A_p) \,\widehat{\otimes}_\pi\, C^\infty(\mathbb{B}))$. A straightforward calculation shows that $G(\lambda)$ is a linear combination of operators of the form

i) $\varphi(\lambda - \widehat{A}_p)^{-1}\psi$, where $\varphi, \psi \in C^\infty([0,1[)$ have disjoint support and either φ or ψ is a cut-off function;
ii) $\varphi_0(\lambda - \Delta^p_{2\mathbb{B}})^{-1}\psi_0$, where $\varphi_0, \psi_0 \in C^\infty(]0,1[)$ have disjoint support;
iii) $\varphi_1\{(\lambda - \widehat{A}_p)^{-1} - (\lambda - \Delta^p_{2\mathbb{B}})^{-1}\}\psi_1$, where $\varphi_1, \psi_1 \in C^\infty(]\varrho, 1[)$.

Both $(\lambda - \widehat{A}_p)^{-1}$ and $(\lambda - \Delta_{2\mathbb{B}})^{-1}$ are, in particular, parameter-dependent pseudodifferential parametrices on $]\varrho, 1[\times X$ to the same operator $\lambda - \Delta$. Hence they coincide modulo smoothing operators, and the terms from iii) are integral operators with a parameter-dependent kernel belonging to $\mathcal{S}(\tilde{\Lambda}_\theta, C^{\infty,\infty}(\mathbb{B}) \,\widehat{\otimes}_\pi\, C^{\infty,\infty}(\mathbb{B}))$ where $C^{\infty,\infty}(\mathbb{B})$ denotes the space of smooth functions on $\mathbb{B}$ vanishing of infinite order at the boundary. The same is true for the terms from ii) due to the disjoint support of φ_0 and ψ_0. Clearly, such integral operators have the required property. It remains to consider terms from i). Since they are located near the boundary, we can describe their kernel in the splitting of coordinates (t, ϑ). It is given by

$$k(\lambda, t, \vartheta, t', \vartheta') = \varphi(t)\psi(t') \int_{\mathbb{R}^2} e^{i((t,\vartheta)-(t',\vartheta'))\eta} \frac{1}{\lambda - |\eta|^2}\, đ\eta,$$

where, for abbreviation, we write $(t, \vartheta) := (t\cos\vartheta, t\sin\vartheta)$. If ψ is a cut-off function (hence $\varphi \in C^\infty_{\mathrm{comp}}(]0,1[)$), this kernel belongs to $\mathcal{S}(\tilde{\Lambda}_\theta, C^{\infty,\infty}(\mathbb{B}) \,\widehat{\otimes}_\pi\, C^\infty(\mathbb{B}))$: indeed, $|(t,\vartheta) - (t',\vartheta')| \geq c > 0$, in view of the fact that the supports of φ and ψ are disjoint. Next suppose φ is a cut-off function and $\psi \in C^\infty_{\mathrm{comp}}(]0,1[)$. A Taylor expansion in

t of the integral shows that $k = k_0 + k_1 + k_2$, where

$$k_0(\lambda, t, \vartheta, t', \vartheta') = \varphi(t)\psi(t') \int e^{-i(t',\vartheta')\eta} \frac{1}{\lambda - |\eta|^2} đ\eta \in \mathcal{S}(\tilde{\Lambda}_\theta, \operatorname{span}(\varphi) \otimes \mathcal{C}^\infty(\mathbb{B})),$$

$$k_1(\lambda, t, \vartheta, t', \vartheta') = \varphi(t)\psi(t')\, t \int e^{-i(t',\vartheta')\eta} (\eta_1 \cos\vartheta + \eta_2 \sin\vartheta) \frac{1}{\lambda - |\eta|^2} đ\eta$$
$$\in \mathcal{S}(\tilde{\Lambda}_\theta, \operatorname{span}(\varphi\, te^{i\vartheta}, \varphi\, te^{-i\vartheta}) \otimes \mathcal{C}^\infty(\mathbb{B})),$$

while

$$k_2(\lambda, t, \vartheta, t', \vartheta') = \varphi(t)\psi(t')\, t^2 \int_0^1 \int e^{i((st,\vartheta)-(t',\vartheta'))\eta} (\eta_1 \cos\vartheta + \eta_2 \sin\vartheta)^2 \frac{1}{\lambda - |\eta|^2} (1-s)\, đ\eta ds.$$

The fact that the supports of φ and ψ are disjoint shows that $|(st, \vartheta) - (t', \vartheta')|$ is bounded away from 0 uniformly in s, hence $k_2 \in \mathcal{S}(\tilde{\Lambda}_\theta, \varphi t^2 \mathcal{C}^\infty(\mathbb{B}) \widehat{\otimes}_\pi \mathcal{C}^\infty(\mathbb{B}))$.

Since $\operatorname{span}(\varphi, \varphi te^{i\vartheta}, \varphi te^{-i\vartheta}) \subset \mathcal{D}(A^p_{\min})$ and $\varphi t^2 \mathcal{C}^\infty(\mathbb{B}) \subset \mathcal{H}_p^{\infty,\gamma_p+2}(\mathbb{B}) \subset \mathcal{D}(A^p_{\min})$, we conclude that $k \in \mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{D}(A_p) \widehat{\otimes}_\pi \mathcal{C}^\infty(\mathbb{B}))$ also in this case. □

PROPOSITION 4.8. *If $R_p(\lambda)$ is as in* (4.7), *then*

$$(\lambda - A_p) R_p(\lambda) - 1 \in \mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{L}(\mathcal{H}_p^{0,\gamma_p}(\mathbb{B}))), \qquad R_p(\lambda)(\lambda - A_p) - 1 \in \mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{L}(\mathcal{D}(A_p))).$$

Here, $\tilde{\Lambda}_\theta$ is the truncated sector as in Lemma 4.7. *In particular, $(\lambda - A_p)^{-1}$ exists for $\lambda \in \tilde{\Lambda}_\theta$ with $|\lambda|$ sufficiently large, and we have*

$$\|(\lambda - A_p)^{-1}\|_{\mathcal{L}(\mathcal{H}_p^{0,\gamma_p}(\mathbb{B}))} = O(|\lambda|^{-1}).$$

PROOF. Let us show the first statement. To this end, write

$$\lambda - A_p = \sigma\, (\lambda + \widehat{\Delta}^p)\, \sigma_0 + (1 - \sigma)\, (\lambda + \Delta^p_{2\mathbb{B}})\, (1 - \sigma_1)$$

with cut-off functions $\sigma, \sigma_j \in \mathcal{C}^\infty_{\text{comp}}([0, 1[)$ satisfying $\sigma\sigma_1 = \sigma_1$, $\sigma_0\sigma = \sigma$, and $\sigma \equiv \sigma_j \equiv 1$ on $[0, \varrho]$. Then

$$(\lambda - A_p) R_p(\lambda) = \sigma\, (\lambda + \widehat{\Delta}^p)\, \sigma_0\, R_p(\lambda) + (1 - \sigma)\, (\lambda + \Delta^p_{2\mathbb{B}})\, (1 - \sigma_1)\, R_p(\lambda).$$

To treat the first summand on the right-hand side, choose a representation of $R_p(\lambda)$ with cut-off function ω such that $\omega\sigma_0 = \sigma_0$. According to the previous Lemma 4.7 and the fact that the operator norm of $\sigma\, (\lambda + \widehat{\Delta}^p)\, \sigma_0 : \mathcal{D}(A_p) \to \mathcal{H}_p^{0,\gamma_p}(\mathbb{B})$ is $O(|\lambda|)$,

$$\sigma\, (\lambda + \widehat{\Delta}^p)\, \sigma_0\, R_p(\lambda) \equiv \sigma\, (\lambda + \widehat{\Delta}^p)\, \sigma_0 \omega\, (\lambda + \widehat{\Delta}^p)^{-1}\, \omega_0 = \sigma\, (\lambda + \widehat{\Delta}^p)(\lambda + \widehat{\Delta}^p)^{-1}\, \omega_0 = \sigma$$

modulo a remainder in $\mathcal{S}(\tilde{\Lambda}_\theta, \mathcal{L}(\mathcal{H}_p^{0,\gamma_p}(\mathbb{B})))$; note that the factor $\sigma_0\omega$ can be omitted due to the locality of $\widehat{\Delta}$. For the second summand choose ω_0 such that $\sigma_1\omega_0 = \omega_0$ to obtain analogously $(1 - \sigma)\, (\lambda + \Delta^p_{2\mathbb{B}})\, (1 - \sigma_1)\, R_p(\lambda) \equiv 1 - \sigma$ modulo a remainder of the same type. The proof of the second statement is analogous. Finally the norm estimate is immediate from the form of R_p in Lemma 4.7. □

COROLLARY 4.9. *Let Δ be the Laplace-Beltrami operator of Example* 3.6. *Then $-\Delta$ is the generator of a holomorphic semigroup.*

Note that this is already sufficient for the solution of certain semilinear evolution equations, cf. e.g. Pazy [**20**, Theorem 6.3.1].

5. Quasilinear parabolic equations

In the previous section we saw that the boundedness of the purely imaginary powers implies the solvability of associated parabolic initial value problems and the maximal regularity of the solution. In turn, the knowledge of maximal regularity is important for the investigation of non-linear equations, as we want to illustrate in this section. Following the concept of Clément and Li [**6**], we will consider examples of quasilinear evolution equations.

Let $E = (E_0, E_1)$ be a couple of Banach spaces such that E_1 is densely and continuously embedded into E_0. For $1 < q < \infty$ denote by

$$E_{1-\frac{1}{q},q} := (E_1, E_0)_{\frac{1}{q},q} = (E_0, E_1)_{1-\frac{1}{q},q}$$

the space given by the real interpolation method $(.,.)_{\vartheta,q}$.

Let $-P \in \mathcal{L}(E_1, E_0)$ be the infinitesimal generator of an analytic semigroup in E_0 with $\mathcal{D}(P) = E_1$. For $T > 0$ and $f \in L_q([0,T]; E_0)$, a function $u \in W_q^1([0,T]; E_0) \cap L_q([0,T]; E_1)$ is called a strict solution of the problem

$$\text{(5.1)}\qquad \begin{cases} \dot{u}(\tau) + Pu(\tau) = f(\tau) \text{ on }]0,T[\\ u(0) = u_0, \end{cases}$$

if u satisfies (5.1) in the $L_q([0,T]; E_0)$ sense. It is known that (5.1) with $f \equiv 0$ has a strict solution if and only if $u_0 \in E_{1-\frac{1}{q},q}$ (see, e.g., [**1**, Theorem 4.10.2]).

We will say that $P \in \mathcal{L}(E_1, E_0)$ belongs to the class $MR(q, (E_0, E_1))$ if for every $f \in L_q([0,T]; E_0)$ and $u_0 \in E_{1-\frac{1}{q},q}$, there exists a unique strict solution $u \in W_q^1([0,T]; E_0) \cap L_q([0,T]; E_1)$ of (5.1) and if there exists $M > 0$, independent of f and u_0, such that

$$\int_0^T \|\dot{u}(\tau)\|_{E_0}^q d\tau + \int_0^T \|Pu(\tau)\|_{E_0}^q d\tau \le M\Big(\int_0^T \|f(\tau)\|_{E_0}^q d\tau + \|u_0\|_{E_{1-\frac{1}{q},q}}^q \Big).$$

Clément and Li considered the quasilinear problem

$$\text{(5.2)}\qquad \begin{cases} \dot{u}(\tau) + A(u)u(\tau) = f(\tau, u(\tau)) + g(\tau) \text{ on }]0,T_0[\\ u(0) = u_0, \end{cases}$$

where $T_0 > 0$ and A, f, and g are supposed to satisfy the following assumptions:

(H1) $A \in \mathcal{C}^{1-}(U, \mathcal{L}(E_1, E_0))$ for some open neighborhood U of u_0 in $E_{1-\frac{1}{q},q}$, and $A(u_0) \in MR(q, (E_0, E_1))$;
(H2) $f \in \mathcal{C}^{1-,1-}([0,T_0] \times U, E_0)$;
(H3) $g \in L_q([0,T_0], E_0)$.

Their main result then is:

THEOREM 5.1. *Under hypotheses* (H1), (H2), *and* (H3) *there exists a* $T_1 \in]0, T_0]$ *and a unique function* $u \in W_q^1([0,T_1]; E_0) \cap L_q([0,T_1]; E_1) \cap \mathcal{C}([0,T_1], E_{1-\frac{1}{q},q})$ *satisfying* (5.2) *on* $]0, T_1[$.

We shall now show how this theorem can be applied to certain equations on manifolds with conical singularities. To this end we shall verify the conditions for some operators A and functions f. In the following, we will fix $E_0 = \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$ and $E_1 = \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})$ with $1 < p, q < \infty$.

5.1. A Lipschitz continuous family of Laplace type operators. By Example 4.5, the operator $-\Delta+c$ on $\mathbb{B}$ admits imaginary powers such that $\|(-\Delta+c)^{iy}\|_{\mathcal{L}(\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}))} \leq Ce^{\theta|y|}$ with $0<\theta<\frac{\pi}{2}$ if c is large enough and $2\max(p,p')-1<n$. We apply a slight extension of the Dore-Venni Theorem 4.1 valid for arbitrary initial data u_0, see [1], Theorem 4.10.7 and conclude that $-\Delta \in MR(q,(\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}),\mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})))$ for any $1<p,q<\infty$.

The situation does not change very much if we replace $-\Delta$ by $-b\Delta$, where b is a smooth positive function on $\mathbb{B}$ which is constant at the boundary $\{t=0\}$, say $b|_{\{t=0\}}=b_0$: the principal symbol and the rescaled symbol of $-b\Delta$ are $-b\sigma_\psi^2(\Delta)$ and $-b_0\tilde{\sigma}_\psi^2(\Delta)$, respectively. They are invertible in the same sector as $-\sigma_\psi^2(\Delta)$ and $-\tilde{\sigma}_\psi^2(\Delta)$, respectively. As condition (E) holds for $-\Delta$, it also holds for $-b\Delta$.

Also the two model cone operators differ only by the constant b_0:

$$-\widehat{b\Delta}=-b_0\widehat{\Delta}.$$

Hence ($\mathrm{E}_{\min}$) holds for $-\widehat{b\Delta}$ in the same sector it holds for $-\widehat{\Delta}$. We may therefore apply Theorem 4.4 and obtain:

PROPOSITION 5.2. *Given a smooth function b on $\mathbb{B}$ which is constant at $\partial\mathbb{B}$, the operator $-b\Delta$ is an element of $MR(q,(\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}),\mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})))$ for any $1<p,q<\infty$ with $2\max(p,p')-1<n$.*

We note the following simple lemma:

LEMMA 5.3. *For $s>\frac{n+1}{q}$ and $\gamma\geq\frac{n+1}{2}$ we have $\mathcal{H}_q^{s,\gamma}(\mathbb{B})\hookrightarrow C_b(\mathbb{B})$, the space of bounded continuous functions on $\mathbb{B}$.*

PROOF. Outside a neighborhood of the boundary, the space $\mathcal{H}_q^{s,\gamma}(\mathbb{B})$ coincides with the standard Sobolev space $H_q^s(\mathbb{B})$, so that our statement follows from the well-known embedding theorem. For functions in $\mathcal{H}_q^{s,\gamma}(\mathbb{B})$ supported near $\partial\mathbb{B}$, we apply the mapping S_γ defined in (2.7). □

Our next step is to study $E_{1-\frac{1}{q},q}$. A precise description of this interpolation space requires the introduction of weighted Besov spaces on $\mathbb{B}$. For our purposes, however, the following embedding statement is sufficient.

LEMMA 5.4. *Let s_0, s_1, γ_0, $\gamma_1\in\mathbb{R}$, $0<\vartheta<1$ and $1<q<\infty$. Then, for arbitrary $\delta,\varepsilon>0$,*

$$(\mathcal{H}_q^{s_0,\gamma_0}(\mathbb{B}),\mathcal{H}_q^{s_1,\gamma_1}(\mathbb{B}))_{\vartheta,q}\hookrightarrow\begin{cases}\mathcal{H}_q^{s,\gamma-\varepsilon}(\mathbb{B}) & \text{if } q\leq 2\\ \mathcal{H}_q^{s-\delta,\gamma-\varepsilon}(\mathbb{B}) & \text{if } q>2\end{cases}$$

with $s=(1-\vartheta)s_0+\vartheta s_1$, $\gamma=(1-\vartheta)\gamma_0+\vartheta\gamma_1$.

PROOF. By definition of the cone Sobolev spaces, cf. (2.8), the statement is true if we can show the following interpolation result for the local spaces (where for notational simplicity we suppress writing $\mathbb{R}^{1+n}$):

$$(H_q^{s_0,\gamma_0},H_q^{s_1,\gamma_1})_{\vartheta,q}\hookrightarrow\begin{cases}H_q^{s,\gamma-\varepsilon} & \text{if } q\leq 2\\ H_q^{s-\delta,\gamma-\varepsilon} & \text{if } q>2,\end{cases} \tag{5.3}$$

where the $H_q^{r,\sigma}$ denotes the weighted space $e^{-\sigma\langle t\rangle}H_q^r(\mathbb{R}^{1+n}_{(t,x)})$. Note that for $\gamma_0=\gamma_1=0$

$$(H_q^{s_0},H_q^{s_1})_{\vartheta,q}=B^s_{qq} \tag{5.4}$$

is a Besov space; in this case (5.3) follows from standard embedding properties (even $\varepsilon=0$ is true), cf. Triebel [**27**].

To prove the general case we need to introduce some notation. The following method appears, e.g., in [**27**], [**4**]. For a Banach space Y and real r we let $l_q^r(Y)$ denote the space of all sequences $(y_k)_{k\in\mathbb{Z}}$ in Y such that

$$\|(y_k)\|:=\Big(\sum_{k\in\mathbb{Z}}(e^{r|k|}\|y_k\|_Y)^q\Big)^{1/q}<\infty.$$

Then, if interpolation of Y_0 and Y_1 makes sense, i.e., (Y_0,Y_1) is an interpolation couple, by Theorem 5.6.2 of [**4**]

$$(l_q^{\gamma_0}(Y_0),l_q^{\gamma_1}(Y_1))_{\vartheta,q}=l_q^{\gamma}((Y_0,Y_1)_{\vartheta,q}). \tag{5.5}$$

Furthermore let us fix a function $\varphi=\varphi(t)\in C^\infty_{\mathrm{comp}}(\mathbb{R})$ supported in $[-1,1]$ and strictly positive in $]-1,1[$ such that $\sum\limits_{k\in\mathbb{Z}}\varphi(\cdot-k)\equiv 1$ on all of $\mathbb{R}$. Then define $\varphi_k\in C^\infty(\mathbb{R}^{1+n})$ by $\varphi_k(t,x)=\varphi(t-k)$. For $u\in H_q^{r,\sigma}$ we can estimate

$$\|\varphi_k u\|_{H_q^r}=\|(\varphi_k e^{-\sigma\langle t\rangle})(e^{\sigma\langle t\rangle}u)\|_{H_q^r}\le C\sup_{k-1\le t\le k+1}e^{-\sigma\langle t\rangle}\|u\|_{H_q^{r,\sigma}}\le Ce^{-\sigma|k|}\|u\|_{H_q^{r,\sigma}},$$

with a constant C independent of k. Then we use the fact that the operator norm of a map $u\mapsto au$ in H_q^r for some $a\in C_b^\infty(\mathbb{R}^{1+n})$ can be estimated by finitely many terms $\|D^\alpha a\|_\infty$. Hence, for any $\sigma'<\sigma$, the map

$$S:u\mapsto(\varphi_k u)_{k\in\mathbb{Z}},\quad H_q^{r,\sigma}\to l_q^{\sigma'}(H_q^r) \tag{5.6}$$

is well-defined and continuous. On the other hand, if $(u_k)_{k\in\mathbb{Z}}\in l_q^{\sigma'}(H_q^r)$ is given and $\psi\in C^\infty_{\mathrm{comp}}(\mathbb{R})$ is chosen in such a way that $\psi\varphi=\varphi$, and we set $\psi_k(t,x)=\psi(t-k)$ then

$$\|\psi_k u_k\|_{H_q^{r,\sigma''}}=\|(\psi_k e^{\sigma''\langle t\rangle})u_k\|_{H_q^r}\le Ce^{\sigma''|k|}\|u_k\|_{H_q^r}$$

by an argument analogous to the above one. This together with Hölder's inequality shows that for any $\sigma''<\sigma'$ the map

$$R:(u_k)_{k\in\mathbb{Z}}\mapsto\sum_{k\in\mathbb{Z}}\psi_k u_k,\quad l_q^{\sigma'}(H_q^r)\to H_q^{r,\sigma''} \tag{5.7}$$

is well-defined and continuous. Clearly $RSu=u$ for any $u\in H_q^{r,\sigma}$, by the choice of ψ. From (5.6), (5.5), and (5.4) we now obtain

$$S:(H_q^{s_0,\gamma_0},H_q^{s_1,\gamma_1})_{\vartheta,q}\to l^{\gamma-\varepsilon/2}(B^s_{qq})\hookrightarrow l^{\gamma-\varepsilon/2}(H_q^{s-\delta}),$$

where $\delta=0$ if $q\le 2$, $\delta>0$ if $q>2$, and $\varepsilon>0$. Applying (5.7) we get that

$$\iota=RS:(H_q^{s_0,\gamma_0},H_q^{s_1,\gamma_1})_{\vartheta,q}\to H_q^{s-\delta,\gamma-\varepsilon}$$

with $\delta,\varepsilon>0$ arbitrarily small is a continuous embedding. □

COROLLARY 5.5. *$(\mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B}),\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}))_{\frac{1}{q},q}\hookrightarrow\mathcal{H}_q^{s,\delta}(\mathbb{B})$ for any $s<\frac{2}{q'}$ and any $\delta<\gamma_p+\frac{2}{q'}$.*

In the sequel, we shall denote by t a smooth, strictly positive function on $\mathbb{B}$ which coincides with the distance to the boundary (i.e. the coordinate t employed above) in a collar neighborhood.

LEMMA 5.6. *Let $c > 0$ and $1 < p, q < \infty$ with $p \geq \frac{n+1}{2+c}$, $q > \max\left(\frac{n+3}{2}, \frac{2p}{(2+c)p-(n+1)}\right)$. Then*

$$E_{1-\frac{1}{q},q} \hookrightarrow t^{-c}\mathcal{C}_b(\mathbb{B}).$$

PROOF. The conditions on p and q allow us to find s, δ with $\frac{(n+1)}{q} < s < \frac{2}{q'}$ and $\frac{n+1}{2} - c \leq \delta < \gamma_p + \frac{2}{q'}$. According to Corollary 5.5 and Lemma 5.3 we have $E_{1-\frac{1}{q},q} \hookrightarrow \mathcal{H}_q^{s,\delta}(\mathbb{B}) \hookrightarrow t^{-c}\mathcal{H}_q^{s,\frac{n+1}{2}}(\mathbb{B}) \hookrightarrow t^{-c}\mathcal{C}_b(\mathbb{B})$. Note that the second inclusion is immediate from Definition 2.2 with interpolation. $\square$

Not for every n it is possible to find p and q satisfying the hypotheses of Lemma 5.6 and the inequality $2\max(p,p') - 1 < n$. However, all these requirements can be fulfilled at the same time when $n \geq 4$, i.e., $\dim \mathbb{B} \geq 5$.

THEOREM 5.7. *Let $n \geq 4$ and $c > 0$. Choose $1 < p, q < \infty$ with $2\max(p,p') < n+1$, $p \geq \frac{n+1}{2+c}$, and $q > \max\left(\frac{n+3}{2}, \frac{2p}{(2+c)p-(n+1)}\right)$. Fix a smooth initial value u_0 which vanishes to infinite order at $\partial\mathbb{B}$ and a bounded neighborhood U of u_0 in $E_{1-\frac{1}{q},q}$. Let a be a smooth, strictly positive function on $\mathbb{C} \cong \mathbb{R}^2$. Then the operator function $A(u) = -a(t^c u)\Delta$, $u \in U$, satisfies* (H1).

PROOF. The function $t^c u_0$ is smooth on $\mathbb{B}$, hence so is $b = a(t^c u_0)$. In addition, b is positive and constant at the boundary. By Proposition 5.2, $A(u_0) = -b\Delta$ belongs to $MR(q, (\mathcal{H}_q^{0,\gamma_p}(\mathbb{B}), \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})))$.

As u varies over a bounded neighborhood U of u_0 in $E_{1-\frac{1}{q},q}$, the functions $t^c u$ vary over a bounded set in $\mathcal{C}_b(\mathbb{B})$. In particular, $a(t^c u)$ is a continuous, bounded, and strictly positive function on $\mathbb{B}$. Hence $A(u)$ is an element of $\mathcal{L}(E_1, E_0)$ for each u. Moreover,

$$\begin{aligned}\|A(u_1) - A(u_2)\|_{\mathcal{L}(E_1,E_0)} &\leq \|a(t^c u_1) - a(t^c u_2)\|_{L_\infty(\mathbb{B})}\|\Delta\|_{\mathcal{L}(E_1,E_0)} \\ &\leq C\|t^c u_1 - t^c u_2\|_{\mathcal{C}_b(\mathbb{B})}\|\Delta\|_{\mathcal{L}(E_1,E_0)} \\ &\leq C\|u_1 - u_2\|_{E_{1-\frac{1}{q},q}}\|\Delta\|_{\mathcal{L}(E_1,E_0)},\end{aligned}$$

where C is the maximum of $|a'(s)|$ as s varies over the bounded set of all values of $t^c u$, $u \in U$. $\square$

REMARK 5.8. *In case $c \in \mathbb{N}$, the initial value u_0 can be chosen in $C^\infty(\mathbb{B})$. Theorem 5.7 extends to the case where c is a smooth real-valued function on $\mathbb{B}$ which is positive and constant at the boundary.*

5.2. Lipschitz continuity of the functions $|u|^\alpha$. Let us now try to find functions f satisfying hypothesis (H2). Here is a first simple example.

EXAMPLE 5.9. (H2) *holds for $f(\tau, u) = h(u)$, with $h \in C(\mathbb{R})$ such that $h(0) = 0$ and $|h(s) - h(s')| \leq M|s - s'|$ for some $M \geq 0$, uniformly in $s, s' \in \mathbb{R}$.*

This follows from the observation that the mapping

$$u \mapsto h(u(.)) \,:\, L_q(\Omega) \to L_q(\Omega)$$

is Lipschitz continuous for any measure space Ω, in particular for $\mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$, which is a weighted L_q-space on $\mathbb{B}$.

As mentioned in the introduction, nonlinearities of the type $|u|^\alpha$ or u^α are relevant for applications. It is then interesting to find out whether a term of this kind fulfills (H2). We shall show the following:

THEOREM 5.10. *The function $f(\tau, u) = |u|^\alpha$ satisfies* (H2) *for all $1 \le \alpha < \alpha^*$, where α^* is determined as follows:*

a) *If $\frac{2p}{q'} < n+1$ then* $\alpha^* = \begin{cases} \frac{n+1}{n+1-2p/q'} & q \ge \frac{n+3}{2} \\ \min\left(\frac{n+1}{n+1-2p/q'}, \frac{n+1}{n+1-2q/q'}\right) & q < \frac{n+3}{2} \end{cases},$

b) *If $\frac{2p}{q'} \ge n+1$ then* $\alpha^* = \begin{cases} \infty & q \ge \frac{n+3}{2} \\ \frac{n+1}{n+1-2q/q'} & q < \frac{n+3}{2} \end{cases}$

COROLLARY 5.11. *Let $n \ge 4$ and $f(t, u) = |u|^\alpha$.*

a) *Hypothesis* (H2) *is satisfied for arbitrary $\alpha \ge 1$, provided we choose $p < \frac{n+1}{2}$ sufficiently close to $\frac{n+1}{2}$ and $q \in]1, \infty[$ sufficiently large. In this case, $2\max(p, p') - 1 < n$, and p and q satisfy also the conditions of Theorem* 5.7.

b) *Given p with $2\max(p, p') - 1 < n$,* (H2) *holds for $1 \le \alpha < \frac{n+1}{n+3-2p}$ with $q = p$.*

c) *Hypothesis* (H2) *is satisfied for $1 \le \alpha < \frac{n+1}{2}$ if $q = p < \frac{n+1}{2}$ is sufficiently close to $\frac{n+1}{2}$.*

PROOF. For $p < \frac{n+1}{2}$ sufficiently close to $\frac{n+1}{2}$, we have $2\max(p, p') - 1 < n$. Conversely, the condition $2\max(p, p') - 1 < n$ implies $2p < n+1$. So the assertions follow from Theorem 5.10a). $\square$

REMARK 5.12. *The statements of Theorem* 5.10 *and Corollary* 5.11 *remain true in case $f(t, u) = u^\alpha$ with a natural number α satisfying the corresponding conditions.*

We shall prove Theorem 5.10 after Theorem 5.15. We first note that, as a consequence of Corollary 5.5, the Lipschitz continuity of a map on bounded subsets of $E_{1-\frac{1}{q},q}$ follows from its Lipschitz continuity on bounded subsets of $\mathcal{H}_q^{s,\delta}(\mathbb{B})$. Let us now introduce $\mathcal{H}_q^{s,\gamma}(X^\wedge)$ as the space of all distributions u on $X^\wedge = \mathbb{R}_+ \times X$ such that $S_\gamma u \in H_q^s(\mathbb{R} \times X)$, where S_γ is the map introduced in (2.7) (with X instead of $\mathbb{R}^n$); the norm is given by $\|u\|_{\mathcal{H}_q^{s,\gamma}(X^\wedge)} = \|S_\gamma u\|_{H_q^s(\mathbb{R}\times X)}$.

Moreover, we denote by $\mathcal{H}_q^{s,\gamma}(X^\wedge)_0$ the subspace of all $u \in \mathcal{H}_q^{s,\gamma}(X^\wedge)$ supported in $[0,1[\times X$.

LEMMA 5.13. *Let $1 < q \le \tilde{q} < \infty$ and $\varepsilon > 0$. Then $\mathcal{H}_{\tilde{q}}^{0,\gamma+\varepsilon}(X^\wedge)_0 \hookrightarrow \mathcal{H}_q^{0,\gamma}(X^\wedge)_0$.*

PROOF. By the definition of $\mathcal{H}_{\tilde{q}}^{0,\gamma+\varepsilon}(X^\wedge)$ we have

$$\|u\|^q_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} = \int_{\mathbb{R}_+\times X} r^{\left(\frac{n+1}{2}-\gamma\right)q} |u(r,x)|^q \frac{dr}{r} dx$$
$$= \int_{]0,1]\times X} r^{\frac{q}{q}-1+\varepsilon q} \left(r^{\left(\frac{n+1}{2}-\gamma-\frac{1}{q}-\varepsilon\right)q} |u(r,x)|^q\right) dr dx.$$

The second factor of the integrand belongs to $L_{\frac{\tilde{q}}{q}}(X^\wedge, dr dx)$. Using Hölder's inequality we can estimate the integral from above by

$$\left(\int_{]0,1]\times X} r^{-1+\varepsilon\frac{q\tilde{q}}{\tilde{q}-q}} dr dx\right)^{\frac{\tilde{q}-q}{\tilde{q}}} \left(\int_{]0,1]\times X} r^{\left(\frac{n+1}{2}-\gamma-\varepsilon\right)\tilde{q}} |u(r,x)|^{\tilde{q}} \frac{dr}{r} dx\right)^{\frac{q}{\tilde{q}}} \le C\, \|u\|^q_{\mathcal{H}_{\tilde{q}}^{0,\gamma+\varepsilon}(X^\wedge)}.$$

This shows the continuity of the stated embedding. □

LEMMA 5.14. *Let $\alpha \ge 1$, $s \ge 0$, and $1 < q < \infty$.*

For $sq < n+1$, $u \mapsto |u|^\alpha$ maps bounded subsets of $\mathcal{H}_q^{s,\varrho}(X^\wedge)$ to bounded subsets of $\mathcal{H}_{\tilde{q}}^{0,\tilde{\varrho}}(X^\wedge)$ for $\tilde{\varrho} = \alpha\varrho - (\alpha-1)\frac{n+1}{2}$ and any $\tilde{q}$ in $[\frac{q}{\alpha}, \frac{q}{\alpha}\frac{n+1}{n+1-sq}] \cap]1,\infty[$.

For $sq \ge n+1$, the same is true for $\tilde{\varrho}$ as before and $\tilde{q} > \max(1, q/\alpha)$.

In both cases there is a positive constant C such that

$$\|\,|u|^\alpha\,\|_{\mathcal{H}_{\tilde{q}}^{0,\tilde{\varrho}}(X^\wedge)} \le C\|u\|^\alpha_{\mathcal{H}_q^{0,\varrho}(X^\wedge)} \qquad \forall u \in \mathcal{H}_q^{0,\varrho}(X^\wedge).$$

PROOF. It is well-known that $H^s_p(\mathbb{R}^n) \hookrightarrow H^t_q(\mathbb{R}^n)$ provided $1 < p \le q < \infty$ and $s - n/p \ge t - n/q$, cf. [**27**, 2.8.1 Remark 2]. Hence we have $H^s_q(\mathbb{R}^{1+n}) \hookrightarrow L_{\alpha\tilde{q}}(\mathbb{R}^{1+n})$ and

$$\|\,|u|^\alpha\,\|_{\mathcal{H}_{\tilde{q}}^{0,\tilde{\varrho}}(X^\wedge)} = \|\,S_{\tilde{\varrho}}|u|^\alpha\,\|_{L_{\tilde{q}}(\mathbb{R}\times X)} = \|\,|S_\varrho u|^\alpha\,\|_{L_{\tilde{q}}(\mathbb{R}\times X)}$$
$$= \|S_\varrho u\|^\alpha_{L_{\alpha\tilde{q}}(\mathbb{R}\times X)} \le C\|S_\varrho u\|^\alpha_{H^s_q(\mathbb{R}\times X)} = C\|u\|^\alpha_{\mathcal{H}_q^{s,\varrho}(X^\wedge)}.$$

□

THEOREM 5.15. *Let $1 < p, q < \infty$ and $\gamma < \delta$. If $0 < s < \frac{n+1}{q}$, then the map*

$$u \mapsto |u|^\alpha : \mathcal{H}_q^{s,\delta}(\mathbb{B}) \longrightarrow \mathcal{H}_q^{0,\gamma}(\mathbb{B})$$

is Lipschitz continuous on bounded sets, whenever

$$(5.8)\qquad 1 \le \alpha < \min\left(\frac{n+1-2\gamma}{n+1-2\delta}, \frac{n+1}{n+1-sq}\right) \qquad \text{if } \delta < \frac{n+1}{2},$$

$$(5.9)\qquad 1 \le \alpha < \frac{n+1}{n+1-sq} \qquad \text{if } \delta \ge \frac{n+1}{2}.$$

For $sq \ge n+1$, the same result is true; the upper bound for α then is $\frac{n+1-2\gamma}{n+1-2\delta}$ in case $\delta < \frac{n+1}{2}$ and ∞ for $\delta \ge \frac{n+1}{2}$.

PROOF. We start with the simple observation that, for every measure space Ω and for every choice of r such that $1 < r \le \alpha r < \infty$, the map

$$v \mapsto |v|^\alpha \;:\; L_{\alpha r}(\Omega) \to L_r(\Omega)$$

is Lipschitz continuous on bounded sets. Indeed, this is a straightforward consequence of Hölder's inequality and the fact that $|x^\alpha - y^\alpha| \le \alpha \max\{x^{\alpha-1}, y^{\alpha-1}\}|x-y|$ for any $x, y \ge 0$ and $\alpha \ge 1$.

The crucial part of the proof concerns the analysis near the boundary, i.e., the Lipschitz continuity of the map

$$u \mapsto |u|^\alpha : \mathcal{H}_q^{s,\delta}(X^\wedge)_0 \longrightarrow \mathcal{H}_q^{0,\gamma}(X^\wedge)_0. \tag{5.10}$$

Assume we have proved this. Combining the above observation with the embedding $H_q^s(2\mathbb{B}) \hookrightarrow L_{\alpha q}(2\mathbb{B})$ which is valid for all $\alpha \geq 1$ in case $sq \geq n+1$ and for $1 \leq \alpha < \frac{n+1}{n+1-sq}$ in case $sq < n+1$, we immediately see that the map $u \mapsto |u|^\alpha$ is Lipschitz continuous from bounded subsets of $H_q^s(2\mathbb{B})$ to $L_q(2\mathbb{B})$. Now, for an arbitrary cut-off function ω, choose cut-off functions σ_1 and σ_2 such that $\sigma_1 \equiv 1$ on the support of ω and $\omega \equiv 1$ on the support of σ_2. Then, by the definition of $\mathcal{H}_q^{0,\gamma}(\mathbb{B})$, we have, for u and v running through bounded subsets,

$$\begin{aligned}
\| \, |u|^\alpha - |v|^\alpha \, \|_{\mathcal{H}_q^{0,\gamma}(\mathbb{B})} &= \|\omega(|u|^\alpha - |v|^\alpha)\|_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} + \|(1-\omega)(|u|^\alpha - |v|^\alpha)\|_{L_q(2\mathbb{B})} \\
&\leq \|\sigma_1^\alpha(|u|^\alpha - |v|^\alpha)\|_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} + \|(1-\sigma_2)^\alpha(|v|^\alpha - |v|^\alpha)\|_{L_q(2\mathbb{B})} \\
&\leq C\big(\|\sigma_1(u-v)\|_{\mathcal{H}_q^{s,\delta}(X^\wedge)} + \|(1-\sigma_2)(u-v)\|_{H_q^s(2\mathbb{B})}\big) \\
&\leq C\|u-v\|_{\mathcal{H}_q^{s,\delta}(\mathbb{B})}.
\end{aligned}$$

We next verify (5.10). In case $sq < n+1$, we set $\tilde{q} = \frac{q}{\alpha}\frac{n+1}{n+1-sq}$; for $sq \geq n+1$ we choose $\tilde{q} > q$ arbitrary. Note that also in the first case, our assumption on α implies $\tilde{q} > q$. For arbitrary $\beta > 0$, we obtain

$$\begin{aligned}
\| \, |u|^\alpha - |v|^\alpha \, \|_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} &\leq C\| \, |u|^\alpha - |v|^\alpha \, \|_{\mathcal{H}_{\tilde{q}}^{0,\gamma+\beta}(X^\wedge)} = \\
&= C\|S_{\gamma+\beta}(|u|^\alpha) - S_{\gamma+\beta}(|v|^\alpha)\|_{L_{\tilde{q}}(\mathbb{R}\times X)} = C\| \, |S_{\tilde{\gamma}}u|^\alpha - |S_{\tilde{\gamma}}v|^\alpha \, \|_{L_{\tilde{q}}(\mathbb{R}\times X)},
\end{aligned} \tag{5.11}$$

where $\tilde{\gamma}$ is defined by $\alpha\tilde{\gamma} - (\alpha-1)\frac{n+1}{2} = \gamma + \beta$, and the first inequality holds in view of Lemma 5.13.

In case $\delta < \frac{n+1}{2}$, we can decrease β and assume $\alpha \leq \frac{n+1-2\gamma-2\beta}{n+1-2\delta}$. This implies $\tilde{\gamma} \leq \delta$. For $\delta \geq \frac{n+1}{2}$, note that $\tilde{\gamma} = \frac{n+1}{2}(1-\frac{1}{\alpha}) + \frac{\gamma+\beta}{\alpha}$. Possibly decreasing β , we obtain $\tilde{\gamma} < \frac{n+1}{2}(1-\frac{1}{\alpha}) + \frac{\delta}{\alpha}$, hence again $\tilde{\gamma} - \delta \leq 0$. But then

$$\|S_{\tilde{\gamma}}u\|_{L_{\alpha\tilde{q}}(\mathbb{R}\times X)} = \| \, |u|^\alpha \, \|^{\frac{1}{\alpha}}_{\mathcal{H}_{\tilde{q}}^{0,\gamma+\beta}(X^\wedge)} \leq C\|u\|_{\mathcal{H}_q^{0,\tilde{\gamma}}(X^\wedge)} \leq C\|u\|_{\mathcal{H}_q^{s,\delta}(X^\wedge)},$$

where the first inequality holds by Lemma 5.14 (with $\tilde{\varrho} = \gamma+\beta$ and $\varrho = \tilde{\gamma}$) and the second one is true due to the boundedness of supp u. Hence, $S_{\tilde{\gamma}}u$ runs through a bounded set of $L_{\alpha\tilde{q}}(\mathbb{R}\times X)$ if u runs through a bounded set of $\mathcal{H}_q^{s,\delta}(X^\wedge)_0$. We employ once more the observation that the map

$$v \mapsto |v|^\alpha \; : \; L_{\alpha\tilde{q}}(\mathbb{R}\times X) \to L_{\tilde{q}}(\mathbb{R}\times X)$$

is Lipschitz continuous on bounded sets and obtain from (5.11)

$$\| \, |u|^\alpha - |v|^\alpha \, \|_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} \leq C\|S_{\tilde{\gamma}}u - S_{\tilde{\gamma}}v\|_{L_{\alpha\tilde{q}}(\mathbb{R}\times X)}$$

for u and v in a bounded set of $\mathcal{H}_q^{s,\delta}(X^\wedge)_0$. Using the embeddings $H_q^s(\mathbb{R}\times X) \hookrightarrow L_{\alpha\tilde{q}}(\mathbb{R}\times X)$ and $\mathcal{H}_q^{s,\delta}(X^\wedge)_0 \hookrightarrow \mathcal{H}_q^{s,\tilde{\gamma}}(X^\wedge)_0$, we arrive at

$$\| \, |u|^\alpha - |v|^\alpha \, \|_{\mathcal{H}_q^{0,\gamma}(X^\wedge)} \leq C\|S_{\tilde{\gamma}}u - S_{\tilde{\gamma}}v\|_{H_q^s(\mathbb{R}\times X)} = C\|u-v\|_{\mathcal{H}_q^{s,\tilde{\gamma}}(X^\wedge)} \leq C\|u-v\|_{\mathcal{H}_q^{s,\delta}(X^\wedge)},$$

showing the desired Lipschitz continuity of (5.10). ∎

We are now ready to prove Theorem 5.10. We know from Corollary 5.5 that $E_{1-\frac{1}{q},q} \hookrightarrow \mathcal{H}_q^{s,\delta}(\mathbb{B})$ for all $0 < s < 2/q'$ and $\delta = \gamma_p + 2/q' - \varepsilon$ with arbitrarily small $\varepsilon > 0$. Theorem 5.15 tells us when $f : \mathcal{H}_q^{s,\delta}(\mathbb{B}) \to \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$ is Lipschitz continuous. In case $2p/q' \leq n+1$, we have $\delta < (n+1)/2$; hence (5.8) – or its simplified version, in case $sq \geq n+1$ – gives the admitted range of α. Similarly, $2p/q' > n+1$ allows us to choose $\delta > (n+1)/2$; this leads to (5.9) – with the corresponding simplification for $sq \geq n+1$. Inserting γ_p and δ in the expressions, letting $\varepsilon \to 0$, and optimizing $0 < s < 2/q'$, we obtain the formula for α^* in 5.10a) for $2p/q' < n+1$ and that in 5.10b) for $2p/q' \geq n+1$. Note that $\frac{2}{q'} < \frac{n+1}{q}$ is equivalent to $q < \frac{n+3}{2}$.

5.3. Conclusion. In order to illustrate the results, let us state one of the possible applications of Proposition 5.2, Theorem 5.7, and Theorem 5.10. Others can be made up easily using Remark 5.8, Example 5.9, and Remark 5.12. As before, $E_0 = \mathcal{H}_q^{0,\gamma_p}(\mathbb{B})$ and $E_1 = \mathcal{H}_q^{2,\gamma_p+2}(\mathbb{B})$ with $1 < p, q < \infty$.

THEOREM 5.16. *Let $n \geq 4$. Given $c > 0$, $\alpha \geq 1$, and $T_0 > 0$, there is a suitable choice of p and q in $]1,\infty[$ and $T_1 > 0$ such that the equation*

$$\dot{u} - a(t^c u)\Delta u = |u|^\alpha + g \quad on\ ]0,T_0[, \qquad u(0) = u_0,$$

has a unique solution $u \in W_q^1([0,T_1];E_0) \cap L_q([0,T_1];E_1) \cap C([0,T_1],E_{1-\frac{1}{q},q})$ on $]0,T_1[$ for every $g \in L_q(]0,T_0[,E_0)$, every smooth, strictly positive function a, and every $u_0 \in C^\infty_{\mathrm{comp}}(\mathrm{int}\,\mathbb{B})$.

Applying Remark 5.8 we see that, for suitable $p,q \in\ ]1,\infty[$, also the Ginzburg-Landau type equation

$$\dot{u} - \Delta u = u - u^3 \quad on\ ]0,T_0[, \qquad u(0) = u_0,$$

has a unique solution u on $]0,T_1[$ in the same space as above for an arbitrary initial value u_0 in $E_{1-\frac{1}{q},q}$.

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