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NON LINEAR OPTIC AND SUPERCRITICAL WAVE EQUATION

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Dedicated to the memory of Pascal Laubin

1. Introduction and results

In this paper, we are interested in the study of the Cauchy problem for the non-linear wave equation in $\mathbb{R}^3$

$$(1.1) \qquad \begin{cases} (\partial_t^2 - \Delta_x)u + u^p = 0 \quad u = u(t,x) \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \\ u_{|t=0} = u_0(x) \in H^1 \cap L^{p+1} \ ; \ \partial_t u_{|t=0} = u_1(x) \in L^2 \end{cases}$$

Here, p is an odd integer, and the function u is assumed to take real values.

The formally conserved energy for (1.1) is

$$(1.2) \qquad E(u) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|\partial_t u|^2 + \frac{1}{2}|\nabla_x u|^2 + \frac{u^{p+1}}{p+1} \right) dx$$

The Sobolev imbedding in $\mathbb{R}^3$, $H^1 \hookrightarrow L^6$, leads to the natural classification in terms of the different values of p

$p = 1$ linear
$p = 3$ subcritical
$p = 5$ critical
$p \geq 7$ super critical

Existence and uniqueness of strong solutions for (1.1) is well known in the subcritical case $p \leq 3$. In the critical case $p = 5$, the Cauchy problem for (1.1) has been solved by Grillakis [G] and Shatah-Struwe [S.S]. We recall here the known global result on strong solutions (see Shatah-Struwe [S.S] and Bahouri-Shatah [B.S]

Theorem 1. ($p = 5$) *For any* $(u_0, u_1) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ *there exist in the space*

$$B = \left\{ \partial_t u \in L^\infty(\mathbb{R}, L^2), \nabla_x u \in L^\infty(\mathbb{R}, L^2), u \in L^5(\mathbb{R}, L^{10}) \right\}$$

a unique solution to the Cauchy problem

$$\Box u + u^5 = 0 \ , \ u_{|t=0} = u_0 \ , \ \partial_t u_{|t=0} = u_1$$

Moreover, a by-product of the work of H. Bahouri and P. Gérard (see [B.G]) gives the uniform continuity of the solution u in terms of the data (u_0, u_1).

Theorem 2. $(p = 5)$ *The solution map*

$$(u_0, u_1) \in \dot{H}^1 \times L^2 \to u \in B$$

is Lipschitz on bounded sets of $\dot{H}^1 \times L^2$.

In the super critical case $p \geq 7$, a classical compactness argument gives the existence of at least one weak solution to the Cauchy problem (1.1), but the uniqueness and the existence of a strong solution for which the energy identity holds true are still open problems.

The aim of this paper is to show that the local behavior of solutions of (1.1) in the super critical case exhibits new phenomena. In particular, we shall verify that the Cauchy-problem is not uniformly well posed in the Hadamard-sens ; more precisely, we prove that theorem 2 is untrue in the super critical case (independantly of the choice of a weak solution to (1.1)).

To get these result we shall study only radial solutions to (1.1), with conormal Cauchy data with respect to the origin. (We recall that existence of strong solutions and uniqueness in the radial case is an open problem).

Following a classical reduction in $\mathbb{R}^3$, for a radial solution $u(t,x) = f(t,|x|)$ of (1.1), we introduce the new function g

$$g(t,\rho) = \rho f(t,\rho) \quad \rho = |x| \tag{1.3}$$

Then, (1.1) is equivalent in $\rho > 0$ to the equation

$$\begin{cases} (\partial_t^2 - \partial_\rho^2) g + \frac{g^p}{\rho^{p-1}} = 0 \\ g_{|t=0} = g_0 \in H_0^1 \cap \rho^{\frac{p-1}{p+1}} L^{p+1} \ , \ \partial_t g_{|t=0} = g_1 \in L^2 \end{cases} \tag{1.4}$$

(Here we use the equality $H_0^1(\mathbb{R}_+) = \{g \in L^2, g' - g/\rho \in L^2\}$).

The formally conserved energy for (1.4) is

$$E(g) = \int_0^\infty \left(\frac{1}{2}|\partial_t g|^2 + \frac{1}{2}|\partial_\rho g|^2 + \frac{g^{p+1}}{(p+1)\rho^{p-1}} \right) d\rho \tag{1.5}$$

which is equal to (1.2) for a radial function u (for $g \in H_0^1$, one has $\int_0^\infty (g')^2 d\rho = \int_0^\infty (g' - g/\rho)^2 d\rho$).

The equation (1.4) is a 1.d non-linear wave equation with a singular coefficient at $\rho = 0$. In particular, we know that (1.4) admits a unique strong solution in the cone $0 \leq |t| < \rho$. The following result shows that the solution map for (1.4), with values in this cone, is unstable , and in particular not lipschitz from the unit ball of the energy space into $L^{p+1}(0 < t < |x|)$.

Theorem 3. *($p \geq 7$, odd) There exist two sequences $\underline{u}^n = (u_0^n, u_1^n)$ and $\underline{v}^n = (v_0^n, v_1^n)$ of radial Cauchy data , and two sequences t_n, σ_n of positive real numbers such that*

$$(1.6)\qquad \begin{cases} \text{support } (\underline{u}_n, \underline{v}_n) \subset \{0 < |x| \leq \sigma_n\} \\ E(\underline{u}_n) \leq 1 \ , \ E(\underline{v}_n) \leq 1 \\ \lim\limits_{n\to\infty} \||x|^{-k}(\underline{u}_n - \underline{v}_n) \ ; \ H^s \times H^{s-1}\| = 0 \quad \forall k, s \\ \lim\limits_{n\to\infty} \frac{t_n}{\sigma_n} = 0 \ , \ \lim\limits_{n\to\infty} \sigma_n = 0 \ . \end{cases}$$

and such that the solutions u_n, v_n of (1.1), in $0 \leq |t| < |x|$ satisfy

$$(1.7)\qquad \liminf_{n\to\infty} \int_{t_n<|x|} |u_n - v_n|^{p+1} dx > 0 \ .$$

To prove this result, we shall concentrate our study on the case of smooth Cauchy data in $\rho > 0$, with asymptotic expansion on $\rho = 0$

$$(1.8)\qquad \begin{cases} g(0,\rho) = g_0 \simeq \rho^\gamma[c_0 + c_1\rho^\beta + \cdots] \\ \partial_t g(0,\rho) = g_1 \simeq \rho^{\gamma-1-\beta}[d_0 + d_1\rho^\beta + \cdots] \end{cases}$$

where γ and β are such that

$$(1.9)\qquad \frac{p-2}{p+1} < \gamma < \frac{p-3}{p-1}$$

$$(1.10)\qquad \beta = \frac{p-3}{2} - (\frac{p-1}{2})\gamma \in]0, \frac{p-5}{2(p+1)}[$$

In (1.8), the asymptotic relation $f \simeq \rho^\alpha \sum\limits_0^\infty c_k \rho^{k\beta}$ is assumed to hold in the C^∞ sense, i.e., for every integer N, $f_N = f - \rho^\alpha \sum\limits_0^N c_k \rho^{k\beta}$ satisfies $|\partial^j f_N| \in \mathcal{O}(\rho^{\alpha+(N+1)\beta-j})$ for all $j < \alpha + (N+1)\beta$.

The range of values for γ in (1.9) is easely understood : the lower bound $\frac{p-2}{p+1} < \gamma$ is given by the energy requirement $g_0 \in H^1 \cap L^{p+1}, g_1 \in L^2$; the upper bound $\gamma < \frac{p-3}{p-1}$ means that we work with super critical data. Notice that the limit case $\gamma = \frac{p-3}{p-1}$ is associated with the self-similar solutions to (1.4). The inequality $p > 5$ (i.e. $\frac{p-2}{p+1} < \frac{p-3}{p-1}$) insure the existence of super critical data in the energy space.

The special value of β given in (1.10) is forced by the homogeneity of the equation and will become clear in §2.

Theorem 3 will be a direct consequence of the following result

Theorem 4. *Let* $h_0 = \rho^\gamma \sum_0^\infty c_k \rho^{k\beta}$, $h_1 = \rho^{\gamma-1-\beta} \sum_0^\infty d_k \rho^{k\beta}$ *be two given asymptotic developpements with* $(c_0, d_0) \neq (0,0)$. *There exist two couples of Cauchy data asymptotics to* $\underline{h} = (h_0, h_1)$

$$\underline{g} = (g_0, g_1) \sim (h_0, h_1) \ ; \ \underline{g}' = (g_0', g_1') \sim (h_0, h_1)$$

positive constants $\varepsilon_0 \in]0, 1/2]$, $c_0, \delta_0, \eta_0, \varepsilon_1$, *a function* $\delta \in]0, \delta_0] \mapsto \mu(\delta)$ *satisfying*

$$M|\text{Log}\delta| \ll \mu(\delta) \ll \delta^{-\nu} \quad (\forall M, \nu) \tag{1.11}$$

such that the solutions g, g' *of (1.4) with Cauchy data* $\underline{g}, \underline{g}'$ *satisfy for any* $\delta = \delta_n = 2^{-n}, n$ *large*

$$\begin{cases} |(g-g')(t_\delta, \rho)| \geq \rho^\gamma \varepsilon_1 [|\cos(\frac{\eta_0}{\rho^\beta})| - \varepsilon_0] \ \ \forall \rho \in I_\delta \\ t_\delta = \delta^{1+\beta}\mu(\delta) \ ; \ I_\delta = [\delta - \frac{c_0\delta}{\mu(\delta)}, \delta + \frac{c_0\delta}{\mu(\delta)}] \end{cases} \tag{1.12}$$

In particular, theorem 4 shows that the asymptotic of the Cauchy data at $t = 0$ doesn't determine the asymptotic of the solution on $t = 0$, even in the case of conormal pointwise singularity.

The paper is organized as follows :

In §2, we use a scalling reduction in order to work on a semi-classical non linear wave equation with a small parameter h, of the form

$$h^2(\partial_t^2 - \Delta_x)u + \frac{\partial F}{\partial u}(x, u) = 0 \tag{1.13}$$

with suitable non linear potential F, and cauchy data

$$\begin{cases} u(0,x) \sim \sum_0^\infty h^k a_k(x) \ , \ a_k \in C^\infty \\ h\partial_t u(0,x) \sim \sum_0^\infty h^k b_k(x) \ , \ b_k \in C^\infty \end{cases} \tag{1.14}$$

We then recall the optical Ansatz associated to (1.13), (1.14) (see Whitham [W]), and we discuss the associated anharmonic oscillator on $L^2(\mathbb{S}^1)$

$$\sigma^2(\frac{d}{d\theta})^2 + \frac{\partial F}{\partial u}(x, .) \tag{1.15}$$

In §3, we solve the whole hierarchy of equations associated to this optical Ansatz.

In §4, we discuss the linearisation of the equation (1.4) at the formal optical solution, and the occurence of instability intervals in the associated Hill equation.

Finally, in §5 we conclude the proof of theorem 4 by non linear perturbation arguments.

Proof of theorem 3 :

To end this introduction, let us show that theorem 4 implies theorem 3. Let $\chi \in C_0^\infty(]-1/2,1/2[)$ be equal to 1 on $[-1/4,1/4]$, $I_\delta = [\delta - \frac{c_0\delta}{\mu(\delta)}, \delta + \frac{c_0\delta}{\mu(\delta)}]$, $|I_\delta| = \frac{2c_0\delta}{\mu(\delta)}$. Let g_δ, g'_δ be the solutions of the Cauchy problem (1.4) with data

$$(1.16) \quad \begin{cases} g_\delta(0,\rho) = \chi(\frac{\rho-\delta}{|I_\delta|})g(0,\rho) \ ; g'_\delta(0,\rho) = \chi(\frac{\rho-\delta}{|I_\delta|})g'(0,\rho) \\ \partial_t g_\delta(0,\rho) = \chi(\frac{\rho-\delta}{|I_\delta|})\partial_t g(0,\rho) \ ; \partial_t g'_\delta(0,\rho) = \chi(\frac{\rho-\delta}{|I_\delta|})\partial_t g'(0,\rho) \end{cases}$$

By finite speed of propagation, one has

$$(1.17) \qquad g = g_\delta \text{ and } g' = g'_\delta \text{ on the set } |\rho - \delta| \leq \frac{1}{4}|I_\delta| - |t|$$

The difference of the Cauchy data satisfies

(1.18) $(g_\delta - g'_\delta)(0,\rho)$ and $(\partial_t g_\delta - \partial_t g'_\delta)(0,\rho)$ is $\mathcal{O}(\delta^\infty)$ in H^s for any s

The energy of g_δ (and g'_δ) is equal to

$$(1.19) \quad E(g_\delta) = \int_0^\infty (\frac{1}{2}|\partial_t g|^2\chi_\delta^2 + \frac{g^{p+1}}{(p+1)\rho^{p-1}}\chi_\delta^{p+1} + \frac{1}{2}\left[\chi_\delta\partial_\rho g + \frac{1}{|I_\delta|}\chi'(\frac{\rho-\delta}{|I_\sigma|})g\right]^2 d\rho$$

with $\chi_\delta = \chi(\frac{\rho-\delta}{|I_\delta|})$

¿From $\gamma < \frac{p-3}{p-1}$, and $|I_\delta| \gg \delta^{1+\nu}(\forall\nu > 0)$, we get $|I_\delta|\delta^{\gamma(p+1)-(p-1)} \gg |I_\delta|^{-1}\delta^{2\gamma}$. Thus we have

$$(1.20) \qquad E(g_\delta) \simeq E(g'_\delta) \approx |I_\delta|\delta^{\gamma(p+1)-(p-1)}$$

Let $\alpha = \frac{p-3}{p-1}$ and $u_{\varepsilon,\delta}, u'_{\varepsilon,\delta}$, the solutions of (1.4)

$$(1.21) \qquad \begin{cases} u_{\varepsilon,\delta}(t,\rho) = \varepsilon^{-\alpha}g_\delta(\varepsilon t,\varepsilon\rho) \\ u'_{\varepsilon,\delta}(t,\rho) = \varepsilon^{-\alpha}g'_\delta(\varepsilon t,\varepsilon\rho) \end{cases}$$

One has

$$(1.22) \qquad E(u_{\varepsilon,\delta}) = \varepsilon^{1-2\alpha}E(g_\delta)$$

Let ε_δ be such that $\varepsilon_\delta^{1-2\alpha}|I_\delta|\delta^{\gamma(p+1)-(p-1)} \approx 1$.
We have $2\alpha - 1 = \frac{p-5}{p-1} > \gamma(p+1) - p + 2$, so we get

$$(1.23) \qquad \delta \ll \varepsilon_\delta \quad \lim_{\delta\to 0}\varepsilon_\delta = 0$$

Let $u_\delta = u_{\varepsilon_\delta,\delta}$, and define u'_δ in the same way. Then we have

$$(1.24) \qquad \begin{cases} E(u_\delta) \simeq E(u'_\delta) \approx 1 \\ \lim_{\delta\to 0}\|\rho^{-k}(\underline{u}_\delta - \underline{u'}_\delta) \ ; \ H^s \times H^{s-1}\| = 0 \text{ for any } k, s \\ \text{support } (\underline{u}_\delta, \underline{u}'_\delta) \subset \{|\rho - \frac{\delta}{\varepsilon_\delta}| \leq \frac{c_0}{\mu(\delta)}\frac{\delta}{\varepsilon_\delta}\} \end{cases}$$

Let $J_\delta = \{|\rho - \delta| \leq \frac{1}{4}|I_\delta| - |t|\}$. We have $t_\delta = \delta^{1+\beta}\mu(\delta) \ll |I_\delta|$, so $|J_\delta| \approx |I_\delta|$. Thus we get using (1.12),for $\delta = \delta_n = 2^{-n}, n$ large and for some $c > 0$

$$
\begin{aligned}
\int_{\frac{t_\delta}{\varepsilon_\delta}<\rho} \frac{|u_\delta - u'_\delta|^{p+1}}{\rho^{p-1}} d\rho &\geq \varepsilon_\delta^{1-2\alpha} \int_{J_\delta} \frac{|g-g'|^{p+1}(t_\delta,\rho)}{\rho^{p-1}} d\rho \\
&\geq 2c\varepsilon_\delta^{1-2\alpha}|J_\delta|\delta^{\gamma(p+1)-(p-1)} \geq c
\end{aligned}
\tag{1.25}
$$

QED.

2. Semi-classical reduction

In order to study the solutions of equations (1.4) with Cauchy data (1.9), we introduce new coordinates and unknown fonction

$$
\begin{cases} t = \hbar s \ , \ \rho = \hbar x \quad h = \hbar^\beta \\ g(\hbar s, \hbar x) = \hbar^\gamma f_h(s,x) \end{cases}
\tag{2.1}
$$

Here $\gamma \in]\frac{p-2}{p+1}, \frac{p-3}{p-1}[$ is a fixed constant, $\beta = \frac{p-3}{2} - (\frac{p-1}{2})\gamma$ and $h \in]0,1]$ is a small parameter. Then $f = f_h$ satisfies the semi-classical non linear wave equation

$$
h^2(\partial_s^2 - \partial_x^2)f + \frac{f^p}{x^{p-1}} = 0
\tag{2.2}
$$

and the Cauchy data for f are deduced from (1.9)

$$
\begin{cases} f_h(0,x) \sim x^\gamma \sum_0^\infty h^k c_k x^{k\beta} \\ h\partial_s f_h(0,x) \sim x^{\gamma-1-\beta} \sum_0^\infty h^k d_k x^{k\beta} \end{cases}
\tag{2.3}
$$

We shall restrict the study of the Cauchy problem (2.2), (2.3) in a small neighborhood of $s = 0, x = 1$; it means that we are looking for values of $g(t,\rho)$ in a small conic neighborhood of the segment $t = 0$, $\rho \in]0,1]$

The parameter $\hbar = h^{1/\beta}$ is then equivalent to the distance to the origin $(t,\rho) = (0,0)$.

In (2.3) the asymptotic relation $f_h \sim \sum_0^\infty h^k a_k(x)$ means that there exist a fixed neighborhood W of $x = 1$ such that $\|\partial_x^j(f_h - \sum_0^N h^k a_k(x))\|_{L^\infty(W)} \in \mathcal{O}(h^{N+1})$ for all j, N.

We shall introduce a non-linear optical ansatz to treat the Cauchy problem (2.2), (2.3). At this level, it appears more conveniant to study the following general local Cauchy problem, with $t \in \mathbb{R}$, $x \in$

$\mathbb{R}^d$, $(t,x) \sim (0,x_0)$

$$(2.4) \quad \begin{cases} h^2(\partial_t^2 - \Delta_x)u + \frac{\partial F}{\partial u}(x,u) = 0 \ , \ u = u_h(t,x) \in \mathbb{R} \\ u_h(0,x) \sim \sum_0^\infty h^k a_k(x) \quad a_k \in C^\infty, u_h(0,x) \in C^\infty \\ h\partial_t u_h(0,x) \sim \sum_0^\infty h^k b_k(x) \quad b_k \in C^\infty, h\partial_t u_h(0,x) \in C^\infty \end{cases}$$

where $F(x,u)$ is a smooth function of $x \in \mathbb{R}^d$, $u \in \mathbb{R}$ defined near $x = x_0$, $u = u_0 = a_0(x_0)$. The formally conserved energy for (2.4) is

$$(2.5) \qquad E(u) = \int \left[\frac{h^2}{2}(|\partial_t u|^2 + |\nabla_x u|^2) + F(x,u)\right] dx$$

By classical results, we know that (2.4) admits a unique smooth solution in a "small" interval $\{|t| \leq C^{te}h\}$. The optical ansatz is a tentative to extend this solution to a "big" interval $\{|t| \leq C^{te}\}$ (as we shall see later, these tentative breaks down for "almost all" Cauchy data...)

The starting point of non linear optics is to search $u_h(t,x)$ as a formal asymptotic expansion (see Whitham [W])

$$(2.6) \qquad u_h(t,x) \simeq \widetilde{U}_0(t,x,\frac{\varphi(t,x)}{h}) + \sum_{k\geq 1} h^k \widetilde{U}_k(t,x,\varphi/h)$$

where $\widetilde{U}_j(t,x,\theta)$ are smooth fonctions, 2π-periodic in the θ variable, and with an unknown phase function $\varphi(t,x)$ encoding the oscillations of the solution, such that

$$(2.7) \qquad \varphi(0,x) \equiv 0$$

It is easy to see that $\theta \mapsto \widetilde{U}_0(t,x,\theta)$ must then satisfy the ordinary differential equation

$$(2.8) \qquad (\varphi_t'^2 - |\nabla_x \varphi|^2)\partial_\theta^2 \widetilde{U}_0 + \frac{\partial F}{\partial u}(x, \widetilde{U}_0) = 0$$

so it is natural to make the assumption (we may use the phase symmetry $\theta \to -\theta$ in (2.8))

$$(2.9) \qquad \frac{\partial \varphi}{\partial t}(0,x_0) > 0$$

We can view (2.8) as a fibration of the phase space $T^*(\mathbb{R}_t \times \mathbb{R}_x^d) \cap \{|\tau| > |\xi|\}$ by an anharmonic oscillator

$$(2.10) \qquad \sigma^2 d_\theta^2 f + \frac{\partial F}{\partial u}(x,f) = 0 \quad \sigma = (\tau^2 - \xi^2)^{1/2}$$

for which the conserved energy is

$$(2.11) \qquad E = \frac{\sigma^2}{2}|f_\theta'|^2 + F(x,f) \ , \ \frac{\partial E}{\partial \theta} = 0$$

and where we consider $\sigma > 0$, $x \sim x_0$ as parameters. In view of the Cauchy data (2.4) of u_k, we have

(2.12) $$u_h \sim a_{0,0} = a_0(x_0) \quad h\partial_t u_h \sim b_{0,0} = b_0(x_0)$$

so we work near the energy level

(2.13) $$E_0 = \frac{b_{0,0}^2}{2} + F(x_0, a_{0,0})$$

The first hypothesis we make on the function F will make sure that (2.10) admits periodic solutions.

$(H.1)$ The connected component of $a_{0,0}$ in $\{F(x_0, u) \leq E_0\}$ is of the form $[u_-^0, u_+^0]$ with $\frac{\partial F}{\partial u}(x_0, u_-^0) < 0$ and $\frac{\partial F}{\partial u}(x_0, u_+^0) > 0$.

If (H.1) is satisfied, then the ordinary differential equation on the line $\frac{d^2 f}{dy^2} + \frac{\partial F}{\partial u}(x_0, f) = 0$ with Cauchy data $f(0) = a_{0,0}, f'(0) = b_{0,0}$ admits a periodic solution with period $\Pi(x_0, E_0)$ with

$$\begin{cases} \Pi(x, E) = \sqrt{2}\int_{u_-(x,E)}^{u_+(x,E)} \frac{du}{(E - F(x,u))^{1/2}} \\ F(x, u_\pm(x, E)) = E \ , \ u_\pm(x_0, E_0) = u_\pm^0 \end{cases}$$

The second hypothesis we made on the function F is that the frequency of the solutions of the equation $\frac{d^2 f}{dy^2} + \frac{\partial F}{\partial u}(x_0, f) = 0$ is a strictly increasing function of the energy parameter

$(H.2)$ $$\frac{\partial}{\partial E}\Pi(x_0, E_0) < 0$$

Remark 2.1. In our case, we have $F(x_0, u) = \frac{u^{p+1}}{(p+1)x_0^{p-1}}$, so (H.1) is satisfied iff $E_0 \neq 0$, i.e. iff

(2.14) $$(a_{0,0}, b_{0,0}) \neq (0, 0)$$

and $\Pi(x_0, E) = (\frac{x_0}{\sqrt{E}})^{\frac{p-1}{p+1}}\sqrt{2}\int_{-\alpha}^{\alpha} \frac{dy}{(1 - \frac{y^{p+1}}{p+1})^{1/2}}$, $\alpha^{p+1} = p + 1$ so (H.2) is satisfied.

If (H.1) and (H.2) are satisfied, the equation on the line, with σ closed to σ_0, such that $\sigma_0 \Pi(x_0, E_0) = 2\pi$

$$\sigma^2 d_\theta^2 f + \frac{\partial F}{\partial u}(x, f) = 0$$

with energy E closed to E_0, and with values $f(\theta)$ closed to $[u_-^0, u_+^0]$, admits 2π-periodic solutions iff σ and E are related by

$$\sigma\Pi(x, E) = 2\pi \text{ (Normalization)} \tag{2.15}$$

and in that case, the space of solutions is a circle parametrized by a phase-shift $\Theta \in [0, 2\pi[$

Definition 2.1. For (σ, x) closed to (σ_0, x_0), let

$$K(\sigma, x; \theta) \tag{2.16}$$

be the 2π-periodic in θ solution of $\sigma^2\partial_\theta^2 K + \frac{\partial F}{\partial u}(x, K) = 0$ such that, with $\sigma\Pi(x, E(\sigma, x)) = 2\pi$

$$\begin{aligned} &K(\sigma, x; 0) = u_+(x, E(\sigma, x)) \ , \\ &\tfrac{\partial K}{\partial \theta}(\sigma, x; 0) = 0 \end{aligned} \tag{2.17}$$

This fundamental solution $K(\cdot; \theta)$ is even in θ ($K(\cdot; -\theta)$ satisfies the same differential equation with the same data at $\theta = 0$), and we have

$$\frac{\sigma^2}{2}|\partial_\theta K(\cdot, \theta)|^2 + F(x, K(\cdot, \theta)) \equiv E(\sigma, x) \quad \forall\theta \tag{2.18}$$

For a 2π-periodic function $f(\theta)$, we shall denote by $\oint f = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$ its mean value, and we define the function $J(\sigma, x)$ by

$$J(\sigma, x) = \frac{1}{2}\oint |\partial_\theta K(\sigma, x; \cdot)|^2 \tag{2.19}$$

Finally, we introduce the linearized equation at K :

$$\mathcal{L}_{(\sigma,x)}(g) = \sigma^2\partial_\theta^2 g + \frac{\partial^2 F}{\partial u^2}(x, K(\sigma, x; \theta))g \tag{2.20}$$

Lemma 2.1. *i) $\mathcal{L}_{(\sigma,x)}$ is self adjoint on $L^2([0, 2\pi])$, and the 2π-periodic kernel of $\mathcal{L}_{(\sigma,x)}$ is the one dimensional subspace span by $\frac{\partial K}{\partial \theta}(\sigma, x; \cdot)$*

ii) The unique solution of $\mathcal{L}(W) = \partial_\theta^2 K$, orthogonal to $\frac{\partial K}{\partial \theta}$ is

$$W = \frac{-1}{2\sigma}\frac{\partial K}{\partial \sigma} + j(x, \sigma)\frac{\partial K}{\partial \theta} \ ; \ j = \frac{1}{4\sigma J}\oint \frac{\partial K}{\partial \sigma}\frac{\partial K}{\partial \theta} \tag{2.21}$$

Proof :

i) Let us denote by $f(\sigma, x, E, y)$ the solution of $\sigma^2 d_y^2 f + \frac{\partial F}{\partial u}(x, f) = 0$ such that

$$\begin{cases} f(\sigma, x, E, 0) = u_+(x, E) \quad F(x, u_+(x, E)) = E \\ \frac{\partial f}{\partial y}(\sigma, x, E, 0) = 0 \end{cases}$$

Then, both $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial E}$ satisfy the second order equation on the line $\sigma^2\frac{d^2 g}{dy^2} + \frac{\partial^2 F}{\partial u^2}(x, f)g = 0$. The fonction f has period $T = \sigma\Pi(x, E)$. By assumption (H.2), we have $\frac{\partial T}{\partial E}(\sigma_0, x_0, E_0) \neq 0$ so $\frac{\partial f}{\partial E}(\sigma_0, x_0, E_0; \cdot)$ is not a 2π-periodic function, and the kernel of $\mathcal{L}_{(\sigma,x)}$ on 2π-periodic functions is thus spaned by $\frac{\partial f}{\partial y}(\sigma, x, E(\sigma, x); .)$.

ii) One has $\mathcal{L}_{(\sigma,x)}(\frac{\partial K}{\partial \sigma}) = -2\sigma\partial_\theta^2 K$, so ii) is obvious.

Lemma 2.2. *Near (σ_0, x_0), the following inequality holds true*

$$J + \sigma\frac{\partial J}{\partial \sigma} > 0 \tag{2.22}$$

Proof : *One has* $2(J + \sigma\frac{\partial J}{\partial \sigma}) = \oint(\partial_\theta K)^2 + 2\sigma\partial^2_{\theta,\sigma}K.\partial_\theta K$

$$= \oint(\partial_\theta K)^2 - 4\sigma^2\partial_\theta V\partial_\theta K = \oint(\partial_\theta K)^2 + 2\sigma^2[V\partial_\theta^2 K - \partial_\theta V\partial_\theta K]$$

with $V = -1/2\sigma\frac{\partial K}{\partial \sigma}$.

Let $g = \partial_\theta K$; Using $\mathcal{L}(g) = 0, \mathcal{L}(V) = \partial_\theta g$, one easely verify that $g^2 + 2\sigma^2(Vg' - V'g)$ is independant of θ ; therefore, we get

$$2(J + \sigma\frac{\partial J}{\partial \sigma}) = 2\sigma^2(VK'')_{|\theta=0} = \frac{1}{\sigma}\frac{\partial K}{\partial \sigma}_{|\theta=0}\frac{\partial F}{\partial u}(x, u_+)$$

We have $\frac{\partial K}{\partial \sigma}_{|\theta=0} = \frac{\partial u_+}{\partial E}\frac{\partial E}{\partial \sigma}$ (by 2.17) and 2.22 follows from $\frac{\partial u_+}{\partial E} > 0$ and from (H.2), since $\sigma\Pi(x, E(\sigma, x)) = 2\pi$ implies $\sigma\frac{\partial E}{\partial \sigma} = -\frac{\Pi}{\partial\Pi/\partial E} > 0$

Remark 2.2. If one starts with a linear equation

$$\sigma^2 d_\theta^2 f + V(x)f = 0$$

then the normalization $\sigma\Pi(E) = 2\pi$, gives $\sigma^2 = V(x)$ which is exactly the eikonale equation for the phase : $\varphi_t'^2 = |\nabla_x\varphi|^2 + V(x)$. The space of 2π-periodic solution is then 2-dimensional. In the linear case, we have thus a 2-dimensional fibration over the caracteristic variety.

In contrast, in the non-linear case, we have a 1-dimensional fibration by circles, parametrized by a phase shift Θ, on the open subset of the cotangent space $\{|\tau| > |\xi|\}$.

In the linear case, the first asymptotic term is $Re(ae^{i\varphi/h}), a = |a|e^{i\Theta}$, and the energy, which is free is given by $E \sim V(x)|a|^2$, so the natural parameters are (E, Θ). In the non linear case, the natural parameters are (σ, Θ), and we recover the energy by the normalization $\sigma\Pi(x, E) = 2\pi$.

3. Non-linear Optic

In this part, we shall study the formal asymptotic solutions of the form

$$u_h(t, x) = \widetilde{U}_h(t, x, \frac{\varphi(t, x)}{h}) \tag{3.1}$$

for the Cauchy-problem (2.4), under the hypothesis (H1), (H2) of §2. We have thus to study the asymptotic equation in h

$$(3.2)\qquad \begin{cases} (\varphi_t'^2 - |\nabla_x\varphi|^2)\partial_\theta^2\widetilde{U}_h + \frac{\partial F}{\partial u}(x,\widetilde{U}_h) \\ \quad +h[2\varphi_t'\frac{\partial}{\partial t} - 2\nabla_x\varphi\nabla_x + \Box\varphi]\partial_\theta\widetilde{U}_h + h^2\Box\widetilde{U}_h = 0 \\ \widetilde{U}_h(0,x,0) = \sum_0^\infty h^k a_k(x) \\ \varphi_t'\partial_\theta\widetilde{U}_h(0,x,0) + h\partial_t\widetilde{U}(0,x,0) = \sum_0^\infty h^k b_k(x) \end{cases}$$

with $\Box = \partial_t^2 - \Delta_x$. In (3.2), we search $\widetilde{U}_h$ as a formal power serie in h

$$(3.3)\qquad \widetilde{U}_h(t,x,\theta) = \sum_0^\infty h^k\widetilde{U}_k(t,x,\theta)$$

We introduce a phase-shift $\Theta(t,x)$, and we define U_k by

$$(3.4)\qquad \widetilde{U}_k(t,x,\theta) = U_k(t,x,\theta+\Theta(t,x))$$

We make the choice

$$(3.5)\qquad U_0(t,x,\theta) = K(\sigma,x,\theta), \quad \sigma = (\varphi_t'^2 - |\nabla_x\varphi|^2)^{1/2}$$

where K is the fundamental solution of $\sigma^2\partial_\theta^2K + \frac{\partial F}{\partial u}(x,K)$ defined in §2.

Let us define the collection of function $Q_n(x,u_0,\cdots,u_n)$ by the identity

$$(3.6)\qquad \frac{\partial F}{\partial u}(x,\sum_0^\infty h^k u_k) = \sum_0^\infty h^n Q_n(x,u_0,\cdots,u_n)$$

We get by the taylor formula

$$(3.7)\qquad \begin{cases} Q_0(x,u_0) = \frac{\partial F}{\partial u}(x,u_0) \\ Q_1(x,u_0,u_1) = \frac{\partial^2 F}{\partial u^2}(x,u_0)u_1 \\ Q_n(x,u_0,\cdots,u_n) = \frac{\partial^2 F}{\partial u^2}(x,u_0)u_n + R_n(x,u_0,\cdots,u_{n-1}) \quad n \geq 2 \end{cases}$$

where the R_n functions are given by

$$(3.8)\qquad \begin{cases} R_2 = \frac{u_1^2}{2}\partial_u^3F(x,u_0) \\ R_3 = u_1u_2\partial_u^3F(x,u_0) + \frac{u_1^3}{6}\partial_u^4F(x,u_0) \\ R_4 = (\frac{1}{2}u_2^2 + u_1u_3)\partial_u^3F(x,u_0) + \frac{1}{2}u_1^2u_2\partial_u^4F(x,u_0) + \widetilde{R}_4(x,u_0,u_1) \\ R_{n\geq 5} = (u_2u_{n-2} + u_1u_{n-1})\partial_u^3F(x,u_0) + \frac{1}{2}u_1^2u_{n-2}\partial_u^4F(x,u_0) \\ \qquad\qquad +\widetilde{R}_n(x,u_0,\cdots,u_{n-3}) \end{cases}$$

(The precise structure of the functions $\widetilde{R}_n(x, u_0, \cdots, u_{n-3})$ will play no role in the sequel).

If we denote by $T = 2\varphi_t'\partial_t - 2\nabla_x\varphi\nabla + \Box\varphi$ the transport operator, the first equation in (3.2) is then equivalent to the hierarchy of equations, with $\sigma^2 = \varphi_t'^2 - |\nabla_x\varphi|^2$

$$(Eq)_0 \qquad \sigma^2\partial_\theta^2\widetilde{U}_0 + F_u'(x, \widetilde{U}_0) = 0$$

$$(Eq)_1 \qquad \sigma^2\partial_\theta^2\widetilde{U}_1 + F_u''(x, \widetilde{U}_0)\widetilde{U}_1 + T\partial_\theta\widetilde{U}_0 = 0$$

$(Eq)_{k\geq 2}$

$$\sigma^2\partial_\theta^2\widetilde{U}_k + F_u''(x, \widetilde{U}_0)\widetilde{U}_k + R_k(x, \widetilde{U}_0, \cdots, \widetilde{U}_{k-1}) + T\partial_\theta\widetilde{U}_{k-1} + \Box\widetilde{U}_{k-2} = 0$$

Moreover the Cauchy data are given by

$$(3.9) \qquad \begin{cases} \widetilde{U}_k(0, x, 0) = a_k(x) \quad \forall k \geq 0 \\ \varphi_t'(0, x)(\partial_\theta\widetilde{U}_0)(0, x, 0) = b_0(x) \\ \varphi_t'(0, x)(\partial_\theta\widetilde{U}_k)(0, x, 0) + (\partial_t\widetilde{U}_{k-1})(0, x, 0) = b_k(x) \quad \forall k \geq 1 \end{cases}$$

We have $\widetilde{U}_0(t, x, \theta) = K(\sigma, x, \theta + \Theta)$, so $(Eq)_0$ is satisfied. Let Z be the vector field

$$(3.10) \qquad Z = 2\varphi_t'\partial_t - 2\nabla_x\varphi.\nabla$$

We have $\partial_{z_j}[f(z, \theta + \Theta(z))] = [(\partial_{z_j}f) + \Theta_{z_j}'(\partial_\theta f)](z, \theta + \Theta(z))]$ so if T_Θ and $\Box_\Theta$ are the operators

$$(3.11) \qquad \begin{cases} T_\Theta = T + Z(\Theta)\partial_\theta \\ \Box_\Theta = (\partial_t + \Theta_t'\partial_\theta)^2 - \sum_j(\partial_{x_j} + \Theta_{x_j}'\partial_\theta)^2 \end{cases}$$

We can rewrite $(Eq)_k$ $k \geq 1$, and the equations for the traces on $t = 0$, as equations involving $\{U_k\}$, using (3.4), with $\mathcal{L} = \sigma^2\partial_\theta^2 + F_u''(x, K)$

$$(Eq)_1 \qquad \mathcal{L}(U_1) + T_\Theta\partial_\theta U_0 = 0$$

$$(Eq)_{k\geq 2} \qquad \mathcal{L}(U_k) + R_k(x, U_0, \cdots, U_{k-1}) + T_\Theta\partial_\theta U_{k-1} + \Box_\Theta U_{k-2} = 0$$

$$(3.12) \qquad \begin{cases} U_k(0, x, \Theta(0, x)) = a_k(x) \\ \varphi_t'(0, x)\partial_\theta U_0(0, x, \Theta(0, x)) = b_0(x) \\ \varphi_t'(0, x)(\partial_\theta U_k)(0, x, \Theta(0, x)) \\ \quad +(\partial_t U_{k-1} + \Theta_t'(0, x)\partial_\theta U_{k-1})(0, x, \Theta(0, x)) = b_k(x) \end{cases}$$

For each $k \geq 1$, we will decompose U_k according to the splitting $L^2(\mathbb{S}_\theta^1) = (\mathrm{Ker}\mathcal{L}) \bigoplus (\mathrm{Im}\mathcal{L})$

$$(3.13) \qquad \begin{cases} U_k(t, x, \theta) = \lambda_k(t, x)\partial_\theta K + V_k(t, x, \theta) \\ \oint V_k\partial_\theta K = 0 \end{cases}$$

We are now ready to study the optical hierarchy.

Level 0

$(Eq)_0$ is satisfied by construction. By (2.9), (2.11), we have $\sigma_{|t=0} = \varphi'_{t|t=0}$ and

$$E(\sigma_{|t=0}, x) = \frac{b_0(x)^2}{2} + F(x, a_0(x)) \tag{3.14}$$

¿From $\sigma\Pi(x, E) = 2\pi$, we find

$$\frac{\partial\varphi}{\partial t}(0, x) = (2\pi)\{\Pi(x, \frac{b_0(x)^2}{2} + F(x, a_0(x)))\}^{-1} = \sigma(0, x) > 0 \tag{3.15}$$

Moreover, the data equations

$$\begin{cases} K(\sigma_{|t=0}, x, \Theta) = a_0(x) \\ \sigma_{|t=0}\partial_\theta K(\sigma_{|t=0}, x, \Theta) = b_0(x) \end{cases}$$

admits a unique solution $\Theta = \Theta(0, x)$ by the assumption (H.1).

The two data $\varphi'_t(0, x)$ and $\Theta(0, x)$ are thus determined.

Level 1

We have $U_1 = \lambda_1 \partial_\theta K + V_1$ and

$$\mathcal{L}(V_1) + T_\Theta \partial_\theta K = 0 \tag{3.16}$$

is solvable iff $\oint (T + Z(\Theta)\partial_\theta)(\partial_\theta K)(\partial_\theta K) = 0$, which is equivalent to, with $J = \frac{1}{2}\oint |\partial_\theta K|^2$

$$\varphi'_t \partial_t J - \nabla_x \varphi \nabla(J) + \Box\varphi J = 0 \tag{3.17}$$

In other words, the phase φ must satisfy

$$\begin{cases} \partial_t[\varphi'_t J] = \mathrm{div}[(\nabla_x\varphi)J] \\ \sigma = (\varphi_t'^2 - |\nabla_x\varphi|^2)^{1/2}; J(\sigma, x) = \frac{1}{2}\oint |\partial_\theta K(\sigma, x, .)|^2 \end{cases} \tag{3.18}$$

The identity $\sigma^2 = \varphi_t'^2 - |\nabla_x\varphi|^2$ allows to rewrite (3.17) in the form of a quasi-linear second order differential equation for the phase φ

$$\frac{\partial_\sigma J}{\sigma}\left[\varphi_t'^2\partial_t^2\varphi - 2\varphi'_t\sum_j \frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_j\partial t} + \sum_{k\,j}\frac{\partial\varphi}{\partial x_j}\frac{\partial\varphi}{\partial x_k}\frac{\partial^2\varphi}{\partial x_j\partial x_k}\right] - \nabla_x\varphi . \nabla_x J + J(\sigma)[\partial_t^2\varphi - \Delta_x\varphi] = 0 \tag{3.19}$$

with data $\varphi_{|t=0} = 0$, $\varphi'_{t|t=0} = \sigma_{|t>0} > 0$.

The restriction of (3.19) on $t = 0$ is equal to

$$(J(\sigma) + \sigma\frac{\partial J}{\partial\sigma}(\sigma))\partial_t^2\varphi - J(\sigma)\Delta_x\varphi \tag{3.20}$$

which is (by lemma 2.2) a strictly hyperbolic equation in time. Therefore, the phase function φ is uniquely determined by the equation (3.17) in the vicinity of $(t, x) = (0, x_0)$, and the $\mathcal{L}$ operator $\sigma^2\partial_\theta^2 + F''_u(x, K(\sigma, x, \theta))$ is known.

Remark 3.1. We can also rewrite (3.18) as a conservation law for the unknowns $p = \nabla_x \varphi$, $q = \varphi'_t J$.

$$\begin{cases} \partial_t q = \operatorname{div}(pJ) \\ \partial_t p = \nabla_x (q/J) \end{cases} \tag{3.21}$$

We have $U_1 = \lambda_1 \partial_\theta K + V_1$, where V_1 solves

$$\mathcal{L}(V_1) + T(\partial_\theta K) + Z(\Theta)\partial_\theta^2 K = 0 \tag{3.22}$$

So if we denote by $\mathcal{L}^{-1}$ the inverse map of $\mathcal{L}$ on $(\ker \mathcal{L})^\perp$, we get, with $W = \mathcal{L}^{-1}(\partial_\theta^2 K)$ (see lemma 2.1)

$$V_1 = -\mathcal{L}^{-1}(T(\partial_\theta K)) - Z(\Theta)W \tag{3.23}$$

Let us denote by

$$\sigma_0(x) = \varphi'_{t|t=0} \ ; \ \Theta_0(x) = \Theta_{|t=0} \tag{3.24}$$

the two Cauchy data determined at level 0. By (3.12), we get

(3.25)

$$\begin{cases} U_1(0, x, \Theta_0) = a_1(x) \\ \sigma_0 \partial_\theta U_1(0, x, \Theta_0) + \Theta'_t(0, x)\partial_\theta K(0, x, \Theta_0) = b_1(x) - \partial_t K(0, x, \Theta_0) \end{cases}$$

If one use $Z(\Theta)_{|t=0} = 2\sigma_0 \Theta'_t(0, x)$, and $U_1 = \lambda_1 \partial_\theta K - \mathcal{L}^{-1}(T(\partial_\theta^2 K)) - Z(\Theta)W$, we obtain that (3.25) is equivalent to a 2×2 system for the unknown $\lambda_1(0, x), \Theta'_t(0, x)$

(3.26)

$$\begin{bmatrix} \partial_\theta K & -2\sigma W \\ \sigma \partial_\theta^2 K & -2\sigma^2 \partial_\theta W + \partial_\theta K \end{bmatrix} (t = 0, x, \theta = \Theta_0(x)) \begin{bmatrix} \lambda_1(0, x) \\ \theta'_t(0, x) \end{bmatrix} = \text{given}$$

The determinant of this system is equal to $\partial_\theta^2 K + 2\sigma^2 (W \partial_\theta(\partial_\theta K) - \partial_\theta W \partial_\theta K) = 2(J + \sigma \frac{\partial J}{\partial \sigma}) \neq 0$. (by the proof of lemma 2.2).

The two data $\lambda_1(0, x), \Theta'_t(0, x)$ are thus determined.

Level 2

We have $U_2 = \lambda_2 \partial_\theta K + V_2$ with

$$\mathcal{L}(V_2) + \frac{1}{2} U_1^2 \partial_u^3 F(x, K) + T_\Theta \partial_\theta U_1 + \Box_\Theta K = 0 \tag{3.27}$$

Using $U_1 = \lambda_1 \partial_\theta K + V_1$, we get the integrability condition for (3.27)

$$\begin{cases} \frac{\lambda_1^2}{2} \oint \partial_u^3 F(x, K)(\partial_\theta K)^3 + \\ \oint \Big[\lambda_1 V_1 \partial_\theta K \partial_u^3 F(x, K) + T(\lambda_1 \partial_\theta^2 K) + \lambda_1 Z(\Theta) \partial_\theta^3 K \Big] \partial_\theta K + \\ \oint \Big[\frac{V_1^2}{2} \partial_u^3 F(x, K) + T_\Theta(\partial_\theta V_1) + \Box_\Theta K \Big] \partial_\theta K = 0 \end{cases} \tag{3.28}$$

¿From $\mathcal{L}(\partial_\theta K) = 0$, we deduce

$$\mathcal{L}(\partial_\theta^2 K) + \partial_u^3 F(x, K)(\partial_\theta K)^2 = 0 \tag{3.29}$$

so in (3.28) the quadratic term in λ_1 vanishes, using the identity
(3.30)

$$\oint \partial_u^3 F(x,K)(\partial_\theta K)^2 \partial_\theta K = -\oint \mathcal{L}(\partial_\theta^2 K)\partial_\theta K = -\oint \partial_\theta^2 K \mathcal{L}(\partial_\theta K) = 0\,.$$

We have $T(\lambda_1 \partial_\theta^2 K) = \lambda_1 T(\partial_\theta^2 K) + Z(\lambda_1)\partial_\theta^2 K$, and $\oint \partial_\theta^2 K \partial_\theta K = 0$, so the second term in (3.28) is equal to

$$(3.31) \qquad \lambda_1 \oint V_1(\partial_\theta K)^2 \partial_u^3 F(x,K) + T(\partial_\theta^2 K)\partial_\theta K + Z(\Theta)\partial_\theta^3 K \partial_\theta K$$

$$= \lambda_1 \oint (-\mathcal{L}(V_1))\partial_\theta^2 K + T(\partial_\theta^2 K)\partial_\theta K - Z(\Theta)(\partial_\theta^2 K)^2$$

which is equal by (3.22) to

$$= \lambda_1 \oint [T(\partial_\theta K) + Z(\Theta)\partial_\theta^2 K]\partial_\theta^2 K + T(\partial_\theta^2 K)\partial_\theta K - Z(\Theta)(\partial_\theta^2 K)^2$$

$$= \lambda_1 Z[\oint \partial_\theta K \partial_\theta^2 K] = 0$$

Therefore, the integrability condition (3.28) does'nt depend on the unknown function λ_1. We have
(3.32)

$$\begin{cases} \oint \frac{V_1^2}{2}\partial_u^3 F(x,K)\partial_\theta K = \oint \frac{V_1^2}{2}\partial_\theta[\partial_u^2 F(x,K)] = -\oint (\partial_\theta V_1)V_1 \partial_u^2 F(x,K) \\ = -\oint \partial_\theta V_1[\mathcal{L}(V_1) - \sigma^2 \partial_\theta^2 V_1] = -\oint \partial_\theta V_1 \mathcal{L}(V_1) = \oint \partial_\theta V_1 T_\Theta(\partial_\theta K) \end{cases}$$

and

$$(3.33)\ \ \Box_\Theta = \Box + (\Box\Theta)\frac{\partial}{\partial\theta} + 2(\Theta_t'\partial_t - (\nabla_x\Theta)\nabla_x)\frac{\partial}{\partial\theta} + (\theta_t'^2 - |\nabla_x\Theta|^2)\partial_\theta^2$$

Therefore, (3.28) is equivalent to

$$(3.34) \quad \begin{cases} \oint (\partial_\theta V_1)T_\Theta(\partial_\theta K) + T_\Theta(\partial_\theta V_1)\partial_\theta K \\ +\oint \Box K \partial_\theta K + \{2(\Theta_t'\partial_t - (\nabla_x\Theta)\nabla_x + \Box\Theta\} \oint (\partial_\theta K)^2 = 0 \end{cases}$$

By (3.23), we have $V_1 = -\mathcal{L}^{-1}(T(\partial_\theta K)) - Z(\Theta)W$, so (3.34) is a differential second order equation for Θ. The non linear part in Θ of this equation is equal to

$$\oint -Z(\Theta)^2 \partial_\theta W \partial_\theta^2 K - Z(\Theta)^2 \partial_\theta^2 W \partial_\theta K \equiv 0$$

so we conclude that (3.34) is in fact a *linear* second order partial differential equation for Θ ; the principal part of this equation is

$$(3.35) \qquad \Box\Theta \oint (\partial_\theta K)^2 - Z(Z(\Theta)) \oint \partial_\theta W \partial_\theta K$$

By lemma 2.1, we have $\oint \partial_\theta W \partial_\theta K = \frac{-1}{2\sigma}\frac{\partial J}{\partial\sigma}$, so the principal symbol of (3.35) is equal to

$$(3.36) \qquad -2J\left[(\tau^2 - \xi^2) + \frac{1}{\sigma}\frac{\partial J/\partial\sigma}{J}(\tau\varphi_t' - \xi.\varphi_x')^2\right]$$

By lemma 2.2, we conclude that (3.34) is a linear second order hyperbolic equation for the phase shift Θ.

Moreover, the data $\Theta_{|t=0}$ and $\partial_t\Theta_{|t=0}$ have been determined at level 0 and 1 respectively, thus the phase-shift function $\Theta(t,x)$ is known.

We have $U_2 = \lambda_2\partial_\theta K + V_2$ and using (3.27), we can expressed V_2 in terms of the still unknown function λ_1.

$$(3.37)\qquad \begin{cases} V_2 = \lambda_1^2 a + \lambda_1 b - Z(\lambda_1)W + H_2 \\ a = -\mathcal{L}^{-1}[\frac{1}{2}\partial_u^3 F(x,K)(\partial_\theta K)^2] = \frac{1}{2}\partial_\theta^2 K \\ b = -\mathcal{L}^{-1}[\partial_u^3 F(x,K)V_1\partial_\theta K + T_\Theta(\partial_\theta^2 K)] \end{cases}$$

where H_2 is independant of λ_1, and known at this level.

The Cauchy data relation (3.12) at level 2 is

(3.38)
$$\begin{cases} U_2(0,x,\Theta_0) = a_2(x) \\ \sigma_0(\partial_\theta U_2)(0,x,\Theta_0) + (\partial_t U_1 + \Theta_t'(0,x)\partial_\theta U_1)(0,x,\Theta_0) = b_2(x) \end{cases}$$

We have $U_1 = \lambda_1\partial_\theta K + V_1$ where V_1 is given by (3.23) and thus is known, and $\lambda_{1|t=0}$ has been determined at level 1 ; we have $U_2 = \lambda_2\partial_\theta K + V_2$, where V_2 is given by (3.37), so $V_2 + 2\sigma_0\partial_t\lambda_1 W_{|t=0}$ is known. From these facts, we conclude that (3.38) is a 2×2 system for the unknown $\lambda_{2|t=0}, \partial_t\lambda_{1|t=0}$, of the form

(3.39)
$$\begin{bmatrix} \partial_\theta K & -2\sigma W \\ \sigma\partial_\theta^2 K & -2\sigma^2\partial_\theta W + \partial_\theta K \end{bmatrix} (t=0, x, \theta=\Theta_0(x)) \begin{bmatrix} \lambda_{2|t=0} \\ \partial_t\lambda_{1|t=0} \end{bmatrix} = \text{given}$$

This is the same system as (3.26) ; the two data $\lambda_{2|t=0}$, $\partial_t\lambda_{1|t=0}$ are thus determined.

Level 3

We have $U_3 = \lambda_3\partial_\theta K + V_3$ with

$$(3.40)\quad \mathcal{L}(V_3) + \frac{1}{6}U_1^3\partial_u^4 F(x,K) + U_1U_2\partial_u^3 F(x,K) + T_\Theta(\partial_\theta U_2) + \Box_\Theta U_1 = 0$$

Using $U_2 = \lambda_2\partial_\theta K + V_2$, we get the integrability condition for (3.40)

(3.41)
$$\begin{cases} \oint Z(\lambda_2)\partial_\theta^2 K\partial_\theta K + \lambda_2[T_\Theta(\partial_\theta^2 K) + U_1(\partial_\theta K)\partial_u^3 F]\partial_\theta K \\ + \oint\{\frac{1}{6}U_1^3\partial_u^4 F(x,K) + U_1V_2\partial_u^3 F(x,K) + T_\Theta(\partial_\theta V_2) + \Box_\Theta U_1\}\partial_\theta K = 0 \end{cases}$$

The first line of (3.41) vanishes [by (3.29) it is equal to $\lambda_2\oint T_\Theta(\partial_\theta^2 K)\partial_\theta K - \mathcal{L}(V_1)\partial_\theta^2 K \equiv 0$, see (3.31)]. Therefore, (3.41) doesn't depend on λ_2, so is an equation for λ_1. By (3.37), this equation has the following structure

(3.42)
$$\begin{cases} \lambda_1^3 \oint [\frac{1}{6}(\partial_\theta K)^3 \partial_u^4 F + a\partial_\theta K \partial_u^3 F]\partial_\theta K \\ +\lambda_1 Z(\lambda_1) \oint 2\partial_\theta a \partial_\theta K - (\partial_\theta K)^2 W \partial_u^3 F \\ +\lambda_1^2 \oint \{\frac{1}{2}(\partial_\theta K)^2 V_1 \partial_u^4 F + (aV_1 + b\partial_\theta K)\partial_u^3 F + T_\ominus(\partial_\theta a)\}\partial_\theta K \\ +\Box\lambda_1 \oint (\partial_\theta K)^2 - Z(Z(\lambda_1)) \oint \partial_\theta W \partial_\theta K \\ +(\text{Linear first order operator})\ (\lambda_1) = \text{ given} \end{cases}$$

In (3.42), the first term vanishes (a is given by (3.37) :

(3.43)
$$\begin{cases} \oint \frac{1}{6}(\partial_\theta K)^4 \partial_u^4 F - a\mathcal{L}(\partial_\theta^2 K) = \oint \frac{1}{6}(\partial_\theta K)^4 \partial_u^4 F + \frac{1}{2}\partial_u^3 F(\partial_\theta^2 K)(\partial_\theta K)^2 \\ = \frac{1}{6} \oint \partial_\theta[(\partial_\theta K)^3 \partial_u^3 F] = 0 \end{cases}$$

The second term in (3.42) vanishes also :

$$(3.44) \quad \begin{cases} \oint 2\partial_\theta a \partial_\theta K - (\partial_\theta K)^2 W \partial_u^3 F = \oint \partial_\theta^3 K \partial_\theta K + W\mathcal{L}(\partial_\theta^2 K) \\ = \oint \partial_\theta^3 K \partial_\theta K + \mathcal{L}(W)\partial_\theta^2 K = \oint \partial_\theta^3 K \partial_\theta K + \partial_\theta^2 K \partial_\theta^2 K = 0 \end{cases}$$

Finally, we get that the third term, in front of λ_1^2 vanishes too ; in fact this term is equal to, using (3.37)

(3.45)
$$* = \oint \frac{1}{2} V_1(\partial_\theta^2 K \partial_u^3 F \partial_\theta K + \partial_u^4 F(\partial_\theta K)^3) - b\mathcal{L}^{-1}(\partial_\theta^2 K) + T_\ominus(\frac{1}{2}\partial_\theta^3 K)\partial_\theta K$$

$$= \oint \frac{1}{2} V_1(3\partial_\theta^2 K \partial_u^3 F \partial_\theta K + \partial_u^4 F(\partial_\theta K)^3) + T_\ominus(\partial_\theta^2 K)\partial_\theta^2 K + T_\ominus(\frac{1}{2}\partial_\theta^3 K)\partial_\theta K$$

¿From $\mathcal{L}(\partial_\theta^2 K) + (\partial_u^3 F)(\partial_\theta K)^2 = 0$, we get

$$(3.46) \qquad \mathcal{L}(\partial_\theta^3 K) + (\partial_u^4 F)(\partial_\theta K)^3 + 3\partial_\theta^2 K \partial_u^3 F \partial_\theta K = 0$$

We have also $\mathcal{L}(V_1) = -T_\ominus(\partial_\theta K)$, so we get

$$* = \oint \frac{1}{2}[T_\ominus(\partial_\theta K)\partial_\theta^3 K + T_\ominus(\partial_\theta^3 K)\partial_\theta K] + T_\ominus(\partial_\theta^2 K)\partial_\theta^2 K$$

$$= \oint \frac{1}{2}[T(\partial_\theta K)\partial_\theta^3 K + T(\partial_\theta^3 K)\partial_\theta K] + T(\partial_\theta^2 K)\partial_\theta^2 K$$

$$= \frac{1}{2} Z \oint [(\partial_\theta K)(\partial_\theta^3 K) + (\partial_\theta^2 K)^2 = 0$$

Therefore (3.41) is a linear hyperbolic equation for λ_1, with principal symbol (3.36) as before. We know that the Cauchy data $\lambda_{1|t=0}$ and $(\partial_t \lambda_1)_{|t=0}$ are determined at level 1 and 2 respectively, so λ_1 is known.

As in level 2, we find that the data equation (3.12) with $k = 3$ gives $\lambda_{3|t=0}$ and $\partial_t \lambda_{2|t=0}$

Level 4

We have $U_4 = \lambda_4 \partial_\theta K + V_4$ with

(3.47)
$$\mathcal{L}(V_4) + (\frac{1}{2}U_2^2 + U_1U_3)\partial_u^3 F(x,K) + \frac{1}{2}U_1^2 U_2 \partial_u^4 F(x,K) + \widetilde{R}_4(x,U_1,U_2) \\ +T_\Theta(\partial_\theta U_3) + \Box_\Theta U_2 = 0$$

Using $U_3 = \lambda_3 \partial_\theta K + V_3$, we get the integrability condition for (3.47)

(3.48)
$$\begin{cases} Z(\lambda_3) \oint \partial_\theta^2 K \partial_\theta K + \lambda_3 \oint U_1 \partial_\theta^2 K \partial_u^3 F + T_\Theta(\partial_\theta^2 K)\partial_\theta K \\ + \oint \left[(\frac{1}{2}U_2^2 + U_1U_3)\partial_u^3 F + \frac{1}{2}U_1^2 U_2 \partial_u^4 F + \widetilde{R}_4 + T_\Theta(\partial_\theta V_3) + \Box_\Theta U_2\right] \partial_\theta K \\ = 0 \end{cases}$$

The first line in (3.48) vanishes, due to

$$(3.49) \qquad \begin{cases} \lambda_3 \oint -\mathcal{L}(U_1)\partial_\theta^2 K + T_\Theta(\partial_\theta^2 K)\partial_\theta K \\ \quad = \lambda_3 \oint T_\Theta(\partial_\theta K)\partial_\theta^2 K + T_\Theta(\partial_\theta^2 K)\partial_\theta K \\ \quad = \lambda_3 Z \oint \partial_\theta K \partial_\theta^2 K = 0 \end{cases}$$

By (3.40), we get

$$(3.50) \qquad V_3 = -Z(\lambda_2)W - \lambda_2 \mathcal{L}^{-1}[U_1 \partial_\theta K \partial_u^3 F + T_\Theta(\partial_\theta^2 K)] + H_3$$

where H_3 is independant of λ_2, and is known at this level so V_3 depends linearly on λ_2, and therefore, (3.48) is a linear equation for λ_2 (we have $\lambda_2^2 \oint (\partial_\theta K)^2 \partial_u^3 F \partial_\theta K = 0$), and the principal part of this equation is still $\Box\lambda_2 \oint (\partial_\theta K)^2 - Z(Z(\lambda_2)) \oint \partial_\theta W \partial_\theta K$, as in (3.35). So we conclude that (3.48) determines λ_2.

As before, the data equation (3.12) with $k = 4$ determines $\lambda_{4|t=0}$ and $\partial_t \lambda_{3|t=0}$.

Level $k \geq 5$

We get for $V_k = U_k - \lambda_k \partial_\theta K$ the equation

$$(3.51) \qquad \mathcal{L}(V_k) + R_k(x, U_0, \cdots, U_{k-1}) + T_\Theta(\partial_\theta U_{k-1}) + \Box_\Theta U_{k-2} = 0$$

We have $U_{k-1} = \lambda_{k-1}\partial_\theta K + V_{k-1}$, and

$$(3.52) \quad V_{k-1} = -Z(\lambda_{k-2})W - \lambda_{k-2}\mathcal{L}^{-1}\left[U_1 \partial_\theta K \partial_u^3 F + T_\Theta(\partial_\theta^2 K)\right] + H_{k-1}$$

where H_{k-1} is independant of λ_{k-2}, and is known at this level. By (3.8), we have

$$R_k = (U_1 U_{k-1} + U_2 U_{k-2})\partial_u^3 F + \frac{1}{2}U_1^2 U_{k-2}\partial_u^4 F + \widetilde{R}_k(x, U_1, \cdots, U_{k-3}).$$

So, as in level 4, we find that the integrability condition for (3.51), is independant of λ_{k-1}, and gives a linear second order equation for λ_{k-2}

with principal symbol (3.36). Using also (3.12), we conclude that this level determines

$$\lambda_{k-2}\ ,\ \lambda_{k|t=0}\ ,\ \partial_t\lambda_{k-1}|_{t=0} \tag{3.53}$$

and gives also the relation

$$V_k = -Z(\lambda_{k-1})W - \lambda_{k-1}\mathcal{L}^{-1}\left[U_1\partial_\theta K\partial_u^3 F + T_\Theta(\partial_\theta^2 K)\right] + H_k \tag{3.54}$$

where H_k is known.

¿From all this fact, we obtain

Proposition 3.1. *The formal Cauchy problem given by the hierarchy $(E_k)_{k\geq 0}$ and Cauchy data (3.9) admits a unique formal solution of the form (3.3), (3.4).*

4. Linearisation and the Hill equation

Let $\underline{g} = (g_0, g_1)$ be a couple of Cauchy data for the non-linear equation (1.4), with asymptotic developpements

$$\begin{cases} g_0 \sim \rho^\gamma \sum_0^\infty c_k\rho^{k\beta}\ ;\ g_1 \sim \rho^{\gamma-1-\beta}\sum_0^\infty d_k\rho^{k\beta} \\ (c_0, d_0) \neq (0,0) \end{cases} \tag{4.1}$$

Let us denote by

$$u_h(t,x) = \sum_{k=0}^\infty h^k U_k(t,x,\frac{\varphi(t,x)}{h} + \Theta(t,x)) \tag{4.2}$$

the formal optical solution of the Cauchy problem near $(t,x) = (0,1)$ (see 2.1, 2.2 and section 3)

$$\begin{cases} h^2(\partial_t^2 - \partial_x^2)u + \frac{u^p}{x^{p-1}} = 0 \\ u(0,x) \sim \hbar^{-\gamma}g_0(\hbar x) \quad (h = \hbar^\beta) \\ h\partial_t u(0,x) \sim \hbar^{-\gamma+1+\beta}g_1(\hbar x) \end{cases} \tag{4.3}$$

The homogeneity of the problem (4.3) shows that $u_h(t,x)$ must be of the form

$$u_h(t,x) = \hbar^{-\gamma}w(\hbar t, \hbar x) \tag{4.4}$$

Moreover, the phase function $\varphi(t,x)$ is odd in time, and homogeneous of degree $-\beta$, and the phase-shift $\Theta(t,x)$ is homogeneous of degree 0

$$\begin{cases} \varphi(t,x) = x^{-\beta}\psi(t/x)\ ,\ \psi \text{ odd, } C^\infty \\ \Theta(t,x) = A(t/x)\ ,\ A \in C^\infty \end{cases} \tag{4.5}$$

Let $K(\sigma,x,\theta)$ be the (2π) periodic solution of $\sigma^2\partial_\theta^2 K + \frac{K^p}{x^{p-1}} = 0$ introduce in definition (2.1). We have

$$K(\sigma,x,\theta) = x\sigma^{\frac{2}{p-1}}G(\theta) \tag{4.6}$$

where $G(\theta)$ is the unique 2π-periodic solution of

$$G'' + G^p = 0 \ ; \ G'(0) = 0 \ , \ G(0) > 0 \tag{4.7}$$

The two constants $\psi'(0) > 0$ and $A(0)$ are given by

$$\begin{cases} (\psi'(0))^{\frac{2}{p-1}} G(A(0)) = c_0 \\ (\psi'(0))^{\frac{p+1}{p-1}} G'(A(0)) = d_0 \end{cases} \tag{4.8}$$

By (4.4), the asymptotic structure of u_h is of the form

$$u_h(t,x) = x^\gamma \sum_{k=0}^{\infty} (\hbar x)^{\beta k} F_k\left(\frac{t}{x}, \frac{\psi(t/x)}{(\hbar x)^\beta} + A(t/x)\right) \tag{4.9}$$

and we have $x^\gamma F_0 = x(\varphi_t'^2 - \varphi_x'^2)^{\frac{1}{p-1}} G(\varphi/h + \Theta)$.

By the Borel sommation lemma, there is a smooth function $F(\varepsilon, u, \theta)$, defined for $\varepsilon \in [0,1], u \sim 0, \theta \in \mathbb{S}^1$, such that

$$F(\varepsilon, u, \theta) \sim \sum_0^\infty \varepsilon^k F_k(u,\theta) \quad (\varepsilon \to 0) \tag{4.10}$$

Let us define the function $f_h^{opt}(t,x)$ by

$$f_h^{opt}(t,x) = x^\gamma F\left((\hbar x)^\beta, t/x, \frac{\psi(t/x)}{(\hbar x)^\beta} + A(t/x)\right) \tag{4.11}$$

Then, by construction, this function satisfies the Cauchy problem with flat right hand side

$$\begin{cases} h^2(\partial_t^2 - \partial_x^2) f_h^{opt} + \frac{(f_h^{opt})^p}{x^{p-1}} = \hbar^{2+2\beta-\gamma} R(\hbar x, t/x) \\ f_h^{opt}(0,x) - \hbar^{-\gamma} g_0(\hbar x) = \hbar^{-\gamma} r_0(\hbar x) \\ h\partial_t f_h^{opt}(0,x) - \hbar^{-\gamma+1+\beta} g_1(\hbar x) = \hbar^{-\gamma+1+\beta} r_1(\hbar x) \end{cases} \tag{4.12}$$

where the functions $R(\rho, u), r_{0,1}(\rho)$ are smooth in $\rho \in [0,1]$, u closed to 0, and flat at $\rho = 0$, with all their derivatives.

Let $\underline{w}(\rho) = (w_0(\rho), w_1(\rho))$ be a couple of Cauchy data, flat at $\rho = 0$ with all their derivates. Let $v_h(t,x)$ be the solution of the Cauchy problem near $t = 0, x = 1$, defined for $h \in]0, h_0], h_0$ small

$$\begin{cases} h^2(\partial_t^2 - \partial_x^2) v_h + \frac{(v_h)^p}{x^{p-1}} = 0 \\ v_h(0,x) = \hbar^{-\gamma}[g_0(\hbar x) + w_0(\hbar x)] \\ h\partial_t v_h(0,x) = \hbar^{-\gamma+1+\beta}[g_1(\hbar x) + w_1(\hbar x)] \end{cases} \tag{4.13}$$

Let $w_h(t,x) = v_h(t,x) - f_h^{opt}(t,x)$. Then, $w_h(t,x)$ satisfies the Cauchy problem, with $\sigma = (\varphi_t'^2 - \varphi_x'^2)^{1/2}$

$$(4.14) \quad \begin{cases} h^2(\partial_t^2 - \partial_x^2)w_h + p\sigma^2 G^{p-1}(\varphi/h + \Theta)w_h \\ \quad +\mathcal{N}(h,t,x,\frac{\varphi}{h} + \Theta; w_h) = -\hbar^{2+2\beta-\gamma}R(\hbar x, t/x) \\ w_h(0,x) = \hbar^{-\gamma}[w_0(\hbar x) - r_0(\hbar x)] \\ h\partial_t w_h(0,x) = \hbar^{-\gamma+1+\beta}[w_1(\hbar x) - r_1(\hbar x)] \end{cases}$$

where the perturbation $\mathcal{N}$ is defined by

$$(4.15) \quad \mathcal{N} = \frac{pw_h}{x^{p-1}}((f_h^{opt})^{p-1} - (x^\gamma F_0)^{p-1}) + \sum_{k=2}^{p} \frac{w_h^k}{x^{p-1}} \frac{p!}{k!(p-k)!}(f_h^{opt})^{p-k}$$

In order to study the Cauchy problem (4.14), we make the change of variables

$$(4.16) \quad \begin{cases} t' = \varphi(t,x) \\ x' = a(t,x) \\ \varphi_t' \frac{\partial a}{\partial t} - \varphi_x' \frac{\partial a}{\partial x} = 0 \ ; \ a(0,x) = x^{-\beta} \end{cases}$$

The functions φ, a are both homogeneous of degree $-\beta$ in (t,x), φ is odd in t, a even in t, and $(t,x) \mapsto (t',x')$ is a diffeomorphism near $(0,1)$. Let σ, q be the two functions

$$(4.17) \quad \sigma = (\varphi_t'^2 - \varphi_x'^2)^{1/2} \quad q = (a_x'^2 - a_t'^2)^{1/2}$$

The wave operator $\Box = \partial_t^2 - \partial_x^2$ in the new coordinates (t', x') is given by

$$(4.18) \quad \Box = \sigma q \partial_{t'}(\frac{\sigma}{q}\partial_{t'}) - \sigma q \partial_{x'}(\frac{q}{\sigma}\partial_{x'})$$

Let $b = -\frac{1}{2}\log(\sigma/q)$. Then we get

$$(4.19) \quad e^{-b}\Box e^b = \sigma^2 \left[\partial_{t'}^2 - \frac{q^2}{\sigma^2}\partial_{x'}^2 + q_1(t',x')\partial_{x'} + q_2(t',x')\right]$$

where the q_j are homogeneous of degree $-j$ in (t',x'). Let B be the function defined by

$$(4.20) \quad B(t'/x') = A(t/x)$$

We introduce the unknown $\tilde{w}_h(t',x') = e^{-b}w_h(t,x)$; on the set $t = 0$, we have $\hbar x = (\frac{h}{x'})^{1/\beta}$, and the Cauchy problem (4.14) for $\tilde{w}_h$ is of the form (we divide by σ^2 the equation)

$$(4.21)\quad \begin{cases} \left[h^2(\partial_{t'}^2 - \frac{q^2}{\sigma^2}\partial_{x'}^2) + pG^{p-1}(\frac{t'}{h} + B(t'/x'))\right] \tilde{w}_h + Q(\tilde{w}_h) \\ +N(\widetilde{w}_h) = \hbar^{2+2\beta-\gamma}\widetilde{R}(h/x', \frac{t'}{x'}) \\ \tilde{w}_h(0,x) = \hbar^{-\gamma}\left[\tilde{w}_0(h/x') - \tilde{r}_0(h/x')\right] \\ h\partial_t\tilde{w}_h(0,x) = \hbar^{-\gamma+1+\beta}\left[\tilde{w}_1(h/x') - \tilde{r}_1(h/x')\right] \end{cases}$$

The data $\tilde{w}_j(u), \tilde{r}_j(u)$, and the right hand side $\widetilde{R}(u, t'/x')$ are flat at $u = 0$. Q is the linear operator

$$(4.22)\qquad Q = hq_1(h\partial_{x'}) + h^2 q_2$$

and the perturbation N is deduced from $\mathcal{N}$ by the rule

$$(4.23)\qquad N(\tilde{w}_h) = \frac{e^{-b}}{\sigma^2}\mathcal{N}(\bullet; e^b\tilde{w}_h)$$

The function $\frac{q^2}{\sigma^2}$ is even and homogeneous of degree 0, so we can write

$$(4.24)\qquad \begin{cases} \frac{q^2}{\sigma^2} = \alpha(t'/x') = \alpha_0 + \alpha_2(t'/x')^2 + \cdots, \alpha_0 > 0 \\ B(t'/x') = b_0 + b_1(t'/x') + \cdots \end{cases}$$

We shall study the Cauchy problem (4.21) for small values of t', of the form

$$(4.25)\qquad t' = hs \quad 0 \le s \le s(h)$$

We choose $s(h)$ such that (the constant μ_0 will be defined in (4.40))

$$(4.26)\qquad \begin{cases} M|\log h| \ll s(h) \ll h^{-\nu} (\forall M > 0, \forall \nu > 0) \text{ when } h \to 0 \\ e^{\mu_0 s(h)}(\widetilde{R}, \tilde{r}_j) \text{ remain flat with all their derivatives when } h \to 0 \end{cases}$$

In this range of values of t', we can expect that the behavior of (4.21) is governed by the linear equation

$$(4.27)\qquad h^2(\partial_{t'}^2 - \alpha_0\partial_{x'}^2) + pG^{p-1}(t'/h + b_0)$$

Taking the Fourier transform $\frac{h}{i}\partial_{x'} = \eta$, we are thus lead to study the Hill equation on the line, with $t' = hs$

$$(4.28)\qquad \partial_s^2 + pG^{p-1}(s + b_0) + \lambda\ ;\ \lambda = \alpha_0\eta^2$$

We shall denote by $\mathbb{M}(\lambda)$ the 2×2 matrix defined by

$$(4.29)\qquad \begin{pmatrix} f(2\pi) \\ f'(2\pi) \end{pmatrix} = \mathbb{M}(\lambda)\begin{pmatrix} f(0) \\ f'(0) \end{pmatrix}\ ;\ f'' + pG^{p-1}(s + b_0)f + \lambda f = 0$$

We have $\det \mathbb{M}(\lambda) = 1$, and the eigenvalues of $\mathbb{M}(\lambda)$ are real iff $|tr(\mathbb{M}(\lambda))| \ge 2$.

Proposition 4.1. i) *For any λ, one has $tr(\mathbb{M}(\lambda)) \ge -2$*

ii) *The instability set, $I = \{tr(\mathbb{M}(\lambda)) \geq 2\}$, is of the form $I = \cup_{k\geq -1} I_k$ with $I_{-1} =]-\infty, \lambda_{-1}]$, $I_0 = [\lambda_0, 0]$, $I_k = [\lambda_k, \lambda'_k]$ for $k \geq 1$, with*

$$\lambda_{-1} < \lambda_0 < 0 < \lambda_1 \leq \lambda'_1 < \lambda_2 \leq \lambda'_2 \cdots \tag{4.30}$$

iii) *There exist $k \geq 1$ such that $\lambda_k \neq \lambda'_k$*

iv) $\overline{lim}_{\lambda\to\infty} tr(\mathbb{M}(\lambda)) = 2$

Proof We refer to [MK-M] for results and references on the Hill equation. First of all, iv) is a general property for the Hill equation.

i) The solution G of (4.7) satisfies $G(\theta + \pi) = -G(\theta)$ so $s \mapsto G^{p-1}(s + b_0)$ is π-periodic. We thus have $\mathbb{M}(\lambda) = \mathbb{N}(\lambda)^2$ with $\mathbb{N}(\lambda) \in \mathcal{M}_{2,2}(\mathbb{R}), \det \mathbb{N}(\lambda) = 1$ ($\mathbb{N}(\lambda)$ is defined by (4.29) at time π). If $\mathbb{N} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get $Tr(\mathbb{M}) = a^2 + 2bc + d^2 = (a+d)^2 - 2 \geq -2$.

ii) Let $V(\theta)$ be a 2π-periodic potential solution of $V'' + F'(V) = 0$; consider the Hill equation $f'' + F''(V)f + \lambda f$. Let $E_0 = \frac{1}{2}V'^2 + F(V)$, $\Pi(E) = \sqrt{2}\int_{u_-}^{u_+} \frac{du}{(E-F(u))^{1/2}}$, and let us assume that F satisfies the hypothesis (H.1) and (H.2) of §2. By lemma (2.1), we know that the kernel of $d_\theta^2 + F''(V)$ acting on 2π periodic functions is one dimensional and span by $\frac{\partial V}{\partial \theta}$. For α small and real $(d_\theta + i\alpha)^2 + F''(V)$ acting on 2π-periodic functions has thus a unique real small eigenvalue $-\lambda(\alpha)$, with a one dimensional eigenspace span by $f(\alpha, \theta)$, with analytic dependance on α. Let us introduce the taylor expansions $\lambda(\alpha) = \mu_1\alpha + \mu_2\alpha^2 + \cdots$ and $f(\alpha, \theta) = \frac{\partial V}{\partial \theta} + \alpha g_1(\theta) + \alpha^2 g_2(\theta) + \cdots$

One easely gets the equation

$$\begin{cases} [\partial_\theta^2 + F''(V)]\, g_1 + (2i\partial_\theta + \mu_1)\frac{\partial V}{\partial \theta} = 0 \\ [\partial_\theta^2 + F''(V)]\, g_2 + (2i\partial_\theta + \mu_1)g_1 + (\mu_2 - 1)\frac{\partial V}{\partial \theta} = 0 \end{cases} \tag{4.31}$$

Let W be the solution of $[\partial_\theta^2 + F''(V)]W = \partial_\theta^2 V$, orthogonal to $\frac{\partial V}{\partial \theta}$. From the first equation in (4.31) we get $\mu_1 = 0$ and $g_1 = C^{te}\frac{\partial V}{\partial \theta} - 2iW$, so the integrability condition for the 2^{de} equation gives

$$(\mu_2 - 1)\oint \left|\frac{\partial V}{\partial \theta}\right|^2 + 4\oint \frac{\partial V}{\partial \theta}\frac{\partial W}{\partial \theta} = 0 \tag{4.32}$$

¿From the fact that $e^{2i\pi\alpha}$ is an eigenvalue of $\mathbb{M}(\lambda(\alpha))$ we get
(4.33)

$$Tr(\mathbb{M})(\lambda(\alpha)) = 2\cos(2\pi\alpha) = 2 - 4\alpha^2\pi^2 + \cdots = 2 - 4\frac{\lambda(\alpha)}{\mu_2}\pi^2 + \cdots$$

$$(4.34) \qquad \frac{d}{d\lambda} Tr(\mathbb{M}(\lambda))|_{\lambda=0} = \frac{-4\pi^2}{\mu_2}$$

In our case, $F(V) = \frac{V^{p+1}}{p+1}$, $V'' + V^p = 0$, so we have $W = \frac{-1}{p-1}V$, and therefore $\mu_2 > 0$. This implies

$$(4.35) \qquad Tr(\mathbb{M}(0)) = 2 \quad \frac{d}{d\lambda} Tr(\mathbb{M}(0)) < 0$$

We now use a deformation argument to achieve the verification of ii). For $\beta \in [0,1]$, let $F_\beta(u) = \beta\frac{u^2}{2} + (1-\beta)\frac{u^{p+1}}{p+1}$, and V_β, be the 2π periodic solution of $V_\beta'' + F'(V_\beta) = 0$ with $V_\beta(0) > 0$, $V_\beta'(0) = 0$. We have $V_0 = G$, $V_1 = \cos(\theta)$

$Tr(\mathbb{M}_0(\lambda)) = Tr(\mathbb{M}(\lambda))$,

$Tr(\mathbb{M}_\beta(0)) = 2$,

$Tr(\mathbb{M}_1(\lambda)) = 2\cos[(\sqrt{1+\lambda})2\pi]$.

As before, we have $V_\beta(\theta + \pi) = -V_\beta(\pi)$, so $F''(V_\beta)$ is π-periodic and all the roots of $Tr(\mathbb{M}_\beta(\lambda)) = -2$ are double.

Let $\lambda_{-1}(\beta), \lambda_0(\beta), \lambda_0'(\beta)$, ... be the roots of $Tr(\mathbb{M}_\beta(\lambda)) = 2$ and $\mu_0(\beta), \mu_1(\beta)$, ... the double roots of $Tr(\mathbb{M}_\beta(\lambda)) = -2$ we have

$$\lambda_{-1}(\beta) < \mu_0(\beta) < \lambda_0(\beta) \leq \lambda_0'(\beta) < \mu_1(\beta) < \lambda_1(\beta) \leq \lambda_1'(\beta) < \mu_2(\beta) \cdots$$

Since 0 belongs to the sequence $\{\lambda_k(\beta), \lambda_k'(\beta)\}$, and $\lambda_{-1}(1) = -1$, $\lambda_0(1) = \lambda_0'(1) = 0$, we get $\mu_0(\beta) < 0$ and $0 \in \{\lambda_0(\beta), \lambda_0'(\beta)\}$ by continuity; therefore ii) is a consequence of (4.35)

iii) By ii) the first instability interval $[\lambda_0, 0]$ is open, and it remains to show that there exist another open instabillity interval. For an Hill equation

$$f'' + V(\theta)f$$

with a potential V 2π-periodic, we know that all the instabillity interval are closed iff $V = C^{te}$, and that there exist only one open instabillity intervall iff V is an elliptic function ; in that case V must satisfied a Weierstrass equation

$$(4.36) \qquad V'' = aV^2 + bV + c \qquad a, b, c \text{ constants}$$

Here $V = pG^{p-1}, p \geq 7$, with $G'' + G^p = 0$, so one verifies that (4.36) doesn't hold true.

We are now ready to study the linear principal part of the non linear hyperbolic equation (4.21).

Take $k \in \mathbb{N}$, $k \in [0, s(h)]$ with $s(h)$ as in (4.26), and let us introduce the Cauchy problem

$$(4.37)\qquad \begin{cases} \partial_s^2 f - h^2\alpha\left(\frac{h(2\pi k+s)}{x'}\right)\partial_{x'}^2 f + pG^{p-1}(s+b_0)f = 0 \\ f(0,x') = f_0(x')\ ;\ \partial_s f(0,x') = f_1(x') \end{cases}$$

Let $Z(k)$ be the solution map at time 2π for (4.37)

$$(4.38)\qquad \begin{bmatrix} f(2\pi, x') \\ \partial_s f(2\pi, x') \end{bmatrix} = Z(k) \begin{bmatrix} f(0,x') \\ \partial_s f(0,x') \end{bmatrix}$$

Let us denote by Z_0 the same object when we replace $\alpha(\frac{h(2\pi k+s)}{x'})$ by α_0 in (4.37).

Then, by the definition (4.29) of $\mathbb{M}$ we have

$$(4.39)\qquad Z_0\begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \frac{1}{2\pi h}\int e^{ix'\eta/h}\mathbb{M}(\alpha_0\eta^2)\begin{bmatrix} \widehat{f}_0(\eta/h) \\ \widehat{f}_1(\eta/h) \end{bmatrix} d\eta$$

where $\hat{f}(\xi) = \int e^{-ix'\xi} f(x')dx'$

Using prop 4.1 iii), we define a constant $\mu_0 > 0$ by

$$(4.40)\qquad e^{2\pi\mu_0} = \sup_{\lambda\geq 0}\{\text{ real eigenvalues of } \mathbb{M}(\lambda)\} > 1$$

We can then select a smooth hermitian matrix $\eta \mapsto Q(\eta)$, such that if we denote by $\mathbb{C}_\eta^2$ the $\mathbb{C}^2$ plane equipped with the norm $\|z\|_\eta^2 \doteq {}^t\bar{z}Q(\eta)z$, then the following holds true

$$(4.41)\qquad \sup_{\eta\in\mathbb{R}} \|\mathbb{M}(\alpha_0\eta^2); \mathbb{C}_\eta^2\| = e^{2\pi\mu_0}$$

$$(4.42)\qquad \lim_{|\eta|\to\infty} \|\mathbb{M}(\alpha_0\eta^2)\ ;\ \mathbb{C}_\eta^2\| = 1$$

$$(4.43)\qquad Q(\eta) = \begin{pmatrix} \alpha_0\eta^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } \eta \text{ large enough}$$

We shall denote by $\mathcal{H}(=\mathcal{H}_h)$ the space $H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$ equipped with the norm

$$(4.44)\qquad \|\underline{f} = (f_0, f_1)\|_{\mathcal{H}}^2 = \frac{1}{2\pi h}\int {}^t\,\overline{\hat{\underline{f}}}(\eta/h)Q(\eta)\hat{\underline{f}}(\eta/h)d\eta$$

Notice that (4.43) implies that $\|\underline{f}\|_{\mathcal{H}}^2$ is uniformly equivalent with respect to h to the usual norm :

(4.45)

$$\exists c_0, \forall h \in]0,1]\ ,\ c_0\|\underline{f}\|_{\mathcal{H}}^2 \leq \int_{\mathbb{R}} |f_0|^2 + |h\partial_{x'}f_0|^2 + |f_1|^2 dx' \leq \frac{1}{c_0}\|\underline{f}\|_{\mathcal{H}}^2$$

The interest of this special choice of Hilbert structure on the set of Cauchy data is that we have now

$$(4.46)\qquad \forall h \in]0,1]\quad \|Z_0; \mathcal{H}\| = e^{2\pi\mu_0}$$

Proposition 4.2. *There exist γ_0, c_1 such that*

1) For $r > 0$ small enough and $\underline{f} \in \mathcal{H}$ with support in $|x' - 1| \leq r$, then $Z(k)\underline{f}$ belongs to $\mathcal{H}$, has support in $|x' - 1| \leq r + 2\pi\gamma_0 h$, and the following holds true

$$\|Z(k)\underline{f}\|_{\mathcal{H}} \leq (e^{2\pi\mu_0} + c_1 h^{3/4})\|\underline{f}\|_{\mathcal{H}} \tag{4.47}$$

2) There exist a bounded operator on $\mathcal{H}$, $S(k)$, such that

$$\partial_{x'} \circ Z(k) = (Z(k) + h^{7/4}S(k)) \circ \partial_{x'} \tag{4.48}$$

with for every $k \in [0, s(h)], h \in]0, 1]$

$$\|S(k); \mathcal{H}\| \leq c_1 \tag{4.49}$$

Proof Let $V(s) = pG^{p-1}(s + b_0)$; we shall consider $\beta = (h, hk)$ as a small parameter in the function $\alpha = \alpha(\frac{2\pi hk+hs}{x'})$; then $(s, x') \mapsto \alpha$ is bounded with all its derivatives uniformly with respect to β and satisfies

$$\forall\alpha, \exists C_\alpha \quad |\partial^\alpha_{s,x'}(\alpha - \alpha_0)| \leq C_\alpha h^{9/10} \quad \forall\beta \tag{4.50}$$

Let us define $E(f, s)$ by

$$E(f, s) = \int \frac{1}{2}|\partial_s f|^2 + \frac{h^2}{2}\alpha|\partial_{x'} f|^2 + |f|^2 dx'$$

The classical energy inequality applied to the Cauchy problem

$$\partial_s^2 f - h^2\alpha\partial_{x'}^2 f + V(s)f = g \;;\; f|_{s=0} = f_0, \partial_s f|_{s=s_0} = f_1 \tag{4.51}$$

gives

$$\left\{\begin{array}{l} \exists C_0, \forall s \in [0, 2\pi], \forall\beta \\ E^{1/2}(f, s) \leq \{E^{1/2}(f, 0) + C_0\int_0^s \|g(s', .)\|_{L^2} ds'\}e^{C_0 s} \end{array}\right. \tag{4.52}$$

This implies, together with the finite speed of propagation for the strictly hyperbolic equation (4.51), that if the data $\underline{f} \equiv (f_0, f_1)$ are in $\mathcal{H}$ and are supported in $|x' - 1| \leq r$, r small, then $(f(s), \partial_s f(s))$ is supported in $|x' - 1| \leq r + \gamma_0 hs$, belongs to $\mathcal{H}$ and we have

$$\exists C_0 \quad \|\underline{f}(s)\|_{\mathcal{H}} \leq C_0\|\underline{f}(0)\|_{\mathcal{H}} \quad \forall\beta, \forall s \in [0, 2\pi] \tag{4.53}$$

Moreover, if f satisfies (4.51), $\partial_{x'} f$ satisfies

$$\left[\partial_s^2 - h^2\alpha\partial_{x'}^2 + V(s) - h\frac{\partial\alpha}{\partial x'}(h\partial_{x'})\right]\partial_{x'} f = \partial_{x'} g \tag{4.54}$$

By (4.50), for $\underline{u} = (u_0, u_1) \in \mathcal{H}$, we have

$$\|h\frac{\partial\alpha}{\partial x'}h\partial_{x'}u_0\|_{L^2(x')} \leq C^{te}h^{7/4}\|\underline{u}\|_{\mathcal{H}};$$

therefore, (4.48) and (4.49) follow from (4.52), (4.53) and Duhamel formula. It remains now to verify (4.47). Let us define the 2×2 matrix kernel $\mathcal{M} = \mathcal{M}(h, \beta, s, x', \eta)$, as the solution of

$$(4.55) \qquad \left[\partial_s + \begin{pmatrix} 0 & -1 \\ -\alpha h^2 \partial_{x'}^2 + V & 0 \end{pmatrix}\right] e^{ix'\eta/h} \mathcal{M} = 0 \ , \ \mathcal{M}|_{s=0} = Id$$

By finite speed of propagation, there exist a fixed neighborhood W of $\{x' = 1, s \in [0, 2\pi]\}$ such that $\mathcal{M}$ is well defined for $(x', s) \in W$, h, β smalls and every $\eta \in \mathbb{R}$; If the data $\underline{f} = (f_0, f_1)$ are supported in $|x' - 1| \leq r$, r small, the solution $\underline{f}(s) = (f, \partial_s f)$ of (4.51) with right hand side $g = 0$ will be given by the formula

$$(4.56) \qquad \underline{f}(s) = \frac{1}{2\pi h} \int e^{ix'\eta/h} \mathcal{M} \hat{\underline{f}}(\eta/h) d\eta$$

We estimate $\mathcal{M}$ separatly in the two cases $|\eta| \leq \Lambda$, $|\eta| \geq \Lambda$, where Λ is a large constant. To do so, we first introduce asymptotics.

1) $|\eta| \leq \Lambda$. In that case, we take $h \to 0$ as small parameter.

Let $\mathcal{L} = \begin{pmatrix} 0 & -1 \\ \alpha\eta^2 + V & 0 \end{pmatrix}$, and let $\sum_{j \geq 0} h^j \mathcal{M}_j^1(\beta, s, x', \eta)$ be the solution of

(4.57)

$$\begin{cases} (\partial_s + \mathcal{L}) \mathcal{M}_j^1 - 2i\alpha\eta \begin{pmatrix} 0 & 0 \\ \partial_{x'} & 0 \end{pmatrix} \mathcal{M}_{j-1}^1 - \alpha \begin{pmatrix} 0 & 0 \\ \partial_{x'}^2 & 0 \end{pmatrix} \mathcal{M}_{j-2}^1 = 0 \\ \mathcal{M}_{0|s=0}^1 = Id; \mathcal{M}_j^1|_{s=0} = 0 \text{ for } j \geq 1 . \end{cases}$$

Then we have

$$(4.58) \qquad \forall \alpha, j \quad \exists C = C(\Lambda, \alpha, j) \quad |\partial_{s,x'}^\alpha \mathcal{M}_j^1| \leq C \quad \forall \beta, \quad \forall \eta, |\eta| \leq \Lambda$$

By the Borel Lemma, there exist $\mathcal{M}^1 \simeq \Sigma_j h^j \mathcal{M}_j^1$ and we have by (4.50)

(4.59)

$$\forall \alpha, \exists C = C(\Lambda, \alpha) |\partial_{x'}^\alpha (\mathcal{M}^1|_{s=2\pi} - \mathbb{M}(\alpha_0 \eta^2))| \leq C h^{9/10} \quad \forall \beta, \forall \eta, |\eta| \leq \Lambda$$

We can choose $\mathcal{M}^1$ of the form

$$\mathcal{M}^1 = \begin{pmatrix} A^1 & B^1 \\ \partial_s A^1 & \partial_s B^1 \end{pmatrix}$$

2) $|\eta| \geq \Lambda$

In that case, we take $\eta \to \infty$ as great parameter. (the negative values of η can be treated in the same way). We then use the classical optical construction of solutions of the form $e^{i\varphi_\pm} \sigma_\pm$ to the equation $\partial_s^2 - h^2 \alpha \partial_{x'}^2 + V(s) = 0$. The phase function $\varphi_\pm$ are given by

$$(4.60) \qquad \varphi_\pm = \frac{x'\eta}{h} + \psi_\pm(\beta, s, x')\eta$$

where $\psi_\pm$ satisfies the eikonal equation

$$(4.61) \qquad \frac{\partial}{\partial s} \psi_\pm = \pm\sqrt{\alpha}(1 + h\partial_{x'}\psi_\pm) \ ; \ \psi_{\pm|s=0} = 0$$

and the symbols $\sigma_\pm \simeq \sum_{j\geq 0} \eta^{-j}\sigma_{\pm,j}(\beta, s, x')$ satisfy the transport equation

(4.62)

$$\begin{cases} T_\pm(\sigma_{\pm,0}) = 0 \ ; \ T_\pm(\sigma_{\pm,j+1}) + P(\sigma_{\pm,j}) = 0 \quad j \geq 0 \\ T_\pm = i\,[2(\partial_s\psi_\pm)\partial_s + \partial_s^2\psi_\pm - 2\alpha h((1 + h\partial_{x'}\psi_\pm)\partial_{x'} + h\partial_{x'}^2\psi_\pm)] \\ P = \partial_s^2 - h^2\alpha\partial_{x'}^2 + V(s) \end{cases}$$

Then for any data $\sigma_{\pm|s=0} = \sigma_\pm^0$, (4.62) admits a unique solution in the space of symbols in η, (with β as parameter). Let $a_\pm, b_\pm$ the symbols solution of (4.62) with data

(4.63)

$$\begin{cases} (a_+ + a_-)_{|s=0} = 1 \ , \ (i\eta(\partial_s\psi_+ a_+ + \partial_s\psi_- a_-) + \partial_s a_+ + \partial_s a_-)_{|s=0} = 0 \\ (b_+ + b_-)_{|s=0} = 0 \ , \ (i\eta(\partial_s\psi_+ b_+ + \partial_s\psi_- b_-) + \partial_s b_+ + \partial_s b_-)_{|s=0} = 1 \end{cases}$$

[We have $\partial_s\psi_{\pm|s=0} = \pm\sqrt{\alpha}$, so (4.63) is solvable]

By the Borel lemma, there exist functions of η (and β, s, x') such that $a_\pm^2 \simeq a_\pm, b_\pm^2 \simeq b_\pm (\eta \to \infty)$ and we define $\mathcal{M}^2$ by

$$\mathcal{M}^2 = \begin{pmatrix} A^2 & B^2 \\ \partial_s A^2 & \partial_s B^2 \end{pmatrix} \tag{4.64}$$

with $A^2 = e^{i\eta\psi_+}a_+^2 + e^{i\eta\psi_-}a_-^2$, $B^2 = e^{i\eta\psi_+}b_+^2 + e^{i\eta\psi_-}b_-^2$.

We have now to estimate the error terms $\mathcal{M} - \mathcal{M}^1, \mathcal{M} - \mathcal{M}^2$. The matrix $\mathcal{M}$ is of the form

$$\mathcal{M} = \begin{pmatrix} A & B \\ \partial_s A & \partial_s B \end{pmatrix}$$

where A, B are solutions of $P_\eta A = P_\eta B = 0$, with

$$\partial_s^2 - \alpha(h\partial_{x'} + i\eta)^2 + V(s) = P_\eta \tag{4.65}$$

We choose the associated energy

$$\mathcal{E}_\eta(f, s) = \int \frac{1}{2}|\partial_s f|^2 + \frac{\alpha}{2}|(h\partial_{x'} + i\eta)f|^2 + |f|^2$$

Then, for the Cauchy problem

$$P_\eta f = g \quad f_{|s=0} = f_0, \partial_s f_{|s=0} = f_1$$

we still have the energy estimate, uniform in η

$$\exists C_0 \quad \mathcal{E}_\eta^{1/2}(f, s) \leq e^{C_0 s}\{\mathcal{E}_\eta^{1/2}(f, 0) + C_0 \int_0^s \|g(s', .)\|_{L^2} ds'\} \tag{4.66}$$

1.) For $|\eta| \leq \Lambda$, the function $f = A - A^1$ is such $f_{|s=0}, \partial_s f_{|s=0}$, $P_\eta f$ are $\mathcal{O}(h^\infty)$ with all their derivatives with respect to (s, x'), uniformly in η, β.

Using commutators of P_η with polynomials in $(h\partial_{x'} + i\eta)$ and η, (4.66), and the Sobolev imbedding $H_{x'}^1 \hookrightarrow L^\infty$, we get

$$\forall\alpha \quad |\partial_{s,x'}^\alpha A - A^1|_{L^\infty} \in \mathcal{O}(h^\infty) \text{ uniformly in } \beta, \eta, |\eta| \leq \Lambda \tag{4.67}$$

2.) For $|\eta| \geq \Lambda$, the function $f = A - A^2$ is such that $f_{|s=0}, \partial_s f_{|s=0}, P_\eta f$ are $\mathcal{O}(\eta^{-\infty})$ with all their derivatives with respect to η, s, x', uniformly in β. Let us prove by induction on k that we have

$$(4.68) \qquad (1, \partial_s, h\partial_{x'})(\partial_{x'}^k f) \text{ are } \mathcal{O}(\eta^{-\infty}) \text{ in } L^\infty(s, L^2(x'))$$

This is true by the energy inequality (4.66) for $k = 0$. We have $[P_\eta, \partial_{x'}^{k+1}] = \sum_{j=0}^{k} g_j(\beta, s, x')(h\partial_{x'} + i\eta)^2 \partial_{x'}^j$ where the g_j are smooth uniformly in β, so we get

(4.69)

$$\begin{cases} P_\eta[\partial_{x'}^{k+1} f] - hg_k(h\partial_{x'} + i\eta)[\partial_{x'}^{k+1} f] \\ \quad = i\eta g_k(h\partial_{x'} + i\eta)\partial_{x'}^k f + \sum_{j<k} g_j(h\partial_{x'} + i\eta)^2 \partial_{x'}^j f + \partial_{x'}^{k+1}[P_\eta f] \end{cases}$$

By induction, the right hand side of (4.69) is $\mathcal{O}(\eta^{-\infty})$ in $L^\infty(s, L^2(x'))$. Applying the energy inequality to the operator $P_\eta - hg_k(h\partial_{x'} + i\eta)$, we get (4.68) at step $k+1$.

Taking derivatives with respect to s, we thus get

$$(4.70) \qquad \forall \alpha, |\partial_{s,x'}^\alpha A - A^2|_{L^\infty} \in \mathcal{O}(\eta^{-\infty}) \quad \forall \beta$$

Let $r > 0$ small, $\chi(x') \in C_0^\infty(\mathbb{R})$ with support closed to $x' = 1$ and equal to 1 near $|x' - 1| \leq r$, and $\varphi(\eta) \in C_0^\infty(]-1,1[)$, equal to 1 in $[-0.9, 0.9]$. Let $\varphi_{1,\lambda}, \varphi_{2,\lambda}$ be the h-pseudo differential operators

$$(4.71) \qquad \varphi_{1,\lambda} = \chi(x')\varphi(\frac{h}{i\lambda}\partial_{x'}) \ , \ \varphi_{2,\lambda} = \chi(x')(1-\varphi)(\frac{h}{i\lambda}\partial_{x'})$$

Let $Z(k) = Z$ and Z_0 the operators defined in (4.38), (4.39). Let us denote by $< | >, \| \ \|$ the scalar product and norm in $\mathcal{H}$.

Lemma 4.3. *Let $\mathcal{H}_r = \{\underline{f} \in \mathcal{H},$ support $(\underline{f}) \subset \{|x'-1| \leq r\}$;Let λ_0 large.*

1) $\forall \lambda \geq \lambda_0$, $\exists c_\lambda, \forall \underline{f} \in \mathcal{H}_r, \forall \beta$ we have

$$(4.72) \qquad |\langle Z\varphi_{1,\lambda}\underline{f} | Z\varphi_{2,\lambda}\underline{f}\rangle - \langle Z_0\varphi_{1,\lambda}\underline{f} | Z_0\varphi_{2,\lambda}\underline{f}\rangle| \leq c_\lambda h^{3/4} \|\underline{f}\|^2$$

$$(4.73) \qquad | \ \|Z\varphi_{1,\lambda}\underline{f}\|^2 - \|Z_0\varphi_{1,\lambda}\underline{f}\|^2| \leq c_\lambda h^{3/4} \|\underline{f}\|^2$$

2) $\forall \lambda \geq \lambda_0$, $\forall h \in]0, h_0]$, $\exists \delta_{\lambda,h}$, $\lim_{\lambda \to \infty, h \to 0} \delta_{\lambda,h} = 0$ s.t. $\forall \underline{f} \in \mathcal{H}_r, \forall k \in [0, s(h)]$ we have

$$(4.74) \qquad |\|Z\varphi_{2,\lambda}\underline{f}\|^2 - \|\varphi_{2,\lambda}\underline{f}\|^2| \leq \delta_{\lambda,h} \|\varphi_{2,\lambda}\underline{f}\|^2$$

$$(4.75) \qquad |\|Z_0\varphi_{2,\lambda}\underline{f}\|^2 - \|\varphi_{2,\lambda}\underline{f}\|^2| \leq \delta_{\lambda,h} \|\varphi_{2,\lambda}\underline{f}\|^2$$

3) $\exists c_0 > 0 \quad \forall \lambda \geq \lambda_0$, $\forall \underline{f} \in \mathcal{H}_r$

$$(4.76) \qquad e^{2\pi\mu_0} \|\underline{f}\|^2 - \|Z_0\underline{f}\|^2 \geq c_0 \|\varphi_{2,\lambda}\underline{f}\|^2$$

Let us first verify that (4.47) (and therefore Prop.4.2) is a consequence of lemma 4.3

For $\underline{f} \in \mathcal{H}_r$ we have, with $\varphi_1 = \varphi_{1,\lambda}, \varphi_2 = \varphi_{2,\lambda}$

$$\begin{aligned}\|Z\underline{f}\|^2 &= \|Z\varphi_1\underline{f} + Z\varphi_2\underline{f}\|^2 \\ &= \|Z\varphi_1\underline{f}\|^2 + \langle Z\varphi_1\underline{f}|Z\varphi_2\underline{f}\rangle + \langle Z\varphi_2\underline{f}|Z\varphi_1\underline{f}\rangle + \|Z\varphi_2\underline{f}\|^2 \\ &\leq \|Z_0\underline{f}\|^2 + 3c_\lambda h^{3/4}\|\underline{f}\|^2 + 2\delta_{\lambda,h}\|\varphi_{2,\lambda}\underline{f}\|^2 \\ &\leq e^{2\pi\mu_0}\|\underline{f}\|^2 - (c_0 - 2\delta_{\lambda,h})\|\varphi_{2,\lambda}\underline{f}\|^2 + 3c_\lambda h^{3/4}\|\underline{f}\|^2\end{aligned}$$

If one choose λ such that $c_0 - 2\delta_{\lambda,h} \geq 0$ for any h small enough, we conclude that (4.47) holds true.

Proof of Lemma 4.3

1) By the energy inequality, both Z and Z_0 are (uniformly in β) bounded operators from $\mathcal{H}_r$ to $\mathcal{H}_{r+2\pi\gamma_0 h}$.

The fourier transform $\widehat{\varphi_{1,\lambda}\underline{f}}(\xi/h)$ is equal to $\int \hat{\chi}(\frac{\xi-\eta}{h})\varphi(\eta/\lambda)\hat{\underline{f}}(\eta/h)d\eta$ and therefore satisfies

$$\forall N\ ,\ \exists C_{\lambda,N}\quad |\widehat{\varphi_{1,\lambda}\underline{f}}(\xi/h)| \leq C_{\lambda,N}(\frac{h}{|\xi|})^N \quad \forall \xi, |\xi| \geq \lambda + 1$$

In particular, $(1 - \varphi(\frac{h\partial_x}{2i\lambda}))\varphi_{1,\lambda}(\underline{f})$ is $\mathcal{O}(h^\infty)$ in $\mathcal{H}$, so in (4.73), we can replace $\varphi_{1,\lambda}\underline{f}$ by $\varphi(\frac{h\partial_x}{2i\lambda})\varphi_{1,\lambda}\underline{f}$. By (4.58) and (4.67), $Z \circ \varphi(\frac{h\partial_x}{2i\lambda})$ is a h-pseudo differential operator ; therefore, using (4.59) we get $\|(Z - Z_0)\varphi(\frac{h\partial_x}{2i\lambda})\varphi_{1,\lambda}\| \leq c_\lambda h^{3/4}$. This implies (4.73).

In order to prove (4.72), it is now sufficient to prove

$$|\langle\varphi(\frac{h\partial_x}{2i\lambda})Z_0\varphi_{1,\lambda}\underline{f}|(Z - Z_0)\varphi_{2,\lambda}\underline{f}\rangle| \leq c_\lambda h^{3/4}\|\underline{f}\|^2$$

Writing $(Z - Z_0) = (Z - Z_0)\varphi(\frac{h\partial_x}{4i\lambda}) + (Z - Z_0)(1 - \varphi(\frac{h\partial_x}{4i\lambda}))$ we are reduce to verify (for $\underline{f} \in \mathcal{H}_r$, support $[(Z - Z_0)\varphi_{2,\lambda}(\underline{f})]$ is compact)

$$\|\varphi(\frac{h\partial_x}{2i\lambda})\psi Z(1 - \varphi(\frac{h\partial_x}{4i\lambda}))\varphi_{2,\lambda}(\underline{f})\| \leq c_\lambda h^{3/4}\|\underline{f}\|^2 \tag{4.77}$$

for $\psi(x') \in C_0^\infty$.

To see this point, we write $Z = Z^2 + S$ where Z^2 is associated to $\mathcal{M}^2$ (see 4.64) ; the coefficients of $\mathcal{M}^2$ are sums of Fourier Integral Operators with phases $(x' - y')\eta + h\psi_\pm(\beta, 2\pi, x')\eta$ (see 4.60) for which the associated canonical transformations in the phase space $\{(x', \eta) \in T^*\mathbb{R}\}$ are closed to Id. More precisely, Z^2 is a sum of operators of the form $Q_\pm(h, x, h\partial_x)J_\pm$, where $Q_\pm$ are h-pseudo differential operators and $J_\pm$ is a change of variables given by $J_\pm(f)(x) = f(x + h\psi_\pm(\beta, 2\pi, x))$. Therefore (4.77) holds true for Z_2. By (4.70), the coefficients of ψS are operators of the form

$$\begin{cases}\frac{1}{2\pi h}\int e^{ix\eta/h}p(\beta, x, \eta)\hat{f}(\eta/h)d\eta = Pf \\ \forall \alpha \quad |\partial_x^\alpha p|_{L^\infty} \in \mathcal{O}(\eta^{-\infty}) \quad \text{uniformly in } \beta\end{cases} \tag{4.78}$$

with p compactly supported in x. Then

$$\widehat{Pf}(\xi/h) = \frac{1}{2\pi h}\int \hat{p}_x(\beta, \frac{\xi-\eta}{h}, \eta)\hat{f}(\eta/h)d\eta$$

and (4.77) follows from the fact that $\hat{p}_x(\beta, \zeta, \eta)$ is rapidly decreasing in (ζ, η) uniformly in β.

2) We have to verify that Z and Z_0 are almost isometries on $\mathcal{H}$ for η large and h small. For solutions of (4.51), with energy $E(f,s)$, and right hand side $g = 0$ we get

(4.79)

$$\begin{cases} |\frac{d}{ds}E(f,s)| \leq C[|\nabla_{s,x'}\alpha|_{L^\infty}E(f,s) + \|f(s,.)\|_{L^2}E^{1/2}(f,s)] \\ |E(f,s) - \|\underline{f}(s)\|^2_{\mathcal{H}}| \leq C\left[|\alpha-\alpha_0|_{L^\infty}\|\underline{f}(s)\|^2_{\mathcal{H}} + \|f(s,.)\|^2_{L^2}\right] \end{cases}$$

Therefore, we have just to prove

(4.80)

$$\sup_{s\in[0,2\pi]} \|\{Z(s)\varphi_{2,\lambda}\underline{f}\}_1\|^2_{L^2} + \|\{Z_0(s)\varphi_{2,\lambda}\underline{f}\}_1\|^2_{L^2} \leq \delta_{\lambda,h}\|\underline{f}\|^2_{\mathcal{H}};\ \lim_{\substack{\lambda\to\infty \\ h\to 0}} \delta_{\lambda,h} = 0$$

where $Z(s), Z_0(s)$ are the solution maps at time s and $\{\}_1$ denotes the first component. As before, we may replace $\varphi_{2,\lambda}\underline{f}$ by $(1-\varphi)(\frac{2h\partial_{x'}}{i\lambda})\varphi_{2,\lambda}\underline{f}$ in (4.80). Then (4.80) is obvious for $Z_0(s)$ (it is a constant coefficient operator in x'), and for $Z(s)$, it follows from (4.70) and from the fact the coefficients of $\mathcal{M}^2$ are sums of Fourier Integral Operators.

3) The inequality (4.76) follows easely from (4.41) (4.42).

Let $0 \leq s_0 \leq s_1 \leq s(h)$, and let $U(s_1, s_0)$ be the solution map for the Cauchy problem

$$\begin{cases} [\partial_s^2 - \alpha(\frac{hs}{x'})h^2\partial^2_{x'} + pG^{p-1}(s+b_0)]f = 0 \\ f(s_0, x') = f_0 \quad \partial_s f(s_0,x') = f_1 \end{cases} \tag{4.81}$$

$$U(s_1,s_0)\begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} f(s_1,x') \\ \partial_s f(s_1,x') \end{bmatrix} \tag{4.82}$$

Let $\eta_0 > 0$ such that

$$\|\mathbb{M}(\alpha_0\eta_0^2); \mathbb{C}^2_{\eta_0}\| = e^{2\pi\mu_0} \tag{4.83}$$

Let $\ell \in \mathbb{Z}$, $\langle\ell\rangle = (1+\ell^2)^{1/2}$ and $\mathcal{H}^1_\ell$ the Hilbert space with norm

$$|\underline{f}|^2_\ell = \langle\ell\rangle^2\|\underline{f}\|^2_{\mathcal{H}} + \|(\partial_{x'} - i\ell\frac{\eta_0}{h})\underline{f}\|^2_{\mathcal{H}} \tag{4.84}$$

Proposition 4.4. *There exist $h_0 > 0$, $C_0 > 0$, $\rho_0 > 0$, such that, for any $h \leq h_0$, $0 \leq s_0 \leq s_1 \leq s(h)$, any data $\underline{f} \in \mathcal{H}^1_{\ell,\rho_0} = \{\underline{f} \in \mathcal{H}^1_\ell$, support $(\underline{f}) \subset |x'-1| \leq \rho_0\}$, one has $U(s_1,s_0)\underline{f} \in \mathcal{H}^1_{\ell,\rho_0+\gamma_0(s_1-s_0)h}$ and*

$$|U(s_1,s_0)\underline{f}|_\ell \leq C_0 e^{\mu_0(s_1-s_0)}|\underline{f}|_\ell \quad \forall \ell \in \mathbb{Z} \tag{4.85}$$

Proof : Using the group property $U(s_2, s_0) = U(s_2, s_1)U(s_1, s_0)$, and the finite speed of propagation, we are reduced to verify that there exist C_1 s.t.

$$(4.86)\qquad |U(s_1, s_0)\underline{f}|_\ell \le C_1|\underline{f}|_\ell \quad \forall \ell, \quad \forall s_1, s_0 \quad |s_1 - s_0| \le 2\pi$$

(4.87)

$$|U(2k\pi + 2\pi, 2k\pi)\underline{f}|_\ell \le (e^{2\pi\mu_0} + C_1 h^{3/4})|\underline{f}_\ell| \quad \forall \ell, \quad \forall h \in [0, s(h)]$$

If f satisfies (4.81), then $(\partial_{x'} - i\ell\frac{\eta_0}{h})f = g$ satisfies

$$(4.88)\quad [\partial_s^2 - \alpha h^2 \partial_{x'}^2 + pG^{p-1}(s+b_0) - h\frac{\partial \alpha}{\partial x'}(h\partial_{x'})]g = i\eta_0\ell(\partial_{x'}\alpha)(h\partial_{x'})f$$

We have $\|i\eta_0\ell(\partial_{x'}\alpha)(h\partial_{x'})f\|_{L^2(x')} \le \langle\ell\rangle C^{te}\|f\|_{\mathcal{H}}$, so (4.86) is a consequence of the energy estimate in the space $\mathcal{H}$. By definition we have $Z(k) = U(2k\pi + 2\pi, 2k\pi)$. Using the proposition (4.2) we get

$$(4.89)\quad (\partial_{x'} - i\ell\frac{\eta_0}{h})Z(k) = (Z(k) + h^{7/4}S(k))(\partial_{x'} - i\ell\frac{\eta_0}{h}) + i\ell\eta_0 h^{3/4}S(k)$$

Together with (4.47), (4.89) implies (4.87).

5. Non-linear perturbation

In this part, we shall study the Cauchy problem (4.21) for $t' \in [0, hs(h)]$, where $s(h)$ satisfies (4.26), and in particular

$$(5.1)\qquad e^{\mu_0 s(h)}(\widetilde{R}, \tilde{r}_j) \text{ are flats with all their derivatives when } h \to 0$$

We rewrite the equation as a 2×2 system, in the coordinates $s = \frac{t'}{h}, x'$

$$(5.2)\qquad \partial_s - A + \underline{\Gamma}\ ;\ A = \begin{pmatrix} 0 & 1 \\ h^2\alpha\partial_{x'}^2 - V(s) & 0 \end{pmatrix}$$

with $V(s) = pG^{p-1}(s + b_0)$, $\underline{\Gamma} = \begin{pmatrix} 0 & 0 \\ \Gamma & 0 \end{pmatrix}$

$$(5.3)\quad \Gamma(f) = p[G^{p-1}(s + B(\frac{hs}{x'})) - G^{p-1}(s + b_0)]f + Q(f) + N(f)$$

Let $\mathcal{E} = \mathcal{E}_h$ be the Hilbert space

$$(5.4)\quad \begin{cases} \mathcal{E} = \{\underline{v} = \sum_{\ell\in\mathbb{Z}} \underline{v}_\ell e^{i\ell\theta}; \underline{v}_\ell e^{i\ell\eta_0 x'/h} \in \mathcal{H}, (\partial_{x'}\underline{v}_\ell)e^{i\ell\eta_0 x'/h} \in \mathcal{H}\} \\ \|\underline{v}\|_{\mathcal{E}}^2 = \sum_\ell \langle\ell\rangle^2(\langle\ell\rangle^2\|\underline{v}_\ell e^{i\ell\eta_0 x'/h}\|_{\mathcal{H}}^2 + \|(\partial_{x'}\underline{v}_\ell)e^{i\ell\eta_0 x'/h}\|_{\mathcal{H}}^2) \end{cases}$$

We extend $\partial_s - A + \underline{\Gamma}$, by the rule $\theta = \eta_0.x'/h$, from $C^\infty(s, x')$ to $\bigoplus_\ell C^\infty(s, x')e^{i\ell\theta}$; if $P = P(s, x', \partial_s, \partial_{x'})$ is a linear differential operator, we extend P in $\widetilde{P}$

$$\widetilde{P}(\sum_\ell v_\ell e^{i\ell\theta}) = \sum_\ell (e^{-i\ell\eta_0 x'/h}P(v_\ell e^{i\ell\eta_0 x'/h}))e^{i\ell\theta}$$

and we extend a power $f \mapsto f^k$ in the natural way, $(\sum_\ell f_\ell e^{i\ell\theta})^k = \sum_\ell \left(\sum_{\ell_1 + + \ell_k = \ell} f_{\ell_1} \cdots f_{\ell_k} \right) e^{i\ell\theta}$.

Let us denote by $\widetilde{A}, \widetilde{\Gamma}, \widetilde{\Gamma}$ the above extensions of $A, \underline{\Gamma}, \Gamma$.

Then, if $\underline{\tilde{f}}(s, x', \theta)$ satisfies

$$(\partial_s - \widetilde{A} + \widetilde{\underline{\Gamma}})\underline{\tilde{f}} = 0 \; ; \; \underline{\tilde{f}}_{|s=0} = \underline{\tilde{f}}_0(x', \theta) \tag{5.5}$$

the 2-vector $\underline{f}(s, x') = \underline{\tilde{f}} = \underline{\tilde{f}}(s, x', \eta_0 . x'/h)$ satisfies

$$(\partial_s - A + \underline{\Gamma})\underline{f} = 0 \quad \underline{f}_{|s=0} = \underline{\tilde{f}}_0(x', \eta_0 . x'/h) \tag{5.6}$$

Proposition 5.1. *There exist C_0, h_0, ρ_0 such that*

i) for any $h \leq h_0$, $0 \leq s_0 \leq s_1 \leq s(h)$, $0 \leq \rho \leq \rho_0$, any data $\underline{\tilde{f}} \in \mathcal{E}_{h,\rho} = \{\underline{v} \in \mathcal{E}_h$, support $(\underline{v} \subset \{|x' - 1| \leq \rho\}\}$, the solution $\widetilde{U}(s, s_0)\underline{\tilde{f}} = \underline{\tilde{f}}(s)$ of $(\partial_s - \widetilde{A})\underline{\tilde{f}}(s) = 0, \underline{\tilde{f}}(s_0) = \underline{\tilde{f}}$ belongs to $\mathcal{E}_h, \rho + \gamma_0(s_1 - s_0)h$ and we have

$$\|\widetilde{U}(s_1, s_0)\underline{\tilde{f}}\|_\mathcal{E} \leq C_0 e^{\mu_0(s_1 - s_0)} \|\underline{\tilde{f}}\|_\mathcal{E} \tag{5.7}$$

ii) for any $h \leq h_0$, $0 \leq s_0 \leq s(h)$, $\rho \leq \rho_0$, and any $\underline{\tilde{f}}, \underline{\tilde{g}} \in \mathcal{E}_{h,\rho}$ such that $\|\underline{\tilde{f}}\|_\mathcal{E} \leq 1$, $\|\underline{\tilde{g}}\|_\mathcal{E} \leq 1$, one has

$$\|\widetilde{\underline{\Gamma}}|_{s=s_0}(\underline{\tilde{f}})\|_\mathcal{E} \leq C_0 \left[h^{3/4} \|\underline{\tilde{f}}\|_\mathcal{E} + \|\underline{\tilde{f}}\|_\mathcal{E}^2 \right] \tag{5.8}$$

$$\|\widetilde{\underline{\Gamma}}|_{s=s_0}(\underline{\tilde{f}}) - \widetilde{\underline{\Gamma}}_{s=s_0}(\underline{\tilde{g}})\|_\mathcal{E} \leq C_0 \|\underline{\tilde{f}} - \underline{\tilde{g}}\|_\mathcal{E} \left[h^{3/4} + \|\underline{\tilde{f}}\|_\mathcal{E} + \|\underline{\tilde{g}}\|_\mathcal{E} \right] \tag{5.9}$$

Proof : Let $A_\ell = e^{-i\ell\eta_0 x'/h} A e^{i\ell\eta_0 x'/h}$; we have $\widetilde{A} = \bigoplus_\ell A_\ell$, and therefore

$$\widetilde{U}(s_1, s_0) = \bigoplus_\ell e^{-i\ell\eta_0 x'/h} U(s_1, s_0) e^{i\ell\eta_0 x'/h}$$

Let $\underline{g}_\ell = \underline{\tilde{f}}_\ell e^{i\ell\eta_0 x'/h}$; we have $e^{i\ell\eta_0 x'/h}\{\widetilde{U}(s_1, s_0)\underline{\tilde{f}}\}_\ell = U(s_1, s_0)\underline{g}_\ell$ and

$$\langle \ell \rangle^2 \|\underline{\tilde{f}}_\ell e^{i\ell\eta_0 x'/h}\|_\mathcal{H}^2 + \|\partial_{x'}(\underline{\tilde{f}}_\ell) e^{i\ell\eta_0 x'/h}\|_\mathcal{H}^2$$

is equal to $\langle \ell \rangle^2 \|\underline{g}_\ell\|_\mathcal{H}^2 + \|(\partial_{x'} - i\ell\frac{\eta_0}{h})\underline{g}_\ell\|_\mathcal{H}^2$. Thus, (5.7) is a consequence of (4.85).

Let $\underline{\tilde{f}} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix}$; we have $\widetilde{\underline{\Gamma}}(\underline{\tilde{f}}) = \begin{pmatrix} 0 \\ \widetilde{\Gamma}(\tilde{f}_0) \end{pmatrix}$.

By (4.15), (4.22), (4.23) and (5.3), the perturbation Γ has the following structure

(5.10)
$$\begin{cases}\Gamma(f) = \frac{hs}{x'}D_0(s,t',x')f + hq_1(t',x')h\partial_{x'}f + h^2q_2(t',x')f\\ +hD_1(s+B(t'/x'),t',x',h)f + \sum_{k=2}^{p} D_k(s+B(t'/x'),t',x',h)f^k\end{cases}$$

The coefficients $D_j(\theta,t',x',h)(j\geq 0)$ are smooth in $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, $t'\sim 0$, $x'\sim 1$, $h\in[0,1]$. Therefore, we get

$$(5.11)\qquad \begin{cases}\widetilde{\Gamma}(\tilde f) = \widetilde{\Gamma}_1(\tilde f) + \widetilde{\Gamma}_2(\tilde f)\\ \widetilde{\Gamma}_1(\tilde f) = (\Gamma - hq_1h\partial_{x'})\tilde f\\ \widetilde{\Gamma}_2(\tilde f) = \bigoplus_\ell hq_1(h\partial_{x'} + i\ell\eta_0)\tilde f_\ell\end{cases}$$

We first verify that (5.8), (5.9) holds true for $\widetilde{\Gamma}_2$. Let $\|w\|^2_{H^1_h} = \int |w|^2 + |h\partial_{x'}w|^2dx'$. By the definition of the $\|\ .\ \|_{\mathcal{E}}$ norm, we just have to prove

(5.12)
$$\begin{cases}\exists C_0\ ,\ \forall\ell,\ \text{for } g=(h\partial_{x'}+i\ell\eta_0)f\ ,\ F=fe^{i\ell\eta_0x'/h}\\ \langle\ell\rangle^2\|g\|^2_{L^2} + \|(\partial_{x'}g)e^{i\ell\eta_0x'/h}\|^2_{L^2} \leq C_0\left[\langle\ell\rangle^2\|F\|^2_{H^1_h} + \|(\partial_{x'} - i\ell\frac{\eta_0}{h})F\|^2_{H^1_h}\right]\end{cases}$$

We have $g = e^{-i\ell\eta_0x'/h}h\partial_{x'}F$, so $\|g\|_{L^2} = \|h\partial_{x'}F\|_{L^2}$, $\|(\partial_{x'}g)e^{i\ell\eta_0x'/h}\|_{L^2} = \|(\partial_{x'} - i\ell\frac{\eta_0}{h})h\partial_{x'}F\|_{L^2}$; this implies (5.12).

It remains to show that (5.8), (5.9) holds true for $\widetilde{\Gamma}_1$. For $0\leq s\leq s(h)$, we have $hs \ll h^{3/4}$. By the structure of Γ_1 given by (5.10), we are reduce to show that (5.8), (5.9) holds true for a power map

$$\begin{pmatrix}\tilde f_0\\ \tilde f_1\end{pmatrix} \mapsto \begin{pmatrix}0\\ \tilde f_0^k\end{pmatrix}\ k\geq 2$$

For $\tilde f = \sum_\ell f_\ell e^{i\ell\theta}$, let us define the $|||\tilde f|||$ norm by

$$(5.13)\qquad |||\tilde f|||^2 = \sum_\ell \langle\ell\rangle^2(\langle\ell\rangle^2\|f_\ell\|^2_{L^2} + \|\partial_{x'}f_\ell\|^2_{L^2})$$

By definition of the $\|\ \|_{\mathcal{E}}$ norm, we have

$$(5.14)\qquad \frac{1}{c_0}\|\begin{pmatrix}0\\ \tilde f\end{pmatrix}\|_{\mathcal{E}} \leq |||\tilde f||| \leq c_0\|\begin{pmatrix}\tilde f\\ 0\end{pmatrix}\|_{\mathcal{E}}$$

so we have just to verify the algebraic inequality

$$(5.15)\qquad |||\tilde f\tilde f'||| \leq C_0|||\tilde f|||\quad |||\tilde f'|||$$

We have, with $\tilde f(x',\theta) = \sum_\ell f_\ell(x')e^{i\ell\theta}$

$$\begin{aligned}&\|f_\ell\|_{L^2(x')} = \langle\ell\rangle^{-2}d_\ell\ ; \quad &&\|d_\ell\|_{\ell^2} \leq |||\tilde f|||\\ &\|f_\ell\|_{H^1(x')} = \langle\ell\rangle^{-1}c_\ell &&\|c_\ell\|_{\ell^2} \leq |||\tilde f|||\end{aligned}$$

¿From the fact that $H^1(x')$ is a subalgebra of $L^\infty(x')$, we get

$$(5.16) \qquad \langle n\rangle \| \sum_{k+\ell=n} f_k f'_\ell \|_{H^1(x')} \leq C^{te} \sum_{k+\ell=n} \frac{\langle n\rangle}{\langle k\rangle\langle \ell\rangle} c_k c'_\ell$$

(5.17)

$$\langle n\rangle^2 \| \sum_{k+\ell=n} f_k f'_\ell \|_{L^2(x')} \leq C^{te} \left[\sum_{\substack{k+\ell=n \\ k\leq n/2}} \frac{\langle n\rangle^2}{\langle k\rangle\langle \ell\rangle^2} c_k d'_\ell + \sum_{\substack{k+\ell=n \\ k> n/2}} \frac{\langle n\rangle^2}{\langle k\rangle^2\langle \ell\rangle} d_k c'_\ell \right]$$

so (5.15) is a consequence of the injection $\ell^1 * \ell^2 \hookrightarrow \ell^2$.

Proposition 5.2. *There exist $C_1, h_0, \rho_0, \varepsilon_0$ such that for any $\varepsilon \in]0, \varepsilon_0]$, $h \in]0, h_0]$, $\rho \in]0, \rho_0]$, the Cauchy problem*

$$(5.18) \qquad (\partial_s - \widetilde{A} + \widetilde{\underline{\Gamma}})\tilde{f} = \tilde{r} \ , \ \tilde{f}_{|s=0} = \tilde{f}_0$$

where the data $\tilde{r}(s, x', \theta, h)$, $\tilde{f}_0(x', \theta, h)$ are such that

$$(5.19) \qquad \begin{cases} \tilde{r}(s,.) \in L^\infty(s; \mathcal{E}_{h,\rho}) \\ \sup_{0\leq s\leq s(h)} \|\tilde{r}(s,.)\|_{\mathcal{E}} \ll \alpha(\varepsilon, h) \end{cases}$$

$$(5.20) \qquad \tilde{f}_0 \in \mathcal{E}_{h,\rho} \ ; \ \|\tilde{f}_0\|_{\mathcal{E}} \leq \alpha(\varepsilon, h)$$

$$(5.21) \qquad \alpha(\varepsilon, h) e^{\mu_0 s(h)} = \varepsilon$$

admits a unique solution $\tilde{f} \in C^0(s; \mathcal{E}_{h,\rho+\gamma_0 sh})$ which satisfies for any $s \in [0, s(h)]$

$$(5.22) \qquad \|\tilde{f}(s)\|_{\mathcal{E}} \leq C_1 \alpha(\varepsilon, h) e^{\mu_0 s}$$

(5.23)

$$\|\tilde{f}(s) - \widetilde{U}(s,0)\tilde{f}_0 - \int_0^s \widetilde{U}(s,\sigma)\tilde{r}(\sigma,.)d\sigma\|_{\mathcal{E}} \leq C_1 \left[\varepsilon + h^{1/2}\right] \alpha(\varepsilon, h) e^{\mu_0 s}$$

Proof : Let $|\cdot|$ be the norm on $C^0([0, s(h)]; \mathcal{E}_{h,\rho+\gamma_0 sh}) = \mathcal{A}$

$$(5.24) \qquad |\tilde{f}| = \sup_{0\leq s\leq s(h)} \|\frac{e^{-\mu_0 s}}{\alpha(\varepsilon, h)} \tilde{f}(s)\|_{\mathcal{E}}$$

Let ϕ be the map from $\mathcal{A}$ to $\mathcal{A}$

$$(5.25) \qquad \phi(\tilde{f}) = \widetilde{U}(s,0)\tilde{f}_0 + \int_0^s \widetilde{U}(s,\sigma)(\tilde{r}(\sigma,.) - \widetilde{\underline{\Gamma}}(\tilde{f}(\sigma,.)))d\sigma$$

¿From proposition (5.1) we get

$$(5.26) \qquad \begin{cases} |\phi(\tilde{f})| \leq C_0[1 + \frac{1}{\mu_0} + s(h)h^{3/4}|\tilde{f}| + \frac{\varepsilon}{\mu_0}|\tilde{f}|^2] \\ |\phi(\tilde{f}) - \phi(\tilde{f}')| \leq C_0|\tilde{f} - \tilde{f}'|[s(h)h^{3/4} + \frac{\varepsilon}{\mu_0}(|\tilde{f}| + |\tilde{f}'|)] \end{cases}$$

We have $s(h)h^{3/4} \ll h^{1/2}$, and we get the result by the fixed point theorem.

To solve the Cauchy problem (4.21) with special data $(\tilde{w}_0, \tilde{w}_1)$, we apply proposition (5.2) with the data

$$\begin{cases} \underline{\tilde{f}}_0 = \alpha(\varepsilon, h)\underline{g}e^{i\theta} - \begin{bmatrix} \hbar^{-\gamma}\tilde{r}_0 \\ \hbar^{-\gamma+1+\beta}\tilde{r}_1 \end{bmatrix} \\ \tilde{r} = \hbar^{2+2\beta-\gamma}\widetilde{R} \end{cases} \tag{5.27}$$

In (5.27), the functions $\tilde{r}_0, \tilde{r}_1, \widetilde{R}$ are those given in (4.21) and are independant of θ, and $\underline{g} = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix}$ is such that

$$||\underline{g}e^{i\eta_0 x'/h}||^2_{\mathcal{H}} + ||\partial_{x'}\underline{g}e^{i\eta_0 x'/h}||^2_{\mathcal{H}} \leq 1 . \tag{5.28}$$

¿From (5.1) and (5.23), we get that the solution $\tilde{f}$ satisfies for any $s \in [0, s(h)]$

$$||\underline{\tilde{f}}(s) - \alpha(\varepsilon, h)\widetilde{U}(s,0)e^{i\theta}\underline{g}||_{\mathcal{E}} \leq C_1(\varepsilon + h^{1/2})\alpha(\varepsilon, h)e^{\mu_0 s} + \mathcal{O}(h^\infty) \tag{5.29}$$

Lemma 5.3. *Let* $\underline{e} = \begin{pmatrix} e_0 \\ e_1 \end{pmatrix} \in \mathbb{C}^2\backslash 0$ *such that* $\mathbb{M}(\alpha_0\eta_0^2)\underline{e} = e^{2\pi\mu_0}\underline{e}$. *let* $\varphi(x') \in C_0^\infty(|x'-1| < \rho_0)$, ρ_0 *small, equal to 1 near* $|x'-1| \leq 3\rho_0/4$, *and* $\psi(x') \in C_0^\infty(|x'-1| \leq \rho_0/2)$. *Let* $\Theta(s)$ *be the solution of the differential equation on the line*

$$(\partial_s^2 + pG^{p-1}(s+b_0) + \alpha_0\eta_0^2)\Theta(s) = 0 \ ; \ \Theta(0) = e_0, \Theta'(0) = e_1 . \tag{5.30}$$

Then the following inequality holds true

(5.31)

$$\begin{cases} \underline{\delta} = \psi(x')e^{-i\eta_0 x'/h}\left[U(s,0)[\varphi(x')e^{i\eta_0 x'/h}\underline{e}] - e^{i\eta_0 x'/h}\begin{pmatrix} \Theta(s) \\ \Theta'(s) \end{pmatrix}\right] \\ \sup\limits_{0\leq s\leq s(h)} (||\underline{\delta}(s,\cdot)||_{\mathcal{H}} + ||\partial_{x'}\underline{\delta}(s,\cdot)||_{\mathcal{H}})e^{-\mu_0 s} \in \mathcal{O}(h^{1/2}) \end{cases}$$

Proof : Let $s = 2k\pi + \sigma$, $\sigma \in [0, 2\pi[$. By the definition of $Z(k)$ given in §4, we get that $e^{-i\eta_0 x'/h}U(s,0)\varphi(x')e^{i\eta_0 x'/h}\underline{e}$ is equal to

$$e^{-i\eta_0 x'/h}U(s, 2k\pi)Z(k-1) \circ \cdots \circ Z(1) \circ Z(0)\varphi(x')e^{i\eta_0 x'/h}\underline{e} . \tag{5.32}$$

Let $\ell \in \{0, 1, \cdots, k\}$. By the definition (4.55) of $\mathcal{M}$, we have in a fixed neighborhood of $x' = 1$

$$Z(\ell)e^{i\eta_0 x'/h}\underline{e} = e^{i\eta_0 x'/h}\mathcal{M}(h, \beta, 2\pi, x', \eta_0)\underline{e} \quad (\beta = (h, h\ell)) \, . \tag{5.33}$$

Using (4.50), (4.57), (4.59), (4.67), we get

$$\begin{cases} Z(\ell)e^{i\eta_0 x'/h}\underline{e} = e^{i\eta_0 x'/h}[e^{2\pi\mu_0}\underline{e} + \underline{\delta}_\ell] \\ \sup_\ell\{||\varphi\underline{\delta}_\ell||_{\mathcal{H}} + ||\varphi\partial_{x'}\underline{\delta}_\ell||_{\mathcal{H}}\} \leq C^{te}h^{5/6} \, . \end{cases} \tag{5.34}$$

Using now proposition (4.2) we get

$$\begin{cases} Z(k-1) \circ \cdots \circ Z(1) \circ Z(0)e^{i\eta_0 x'/h}\underline{e} = e^{i\eta_0 x'/h}e^{2\pi k\mu_0}[\underline{e} + \underline{\delta}] \\ ||\varphi\underline{\delta}||_{\mathcal{H}} + ||\varphi\partial_{x'}\underline{\delta}||_{\mathcal{H}}\} \leq C^{te}h^{3/4} \, . \end{cases} \tag{5.35}$$

We also have $(\Theta(2\pi k), \Theta'(2\pi k)) = e^{2\pi k\mu_0}\underline{e}$, so we get (5.31) for the values $s \in 2\pi\mathbb{N}$. We obtain the general case, applying $U(2k\pi + \sigma, 2k\pi)$ to (5.35), using $U(2k\pi + \sigma, 2k\pi)e^{i\eta_0 x'/h}\underline{e} = e^{i\eta_0 x'/h}\mathcal{M}(k, \beta, \sigma, x', \eta_0)\underline{e}$ $(\beta = (h, hk))$ and (4.50), (4.57).

Lemma 5.4. *Let $\tilde{w}_h(t', x')$ be the solution of the Cauchy problem (4.21) with data ($a \in [0, 1]$)*

$$\begin{cases} \tilde{w}_0(h/x') = a\hbar^\gamma\alpha(\varepsilon, h)\cos(\eta_0 x'/h)e_0 \\ \tilde{w}_1(h/x') = a\hbar^{\gamma-1-\beta}\alpha(\varepsilon, h)\cos(\eta_0 x'/h)e_1 \, . \end{cases} \tag{5.36}$$

Then the following inequality holds true, uniformly for $t' \in [0, hs(h)]$

$$\begin{aligned} &|\tilde{w}_h(t', x') - a\cos(\eta_0 x'/h)\Theta(t'/h)\alpha(\varepsilon, h)|_{L^\infty(|x'-1| \leq \frac{\rho_0}{2})} \\ &\quad \leq C^{te}[\varepsilon + h^{1/2}]\alpha(\varepsilon, h)e^{\mu_0 t'/h} + \mathcal{O}(h^\infty) \, . \end{aligned} \tag{5.37}$$

Proof : Let $\underline{\tilde{f}}(s, x', \theta) = \sum_\ell \underline{f}_\ell(s, x')e^{i\ell\theta}$. By (5.4), we get with $\underline{f}_\ell = \begin{pmatrix} f_{0,\ell} \\ f_{1,\ell} \end{pmatrix}$

$$\begin{cases} <\ell>^2 ||f_{0,\ell}||^2_{L^2_{x'}} + ||\partial_{x'} f_{0,\ell}||^2_{L^2_{x'}} = b_\ell^2 \\ \sum_\ell <\ell>^2 b_\ell^2 \leq C^{te}||\underline{\tilde{f}}(s, \cdot)||^2_{\mathcal{E}} \, . \end{cases} \tag{5.38}$$

We have

$$(5.39) \qquad ||f_{0,\ell}||_{L^\infty(|x'-1|\le\rho_0)} \le C^{te}\langle\ell\rangle^{-1/2}b_\ell$$

so we get

$$(5.40) \qquad ||\sum_\ell f_{0,\ell}(s,x')e^{i\ell\eta_0 x'/h}||_{L^\infty(|x'-1|\le\rho_0)} \le C^{te}||\underline{\tilde{f}}(s,\cdot)||_{\mathcal{E}} .$$

Therefore, (5.37) is a consequence of (5.29) and Lemma (5.3)

We now come back to the Cauchy problem (4.13).

Let us denote by $v_h^a(t,x), a = 0,1$ the solutions of

$$(5.41) \qquad \begin{cases} h^2(\partial_t^2 - \partial_x^2)v_h^a + \frac{(v_h^a)^p}{x^{p-1}} = 0 \\ v_h^a(0,x) = \hbar^{-\gamma}[g_0(\hbar x) + aw_0(\hbar x)] \\ (h\partial_t)v_h^a(0,x) = \hbar^{-\gamma+1+\beta}[g_1(\hbar x) + aw_1(\hbar x)] \end{cases}$$

where $\{w_0, w_1\}$ are the data obtained by the transformation $\tilde{w}_k(t',x') = e^{-b}w_k(t,x)$ from $\{\tilde{w}_0, \tilde{w}_1\}$ given by 5.36. We have $\alpha(\varepsilon,h) = \varepsilon\alpha(h)$ with

$$(5.42) \qquad \alpha(h)e^{\mu_0 s(h)} = 1$$

and (4.26) implies $(s(h) \gg M|\log h|\ \forall M)$

$$(5.43) \qquad \alpha(h) \in \mathcal{O}(h^\infty) .$$

Thus, the differences of the Cauchy data of the v_h^a are flat when $h \to 0$ in a fixed neighborhood of $x' = 1$

$$(5.44) \qquad (v_h^0 - v_h^1)_{t=0} \text{ and } h\partial_t(v_h^0 - v_h^1)_{t=0} \text{ are } \mathcal{O}(h^\infty) \text{ in } C^\infty_{x'} .$$

We have $t' = \varphi(t,x)$ and $x' = x^{-\beta}(1 + \mathcal{O}(t/x)^2)$.

¿From $s(h) \ll h^{-\nu}(\forall\nu > 0)$, we deduce

$$(5.45) \qquad |\cos(\eta_0 x'/h) - \cos(\frac{\eta_0}{hx^\beta})| \le C^{te}h^{1/2} \text{ for } t' \in [0, hs(h)]$$

By (5.37), we get for $\varphi(t,x) \in [0, hs(h)], |x'-1| \le \frac{\rho_0}{2}$

$$(5.46) \qquad |(v_h^0 - v_h^1)(t,x) - \cos(\frac{\eta_0}{hx^\beta})\Theta(\varphi(t,x)/h)\varepsilon\alpha(h)| \le C^{te}(\varepsilon\alpha(h)(\varepsilon + h^{1/2})e^{\mu_0\varphi(t,x)/h} + \mathcal{O}(h^\infty)) .$$

We are now ready to achieve the verification of Theorem 4.
We denote by $T = \hbar t, \rho = \hbar x$ the time and space variables in Theorem 4, and we define the two functions g and g' as follows.
By homogeneity, there exist $g(T, \rho)$ solution of (1.4), such that

$$v_h^0(t,x) = \hbar^{-\gamma} g(\hbar t, \hbar x) \tag{5.47}$$

In order to define $g'(T,\rho)$, we introduce

$$\begin{cases} g^{opt}(\hbar t, \hbar x) &= \hbar^{\gamma} f_h^{opt}(t,x) \\ \tilde{g}(\hbar, \hbar t, \hbar x) &= \hbar^{\gamma} v_h^1(t,x) \end{cases} \tag{5.48}$$

Let $\tilde{g}_{0,1}(\hbar,\rho)$ (resp. $g_{0,1}^{opt}(\rho)$) the Cauchy data of $\tilde{g}$ (resp. g^{opt}) and $\hbar_n = 2^{-n}$. Let $\psi(u) \in C_0^\infty(]-1/4, 1/4[)$ equal to 1 in the set $|u| \leq 1/8$. Let us define $g'(T,\rho)$ as the solution of (1.4) with Cauchy data

$$g'_{0,1}(\rho) = g_{0,1}^{opt}(\rho) + \sum_{n=1}^{\infty} \psi\left(\frac{\rho - \hbar_n}{\hbar_n}\right)[\tilde{g}_{0,1}(\hbar_n, \rho) - g_{0,1}^{opt}(\rho)] \tag{5.49}$$

By finite speed of propagation, one has $g'(T,\rho) = \tilde{g}(\hbar_n, T, \rho)$ in the set $|\rho - \hbar_n| \leq \frac{\hbar_n}{8} - |T|$. We have $\partial_\rho^\beta (g_{0,1} - g'_{0,1}) \in \mathcal{O}(\rho^\infty)$, and the Cauchy data of g,g', and g^{opt} have the same asymptotic expansion on $\rho = 0$.

In order to verify (1.12) we take $\delta = \delta_n = \hbar_n = 2^{-n}$. Using (4.5),we get for $T = \hbar t = \delta^{1+\beta}\mu(\delta)$ and $\rho = \hbar x \in I_\delta = [\delta - \frac{c_0 \delta}{\mu(\delta)}, \delta + \frac{c_0\delta}{\mu(\delta)}]$

$$\frac{\varphi(t,x)}{h} = c_1 \mu(\delta) + z \tag{5.50}$$

with $c_1 = \psi'(0)$ and $z \in [-c_2, c_2]$, with c_2 small if c_0 and δ are small.

Using (5.46), it remains to show that we can choose the function μ such that for any $z \in [-c_2, c_2]$, one has for any $\delta = 2^{-n}$, n large

$$e^{-\mu_0 s(\delta^\beta)} \Theta(c_1\mu(\delta) + z) \geq \varepsilon_1 \tag{5.51}$$

and

$$c_1\mu(\delta) + c_2 \leq s(\delta^\beta) \tag{5.52}$$

The function $e^{-\mu_0 y}\Theta(y)$ is 2π-periodic ; it is thus sufficient to take $c_1\mu(\delta) = s(\delta^\beta) - l(\delta)$ where for each n large, we choose $l(2^{-n}) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ in order to maximize the left hand side of (5.51).

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