A GENERALIZATION OF TIETZE'S THEOREM ON LOCAL CONVEXITY FOR OPEN SETS

J. CEL

Abstract

Let $S$ be a nonempty subset of a real topological linear space $L$ and $s$ a point in $\text{cl}S$. A point $s$ of weak local $C$-convexity of $S$ is defined as follows: if there exists a neighbourhood $N$ of $s$ such that $s \in \text{cl}C_s$, where $C_s$ is a component of $S \cap N$, then $[x, y] \subseteq S$ for each pair of points $x, y \in C_s$, otherwise $[x, y] \not\subseteq S$ for each pair of points $x, y$ in any component of $S \cap N$. It is proved that an open connected subset $S$ of $L$ whose boundary consists exclusively of $C$-wlc points of $S$ is convex. This is a version of the Sacksteder-Straus-Valentine generalization of Tietze's local characterization of convexity for open sets.

Key words: weak local $C$-convexity point, Tietze-type theorem.


Let $S$ be a nonempty subset of a real topological linear space $L$. A point $s$ in $\text{cl}S$ is said to be a point of weak local convexity of $S$ if and only if there is some neighbourhood $N$ of $s$ such that for each pair of points $x, y \in S \cap N$, $[x, y] \subseteq S$ [2, Def 4.2]. A point $s$ of weak local $C$-convexity of $S$ is defined as follows: if there exists a neighbourhood $N$ of $s$ such that $s \in \text{cl}C_s$, where $C_s$ is a component of $S \cap N$, then $[x, y] \subseteq S$ for each pair of points $x, y \in C_s$, otherwise $[x, y] \not\subseteq S$ for each pair of points $x, y$ in any component of $S \cap N$ (cf. 2, Def 4.5). Furthermore [1],[2, Def 4.3], a point $s$ in $\text{cl}S$ is said to be a point of strong local convexity ($C$-convexity) if and only if $S \cap N$ (each component of $S \cap N$) is convex for some neighbourhood $N$ of $s$ in $L$. For the sake of brevity, we call points of weak and strong local convexity ($C$-convexity) of $S$, respectively, wlc and slc ($C$-wlc and $C$-slc) points of $S$. $(xyz)$ will represent the two-dimensional flat determined by three noncollinear points $x, y, z$.

Tietze's famous characterization of convexity states that a closed connected subset $S$ of $L$ consisting exclusively of wlc points is convex [2, Th 4.4]. In [1], a generalization was proved that a connected compact subset $S$ of a complete locally convex real topological linear space consisting exclusively of $C$-slc points is convex. In [3, Cor 2.3], the author proved essentially that an open connected subset $S$ of $L$ whose boundary consists exclu-
sively of wlc points is convex. The purpose of this note is to prove a generalization of this result kept in the spirit of [1]. The straightforward argument differs from that in [3] and thanks to the assumption of the openness of S is simpler.

**Theorem.** If S is an open connected subset of a real topological linear space L with the boundary consisting exclusively of C-wlc points of S, then S is convex.

**Proof.** Since S is open and connected, and L is locally starshaped [2, Th.1.4], an easy argument reveals that S must be polygonally connected. For this, we fix a point $x_0$ in S and consider the subset $W$ of S consisting of points which can be joined to $x_0$ via polygonal paths in S. The local starshapedness of S implies immediately that W is simultaneously open and closed in S, so that it must coincide with connected S. Select arbitrarily distinct points $x, y$ of S. Let $[x, y] \not\subseteq S$. Hence, there exists in S a simple polygonal path $P = [x_0, x_1] \cup \ldots \cup [x_n, x_{n+1}] (n \geq 1, x_0 = x, x_{n+1} = y)$ with the minimal number $n + 1$ of nondegenerate line segments. Consider the subpath $[x_0, x_1] \cup [x_1, x_2]$. Of course, the points $x_0, x_1, x_2$ are noncollinear. By [2, Th.1.8], we can identify $(x_0, x_1, x_2)$ in the topology induced from L with the Euclidean plane $\mathbb{R}^2$. Without loss of generality, assume that $x_1$ is the origin of L. Since $[x_1, x_2]$ is compact and $S \cap (x_0, x_1, x_2)$ is relatively open in $(x_0, x_1, x_2)$, there exists a relatively open circle $B$ in $(x_0, x_1, x_2)$ centered at $x_1$ such that $[x_1, x_2] + B \subseteq S$. If $x_0 \in B$, then $[x_0, x_2] \subseteq S$ and $P$ can be replaced by a path consisting of line segments, a contradiction. Denote thus $(t, x_1) = B \cap [x_0, x_1]$. Then $conv([t, x_1] \cup [x_1, x_2]) \subseteq S$. Since S is open, there is a point $x_0' \in S$ such that $x_0 \in (x_0', x_1)$. Suppose, to reach a contradiction, that $conv((x_0', x_1) \cup [x_1, x_2]) \not\subseteq S$. Then there exists the largest subsegment $[w, x_1]$ of $[x_0', x_1]$ such that $conv((w, x_1) \cup [x_1, x_2]) \subseteq S$. Suppose that $[x_2, u] \not\subseteq S$. Since $x_2 + B \subseteq S$, there exists a largest subsegment $[x_2, u]$ of $[x_2, w]$ contained in $S$. $u \in \partial S$, so that, by initial assumption, $u$ is a C-wlc point of S. Consequently, there exists a relatively open circle $D$ in $(x_0, x_1, x_2)$ centered at $u$ such that for the component $C_u$ of $D \cap S$ for which $u \in \partial C_u$, if $p, q \in C_u$, then $[p, q] \subseteq S$. Pick out a point $a \in D \cap (x_2, u)$. $a \in S$ and S is open, so that there is a relatively open circle $D_a$ in $(x_0, x_1, x_2)$ centered at $a$ and contained in $D \cap S$. $D_a$ and $D \cap conv((w, x_1) \cup [x_1, x_2])$ lie in the same component of $D \cap S$ having $u$ in its closure, so that by assumption $u \in conv(D_a \cup (D \cap conv((w, x_1) \cup [x_1, x_2]))) \subseteq S$ and the segment $[x_2, u]$ can be extended beyond $u$ in $S$, a contradiction. Hence, $[x_2, u] \subseteq S$. But S is open and $[x_2, u]$ is compact, so that there exists a point $w' \in (x_2, u)$ such that $conv((w', x_1) \cup [x_1, x_2]) \subseteq S$, contradictory to the choice of $w$. Hence, $[x_0, x_2] \subseteq conv((x_0', x_1) \cup [x_1, x_2]) \subseteq S$. Thus we can replace the path $P$ by the path $[x_0, x_2] \cup \ldots \cup [x_n, x_{n+1}]$ consisting of at most $n$ line segments, contradictory to the choice of $P$. Hence, $[x, y] \subseteq S$ and S is convex, by the arbitrary choice of $x, y$.

The proof is complete. \(\square\)

It is still an open question if the assumptions of the theorems in [1] (cf. [2, Ths. 4.5 and 4.6]) can be weakened in any way.

**Acknowledgment**

Thanks are due to the Mathematical Institute of the Polish Academy of Sciences and the Circuit Theory Division of the Technical University of Łódź for the support during the preparation of this paper.
References


J. Cel
Warszawska 24c/20
26-200 Końskie
Poland