Optimization by \( n \)-homogeneous polynomial perturbations *

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We deal with infinite dimensional optimization in Banach spaces, finding an existence result for maximum (or minimum) points for a certain type of functions.

A remarkable result in this direction is the Stegall variational principle [10]: if \( C \) is a nonempty, closed, bounded and convex subset of a Banach space, \( C \) has the Radon-Nikodym property and \( f \) is an upper bounded upper semicontinuous real-valued function on \( C \), then there exists an arbitrarily small linear continuous perturbation \( \varphi \) such that \( f + \varphi \) attains its strong maximum on \( C \). Our aim in this note is to obtain a Stegall’s type result showing that we have an arbitrarily small continuous \( n \)-homogeneous polynomial perturbation (\( n \)-odd natural number) with the same property.

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There are many other variational principles, for example, Ekeland's variational principle [6], which has the same hypotheses on the function to be maximized but without assuming the Radon-Nikodym property, and gives a small perturbation which is only Lipschitz. A general question in the optimization theory is to find good perturbations (assuring existence of a point of minimum of the perturbed function) in some class of perturbations. In many cases the set of good perturbations is dense (even $G_δ$) in the set of all perturbations. For further information about variational principles and perturbed optimization see for example : [3-10].

Our result in this note is related, as well, to the paper of R.Aron, C.Finet and E.Werner [2], where an extension of the Bishop-Phelps theorem is proved for $n$-linear continuous forms in some spaces. Their arguments show also the denseness of norm-attaining $n$-homogeneous polynomials in a Banach space with the Radon-Nikodym property. M.D.Acosta, F.J.Aguirre and R.Paya [1] constructed examples, showing that the results in [2] are not valid in arbitrary Banach spaces.

Let $(X, \|\|)$ be a Banach space. We shall recall the following definitions.

**Definition 1** (a) Suppose that $C \subset X$ is a nonempty set and that $f$ is an upper bounded real-valued function on $C$. For each $\alpha \geq 0$ define the slice of $C$ by

$$S(f, \alpha) = \{x \in C : f(x) \geq \sup C - \alpha\}$$

(b) A nonempty subset $A$ of $X$ is said to be dentable provided it admits slices of arbitrarily small diameter, that is, for every $\varepsilon > 0$ there exists $x^* \in E^*$ and $\alpha > 0$ such that $\text{diam} S(x^*, A, \alpha) < \varepsilon$.

**Definition 2** A subset $A$ of $X$ is said to have the Radon-Nikodym property if every nonempty bounded subset of $A$ is dentable.

We say that $f$ attains its strong maximum at $x$ over $C$, if $f(x) = \sup f(C)$ and $\|x - x_n\| \to 0$ whenever $f(x_n) \to f(x)$.
Let us denote by $F_n(X)$ the space of all continuous and symmetric $n$-linear forms on $X \times \ldots \times X$ into $\mathbb{R}$ endowed with the norm

$$
\|A\| = \sup\{|A(x_1, \ldots, x_n)| : \|x_i\| \leq 1, \quad i = 1, \ldots, n\}.
$$

If $A \in F_n(x)$, then consider the function $\varphi_A : X \to \mathbb{R}$ defined by $\varphi_A(x) = A(x, \ldots, x); \varphi_A$ is called $n$-homogeneous polynomial on $X$. We denote by $P_n(X)$ the Banach space of all continuous $n$-homogeneous polynomials on $X$, endowed with the norm:

$$
\|\varphi_A\| = \sup\{|\varphi_A(x)|, \|x\| \leq 1\}
$$

**Theorem 3** Suppose that $C \subset X$ is a nonempty, bounded, closed and convex set with the Radon-Nikodym property. Let $f$ be an upper bounded upper semicontinuous real-valued function on $C$. Then for every odd integer $n$ there exists a dense $G_\delta$ subset $Q_n$ of $P_n(X)$ such that for every $\varphi \in Q_n, f + \varphi$ attains its strong maximum on $C$.

Our proof follows the ideas of the proof of Stegall's variational principle [10], given in the book of Phelps [8]. We need the following lemmas.

**Lemma 4** ([8]). Suppose that $\{A_n\}_{n=1}^\infty$ is a sequence of nonempty subsets of $X$ with the following property: there exist constants $\epsilon > 0$ and $\lambda > 0$ such that for all $x \in \text{co}A_n$ and $y \in X$

$$
\text{dist}[x, \text{co}(A_{n+1} \setminus B(y; \epsilon))] \leq \lambda/2^n.
$$

Then the set

$$
A := \cap_{n=1}^\infty \bigcup_{j \geq n} \text{co}A_j
$$

is nonempty and not dentable.

The proof of the following lemma is straightforward and is omitted.

**Lemma 5** ([8]). Suppose that the real-valued function $f$ is bounded above on the nonempty subset $C$ of $X$. Then for every $\alpha > 0$, there exists $\beta > 0$ such that $S(f + \varphi, \beta) \subset S(f, \alpha)$, whenever $\varphi \in P_n(X)$ and $\|\varphi\| < \beta$. 

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Lemma 6. Let $C$ be a nonempty, bounded, closed and convex subset of $X$ with the Radon-Nikodym property, $f$ be an upper semicontinuous, real-valued, bounded above function on $C$ and $n$ be an odd integer number. Then for any $\varepsilon > 0$ there exist $\varphi \in P_n(X)$, $\|\varphi\| < \varepsilon$, and $\alpha > 0$ such that $\text{diam } S(f + \varphi, \alpha) \leq 2\varepsilon$.

**Proof.** Proceeding by contradiction, suppose that there exists $\varepsilon > 0$ such that for every $\varphi \in P_n(X)$, $\|\varphi\| < \varepsilon$ and each $\alpha > 0$, we have
\[
\text{diam } S(f + \varphi, \alpha) > 2\varepsilon
\]
For each $m$ let
\[
A_m = \bigcup \{S(f + \varphi, \frac{1}{4mn}) : \varphi \in P_n(X), \|\varphi\| \leq \varepsilon - \frac{1}{2m}\}.
\]
The set $A_m$ is nonempty. Take $\lambda = 5/2$. We will show that the sequence $\{A_m\}_{m=1}^{\infty}$ satisfies the hypothesis of Lemma 4, which will give us a contradiction, since $C$ has the Radon-Nikodym property. We want to show that, for any natural $m$ and $y \in X$,
\[
\text{co}A_m \subset \text{co}(A_{m+1} \setminus B(y; \varepsilon)) + \frac{\lambda}{2m}B(0; 1).
\]
Let $m$ be fixed natural number. Since the set of the right side is convex, it suffices to prove that it contains $A_m$. Suppose that $x \in A_m$ but for some $y \in X$, it is not in the right hand side of (1).

By the separation theorem, there exists $y^* \in X^*$, $\|y^*\| = 1$ such that
\[
\langle y^*, x \rangle > \sup \{\langle y^*, a \rangle : a \in A_{m+1} \setminus B(y; \varepsilon)\} + \frac{\lambda}{2m}
\]
Then
\[
\langle y^*, x \rangle^n > \left(\langle y^*, a \rangle + \frac{\lambda}{2m}\right)^n \forall a \in A_{m+1} \setminus B(y; \varepsilon) \quad (2)
\]
Now suppose that $x \in S(f + \varphi, \frac{1}{4mn})$ with $\varphi \in P_n(X)$ and $\|\varphi\| \leq \varepsilon - \frac{1}{2m}$. Consider
\[
P(x) = \varphi(x) + r(y^*, x)^n
\]
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with \( r = \frac{1}{2^{m+1}} \). As \( \varphi \in P_n(X) \), let \( H \) in \( F_n(X) \) be the corresponding symmetric form. Then 
\[
P(z) = L(z, \ldots, z),
\]
where 
\[
L(x_1, \ldots, x_n) = H(x_1, \ldots, x_n) + r\langle y^*, x \rangle \langle y^*, x \rangle.
\]
Therefore \( L \in F_n(X) \), \( P \in P_n(X) \) and 
\[
\|P\| \leq \|\varphi\| + r \leq \varepsilon - \frac{1}{2^{m+1}}.
\]
Then \( S(f + P, \frac{1}{2^{m+1}}) \) is contained in \( A_{m+1} \). But as \( P \in P_n(X) \) and \( \|P\| < \varepsilon \), we have 
\[
\text{diam } S(f + P, \frac{1}{2^{m+1}}) > 2\varepsilon.
\]
Then there exists \( z \) in \( S(f + P, \frac{1}{2^{m+1}}) \setminus B(y, \varepsilon) \) and
\[
(f + P)(z) \geq \sup(f + P)(C) - \frac{1}{2^{m+1}}
\]
\[
\geq (f + \varphi)(x) - \frac{1}{2^{m+1}}
\]
\[
= (f + \varphi)(x) + r\langle y^*, z \rangle - \frac{1}{2^{m+1}} \quad \text{(by definition of } P \text{)}
\]
\[
\geq \sup(f + \varphi)(C) - \frac{1}{2^{m+1}} + r\langle y^*, x \rangle - \frac{1}{2^{m+1}}
\]
\[
\text{(as } x \in S(f + \varphi, \frac{1}{2^{m+1}}))
\]
\[
> (f + \varphi)(z) - \frac{1}{2^{m+1}}(1 + \frac{1}{4}) + r\langle (y^*, z) + \frac{1}{2^{m+1}} \rangle
\]
\[
= (f + P)(z) - r\langle y^*, z \rangle - \frac{1}{2^{m+1}}(1 + \frac{1}{4}) + r\langle (y^*, z) + \frac{1}{2^{m+1}} \rangle.
\]

It follows that
\[
2^{m(1-n)+1}(1 + \frac{1}{4^n}) > 2^{mn}((\langle y^*, z \rangle + \frac{1}{2^m})^n - \langle y^*, z \rangle^n).
\]

Case 1. \( n = 1 \). 
Then by (3) we obtain \( \frac{1}{2} > \lambda \), a contradiction.
Case 2. \( n \geq 3 \) and \( \langle y^*, z \rangle \neq 0 \).
Consider the real function 
\[
\xi(t) = (at + \lambda)^n - (at)^n,
\]
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where $a = \langle y^*, z \rangle$ (recall that $n$ is an odd natural number). It is easy to see that $\lim_{t \to \pm \infty} = +\infty$, $\xi$ attains its global minimum over $R$ at the point $t_0 = -\frac{\alpha}{2\bar{a}}$ and $\xi(t_0) = 2(\frac{1}{2})^n$. Then the left side of (3) is less than $2(\frac{1}{2})^n$, a contradiction with (3).

Case 3. $n \geq 3$ and $\langle y^*, z \rangle = 0.$

By (3) we obtain a contradiction. The lemma is proved.

**Proof of Theorem 3.** Fix $n$ and for every natural $m$ define

$$Q^m_n = \{\varphi \in P_n(X) : \text{diam } S(f + \varphi, \alpha_m) < \frac{1}{m} \text{ for some } \alpha_m > 0\}.$$  

By Lemma 6, for each $m, Q^m_n$ is dense in $P_n(X)$ and it is open by Lemma 5. By the Baire category theorem, the set

$$Q_n = \cap_m Q^m_n$$

is a dense $G_\delta$ subset of $P_n(X)$.

Since $f$ is upper semicontinuous, the set $S(f + \varphi, \alpha_m)$ is a closed set. Let $x_0$ be the unique common point of the sets $S(f + \varphi, \alpha_m)$, $m \geq 1$ and let $\varphi \in Q_n, \{x_n\} \subset C$, $(f + \varphi)(x_n) \to \sup_{x \in C}(f + \varphi)(x)$. Then for every $m \geq 1$ there exists $\nu$ such that $x_n \in S(f + \varphi, \alpha_m)$ for every $n > \nu$. This means $\|x_n - x_0\| < \frac{1}{m}$, i.e., $x_n \to x_0$ and the theorem is proved.

It is easy to see that Theorem 3 is not valid for even $n$: take, for example, $X = R, C = [0, 1], n = 2$ and $f(x) = x^2$.

**REFERENCES**


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