Bulletin de la Société Royale des Sciences de Liège, Vol. 67, 1-2, 1998, pp. 35-42

FACTORIZATION OF FINITE-RANK OPERATORS IN BANACH OPERATOR IDEALS

Klaus FLORET

Abstract

A condition is given which implies that every finite-rank operator between arbitrary Banach spaces factors through an operator between Banach spaces of finite dimension under control of its norm in a given Banach operator ideal $\mathcal{A}$. This property of $\mathcal{A}$ is an operator-theoretic version of Grothendieck's notion of total accessibility of tensor norms.

1 The result

1.1. Given a quasi-normed operator ideal $(\mathcal{A}, \boldsymbol{A})$ (see [7] and [2] for the definitions and notation) it is interesting to know under which circumstances every $T \in \mathcal{F}(E;F)$ (where E, F are Banach spaces, $\mathcal{F}$ the ideal of finite-rank operators) and $\varepsilon > 0$ there are a closed finite-codimensional subspace $L \subset E$ (notation: $L \in \mathrm{COFIN}(E)$), a finite-dimensional subspace $N \subset F$ (notation: $N \in \mathrm{FIN}(F)$) and an operator $T_0 \in \mathcal{L}(E/L; N)$ such that

$$\begin{array}{ccc} E & \xrightarrow{\quad T \quad} & F \\ Q_L^E \downarrow & \circlearrowleft & \uparrow I_M^E \\ E/L & \xrightarrow{\quad T_0 \quad} & N \end{array} \qquad \text{and } \boldsymbol{A}(T_0) \le (1+\varepsilon)\boldsymbol{A}(T) \qquad (*)$$

If E' and F have the metric approximation property (m.a.p. for short) this is possible: for a proof use that if F (resp. E') has m.a.p., then for every $T \in \mathcal{F}(E;F)$ and $\varepsilon > 0$ there is an $R \in \mathcal{F}(F;F)$ (resp. $\in \mathcal{F}(E;E)$) with $\|R\| \le 1 + \varepsilon$ and $R \circ T = T$ (resp. $T \circ R = T$); this *lemma* (see [2, 16.9.] for a proof) will be used various times in this note.

1.2. A quasi-normed operator ideal $(\mathcal{A}, \boldsymbol{A})$ is called *totally accessible*, if $(*)$ holds for all Banach spaces E, F; it is called *right-accessible* if $(*)$ holds for all

présenté par J. Schmets
accepté le 14 mai 1998

F and finite-dimensional E and *left-accessible* if $(*)$ holds for all E and finite-dimensional F. An *accessible* operator ideal is, by definition, right- and left-accessible. In 1992 Pisier proved the existence of a maximal Banach operator ideal which is neither right- nor left-accessible (see [2, 31.6.]). Minimal p-Banach operator ideals are always accessible, but not necessarily totally accessible ([2, 25.3.]).

1.3. The various notions of accessibility were originally introduced for tensor norms by Grothendieck [5] in 1956; in 1986 Defant [1] "translated" them to quasi-Banach operator ideals. Accessible ideals were used before by Reisner [8] under the name "semi-tensorial". More information about accessible operator ideals can be found in [2] and [3]; in particular, [2, 21.4.] is a useful criterion for total accessibilty of composition ideals.

1.4. The following – more or less known – characterization shows how trace-duality and accessibility are linked together. Recall that for a quasi-normed ideal $(\mathcal{A}, \boldsymbol{A})$ the *adjoint* Banach operator ideal $(\mathcal{A}^*, \boldsymbol{A}^*)$ is defined to be the class of all operators such that

$$\boldsymbol{A}^*(T : E \longrightarrow F) := \sup \left\{ \left| \operatorname{tr} \begin{array}{ccc} E & \xrightarrow{T} & F \\ \uparrow & & \downarrow \\ M & \xleftarrow{U} & F/L \end{array} \right| \;\middle|\; \begin{array}{c} M \in \mathrm{FIN}(E),\ L \in \mathrm{COFIN}(F) \\ \boldsymbol{A}(U : F/L \longrightarrow M) \leq 1 \end{array} \right\}$$

is finite. Note that $(\mathcal{A}^{**}, \boldsymbol{A}^{**}) = (\mathcal{A}, \boldsymbol{A})$ if and only $(\mathcal{A}, \boldsymbol{A})$ is maximal and normed (and hence Banach). It is clear from the definition that for finite-dimensional Banach spaces N, M (notation: $N, M \in \mathrm{FIN}$) the following inequality holds:

$$|\operatorname{tr}_N(N \xrightarrow{S} M \xrightarrow{T} N)| \leq \boldsymbol{A}(S)\boldsymbol{A}^*(T). \qquad (**)$$

PROPOSITION. *For each maximal normed operator ideal* $(\mathcal{A}, \boldsymbol{A})$ *the following statements are equivalent:*

(a) $(\mathcal{A}, \boldsymbol{A})$ *is right-accessible.*

(b) $(\mathcal{A}^*, \boldsymbol{A}^*)$ *is left-accessible.*

(c) *For each* $N \in \mathrm{FIN}$, *Banach space* E, *operators* $S \in \mathcal{L}(N; E)$ *and* $T \in \mathcal{L}(E; N)$ *the inequality*

$$|\operatorname{tr}_N(N \xrightarrow{S} E \xrightarrow{T} N)| \leq \boldsymbol{A}(S)\boldsymbol{A}^*(T)$$

holds.

PROOF: The equivalence of (a) and (b) is well-known ([2, 21.3., 15.6. and 17.9.]). It follows immediately from (**) and the ideal property that (a) implies (c). To show the converse one has to verify (by Örtel's result [2, 25.4.]) that $\mathcal{A}^* \circ \mathcal{A} \subset \mathcal{I}$ and $\boldsymbol{I} \leq \boldsymbol{A}^* \circ \boldsymbol{A}$ where $(\mathcal{I}, \boldsymbol{I})$ denotes the maximal ideal of integral operators; using $(\mathcal{I}, \boldsymbol{I}) = (\mathcal{L}, \|\ \|)^*$ and (c), this follows (notation as in the definition of the adjoint ideal) from

$$\begin{aligned} |\mathrm{tr}(M \hookrightarrow E \xrightarrow{S} G \xrightarrow{T} F \twoheadrightarrow F/L \xrightarrow{U} M)| &\leq \boldsymbol{A}(S \circ I_M^E)\boldsymbol{A}^*(U \circ Q_L^F \circ T) \\ &\leq \boldsymbol{A}(S)\boldsymbol{A}^*(T)\|U\|\,. \end{aligned}$$

■

1.5. To facilitate the statement and use of the main result it is helpful to say that a quasi-normed operator ideal $(\mathcal{B}, \boldsymbol{B})$ satisfies the *m.a.p. factorization property* if for all $T \in \mathcal{B}(E;F)$ and $\varepsilon > 0$ there are Banach spaces G_1 and G_2 both with m.a.p. and operators $T_0 \in \mathcal{B}$ and $U, V \in \mathcal{L}$ such that

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \hookrightarrow & F'' \\ {\scriptstyle U}\downarrow & & \circlearrowleft & & \uparrow{\scriptstyle V} \\ G_1 & & \xrightarrow{T_0} & & G_2 \end{array} \qquad \text{and } \|U\|\,\|V\|\boldsymbol{B}(T_0) \leq (1+\varepsilon)\boldsymbol{B}(T)\,. \qquad (*\!*\!*)$$

THEOREM. *Let $(\mathcal{A}, \boldsymbol{A})$ be a normed operator ideal. If the adjoint ideal $(\mathcal{A}^*, \boldsymbol{A})$ has the m.a.p. factorization property, then $(\mathcal{A}, \boldsymbol{A})$ is totally accessible and $(\mathcal{A}^*, \boldsymbol{A})$ is accessible.*

I do not know whether the theorem holds for quasi-normed or p-normed ideals. In section 2 the result will be mostly used in the following form: *Let $\mathcal{A}$ be maximal and normed, then the m.a.p. factorization of $\mathcal{A}$ implies that $\mathcal{A}^*$ is totally accessible.* Unfortunately, I do also not know whether the converse of this is true (see also 2.7. below); the validity or non-validity may give some more information on the metric approximation property.

1.6. The proof is postponed to 1.7. since actually a slightly finer result holds; for its formulation define that $(\mathcal{B}, \boldsymbol{B})$ has the *right (*resp. *left) m.a.p. factorization property* if $(*\!*\!*)$ holds with only G_2 (resp. G_1) having m.a.p.

PROPOSITION. *Let $(\mathcal{A}, \boldsymbol{A})$ be a normed operator ideal. If $(\mathcal{A}^*, \boldsymbol{A}^*)$ has the left m.a.p. factorization property and is right-accessible or: if $(\mathcal{A}^*, \boldsymbol{A}^*)$ has the right m.a.p. factorization property and is left-accessible, then $(\mathcal{A}, \boldsymbol{A})$ is totally accessible and $(\mathcal{A}^*, \boldsymbol{A}^*)$ is accessible.*

PROOF: It is straightforward from the definition that a (quasi-) normed operator ideal is totally accessible if its maximal hull is. Therefore one may assume that $(\mathcal{A}, \boldsymbol{A})$ is a maximal normed operator ideal. Let α be the associated finitely generated tensor norm of $(\mathcal{A}, \boldsymbol{A})$ in the sense of [2, §17]. The claim about $\mathcal{A}$ is now equivalent to α being totally accessible ([2, 21.3.]) – and this is equivalent to

$$I : E \otimes_\alpha F \hookrightarrow \mathcal{A}(E'; F)$$

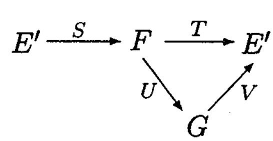
being isometric for all Banach spaces E, F; for a proof of this use [2] – more precisely: the definition 15.6. of totally accessible and the embedding theorem 17.6.. It is also clear from these results that $\|I\| \leq 1$ holds always and therefore it is enough to show that for every $z \in E \otimes F$ with associated $S \in \mathcal{F}(E'; F) \subset \mathcal{A}(E'; F)$

$$\alpha(z; E, F) \leq \boldsymbol{A}(S : E' \longrightarrow F)$$

holds. For this take φ in the unit ball of $(E \otimes_\alpha F)' = \mathcal{A}^*(F; E')$ (representation theorem!) with associated operator T such that

$$\alpha(z; E, F) = \langle \varphi, z \rangle = \operatorname{tr}_{E'}(T \circ S)$$

and $\boldsymbol{A}^*(T : F \longrightarrow E') \leq 1$, hence there is a factorization



according to the right or left m.a.p. factorization property. Since G has m.a.p. and $U \circ S$ has finite rank the lemma in 1.1. implies that there are $M \in \mathrm{FIN}(G)$ and $R \in \mathcal{L}(G; M)$ with $\|R\| \leq 1 + \varepsilon$ and $US = I_M^G RUS$.

Now suppose (using the left m.a.p. factorization property of $\mathcal{A}^*$) that $\|U\| \leq 1$ and $\boldsymbol{A}^*(V) \leq (1+\varepsilon)\boldsymbol{A}^*(T) \leq (1+\varepsilon)$; then proposition 1.4. implies that

$$\alpha(z; E, F) = \operatorname{tr}_{E'}(V I_M^G RUS) = \operatorname{tr}_M(RUS \circ V I_M^G) \leq$$
$$\leq \boldsymbol{A}^*(V I_M^G)\boldsymbol{A}(RUS) \leq (1+\varepsilon)^2 \boldsymbol{A}(S)\,.$$

For the other condition a similar argument gives the claim. Finally recall that α^* (and hence $\mathcal{A}^*$) is accessible if α is (see [2, 15.6.]). ■

It is worthwhile to note that, in terms of the associated tensor norm α of $\mathcal{A}^{\max}$ it was shown that $\alpha \leq \overleftarrow{\alpha}$ where $\overleftarrow{\alpha}$ is the cofinite hull of α (see [2, 12.4.]).

1.7. PROOF of theorem 1.5.: It is apparently enough to show that an operator ideal is right-accessible if it satisfies the (right) m.a.p. factorization property; but using the lemma in 1.1. and the principle of local reflexity (see e.g. [2, 6.6.]) this is immediate. ■

2 Examples and comments

2.1. The maximal Banach ideal $\mathcal{I}_p$ of p-integral operators ($1 \leq p \leq \infty$) has the m.a.p. factorization property hence its adjoint operator ideal $\mathcal{P}_{p'} = \mathcal{I}_p^*$ of *absolutely p'-summing operators* is totally accessible. More general, the ideal $\mathcal{L}_{p,q}$ of (p,q)-factorable operators (for $\frac{1}{p}+\frac{1}{q} \geq 1$) has the m.a.p. factorization property (see [2, 18.11.]) hence the adjoint operator ideal $\mathcal{D}_{p',q'} = \mathcal{L}_{p,q}^*$ of *(p',q')-dominated operators* is totally accessible. This result is seemingly due to Gilbert-Leih [4]; another proof was given by Defant in [1] (see also [2, 21.4.]) using Kwapien's factorization theorem $\mathcal{D}_{p',q'} = \mathcal{P}_{q'}^{\text{dual}} \circ \mathcal{P}_{p'}$.

2.2. Lopez Molina and Sanchez Pérez proved that the adjoint of Matter's interpolation ideal $\mathcal{P}_{p\sigma}$ of (p,σ) *absolutely summing operators* ($1 < p < \infty$ and $0 < \sigma < 1$) satisfies the m.a.p. factorization property ([6, theorem 16] with $G_1 = L_\infty$ and $G_2 := L_1 + L_p$), hence $\mathcal{P}_{p\sigma}$ is totally accessible. This is also proved in [6] using (p,σ)-nuclear operators and the RNP.

2.3. For $1 \leq r \leq \infty$ denote by $(\mathcal{L}_r, L_r)$ the ideal of *r-factorable operators*. If $\frac{1}{p}+\frac{1}{q} \geq 1$ the regular hull of the *composition ideal* $\mathcal{L}_p \circ \mathcal{L}_{q'}$ is maximal and normed (see [2, 28.7.]) and has obviously the m.a.p. factorization property hence $(\mathcal{L}_p \circ \mathcal{L}_{q'})^*$ is totally accessible; in [2, 28.7.] there is a description of these operators in terms of matrix inequalities. The same way (use [2, Ex. 28.7.]) one obtains that $(\mathcal{L}_p \circ \mathcal{I}_{q'})^*$ is totally accessible. Defant/Junge [11] show that $(\mathcal{L}_p \circ \mathcal{I}_{q'})^* = \mathcal{L}_{q,p}^{\text{inj,sur}} \circ \mathcal{P}_{p'}$. Note that in these examples the factorization is easy – the problem was to show for which (p,q) the ideals $(\mathcal{L}_p \circ \mathcal{L}_{q'})^{\text{reg}}$ and $(\mathcal{L}_p \circ \mathcal{I}_{q'})^{\text{reg}}$ are normed (and maximal); here $\mathcal{A}^{\text{reg}}$ denotes the regular hull of $\mathcal{A}$.

2.4. It is not known whether the ideals $\mathcal{L}_p$ of *p-factorable operators* are totally accessible for $p \neq 2$. They were if the adjoint ideal $\mathcal{L}_p^* = \mathcal{P}_p^{\text{dual}} \circ \mathcal{P}_{p'}$ (Kwapien's theorem) would have the m.a.p. factorization property – or at least the left or right one since $\mathcal{L}_p^*$ is accessible. In particular: Does the ideal of absolutely summing operators $\mathcal{P}_1 = \mathcal{L}_\infty^*$ have any of these factorization properties?

2.5. If in the factorization property of $\mathcal{A}^*$ in Theorem 1.5. the *metric* approximation property is substituted by the *bounded* approximation properties with constant $\leq c$, and only a fixed $\varepsilon_0 > 0$, then a check of the proof gives $\alpha \leq c^2(1+\varepsilon_0)\overleftarrow{\alpha}$

$$\alpha(z; E, F) \leq c^2(1+\varepsilon_0)\boldsymbol{A}(S_z : E' \longrightarrow F)$$

which gives an isomorphic version of total accessibility.

Recall from [2, 21.7.] that a Banach space E has the *bounded α-approximation property with constant* c if $\alpha \leq c\overleftarrow{\alpha}$ on $F \otimes E$ for all Banach spaces F. The usual bounded approximation is obtained by taking $\alpha = \pi$ (see [2, 16.2.]).

2.6. The results have also a *negative flavour:* if a normed ideal $(\mathcal{A}, \boldsymbol{A})$ is not totally accessible, then $\mathcal{A}^*$ does *not satisfy* the m.a.p. factorization property. Example: The ideal $\mathcal{I}_p$ of p-integral operators is not totally accessible for $1 \leq p < \infty$ and $p \neq 2$ (for $p = 1$ this is a consequence of the existence of spaces without the bounded approximation property, and for $p > 1$ this is due to the existence of spaces without the bounded g_p-approximation property given by Saphar; see [2, §21] for the arguments). It follows that for every $q \in]1, \infty]$ with $q \neq 2$ the ideal $\mathcal{P}_q = \mathcal{I}_{q'}^*$ of absolutely q-summing operators does not have the m.a.p. factorization property. Using the ideas of 2.5. it follows even that there are no constants $c, d \geq 1$ such that for every $T \in \mathcal{P}_q(E; F)$ there is a factorization

$$\begin{array}{ccccc} E & \xrightarrow{T} & F & \hookrightarrow & F'' \\ {\scriptstyle U}\downarrow & & \cdot/\cdot & & \uparrow{\scriptstyle V} \\ G_1 & & \xrightarrow{\quad T_0 \quad} & & G_2 \end{array}$$

with $\|U\| \, \|V\| \boldsymbol{P}_q(T_0) \leq d\boldsymbol{P}_q(T)$ and G_1 and G_2 having the bounded approximation property with constant $\leq c$.

2.7. If the adjoint $\mathcal{A}^*$ of a maximal normed operator ideal $(\mathcal{A}, \boldsymbol{A})$ is totally accessible, then every Banach space E with $\mathrm{id}_E \in \mathcal{A}$ has the bounded approximation with constant $\boldsymbol{A}(\mathrm{id}_E)$ (see [2, 21.6.]). This is a certain converse of the theorem 1.5..

2.8. It is not true that if a (not necessarily maximal) Banach operator ideal $\mathcal{A}$ satisfies the m.a.p. factorization property, the adjoint ideal $\mathcal{A}^*$ is totally accessible. For an example take $\mathcal{A} := \overline{\mathcal{F}}$ the ideal of approximable operators (the adjoint $\overline{\mathcal{F}}^* = \mathcal{L}^* = \mathcal{I}$, the integral operators, is not totally accessible) and observe that every $T \in \overline{\mathcal{F}}$ factors through an $\ell_2((N_n)_{n \in \mathbb{N}})$ with $T = UV$ and $\|U\| \, \|V\| \le 1 + \varepsilon$ and that $\overline{\mathcal{F}} \stackrel{1}{=} \overline{\mathcal{F}} \circ \overline{\mathcal{F}} \circ \overline{\mathcal{F}}$ (where all $N_n \in$ FIN).

2.9. The proof of the theorem suggests that there is a related result for bilinear forms. This is true and reads as follows:

PROPOSITION. *Let α be a finitely generated tensor norm such that for all $\varphi \in (E_1 \otimes_\alpha E_2)'$ and $\varepsilon > 0$ there are Banach spaces G_j with the metric approximation property, $R_j \in \mathcal{L}(E_j; G_j)$ of norm ≤ 1 and $\psi \in (G_1 \otimes_\alpha G_2)'$ such that $\varphi = \psi \circ (R_1 \otimes R_2)$ and*

$$\|\psi\|_{(G_1 \otimes_\alpha G_2)'} \le (1 + \varepsilon) \|\varphi\|_{(E_1 \otimes_\alpha E_2)'} ,$$

then α is totally accessible.

PROOF: One has to show that $\alpha \le \overleftarrow{\alpha}$ (the cofinite hull of α). For this take $z \in E_1 \otimes E_2$ and $\varphi \in (E_1 \otimes_\alpha E_2)'$ with norm 1 and $\alpha(z; E, F) = \langle \varphi, z \rangle$. Take, for given $\varepsilon > 0$, a factorization as in the assumption, then

$$\begin{aligned} \alpha(z; E, F) &= \langle \varphi, z \rangle = \langle \psi, R_1 \otimes R_2(z) \rangle \le \\ &\le (1+\varepsilon) \alpha\Big(R_1 \otimes R_2(z); G_1, G_2\Big) = (1+\varepsilon) \overleftarrow{\alpha}\Big(R_1 \otimes R_2(z); G_1, G_2\Big) \le \\ &\le (1+\varepsilon) \|R_1\| \, \|R_2\| \overleftarrow{\alpha}(z; E_1, E_2) \le (1+\varepsilon) \overleftarrow{\alpha}(z; E_1, E_2) \end{aligned}$$

by the approximation lemma [2, 13.1.] (here the m.a.p. of the G_j is used) and the fact that $\overleftarrow{\alpha}$ is also a tensor norm hence satisfies the metric mapping property. ∎

The translation of this result to maximal normed operator ideals as in 1.5., however, would require a factorization with G_1 and the *dual* G_2' having the m.a.p..

Acknowledgement: I thank Andreas Defant for various very helpful conversations on the topic.

References

[1] DEFANT, A.: *Produkte von Tensornormen;* Habilitationsschrift, Oldenburg, 1986.

[2] DEFANT, A. –K. FLORET: *Tensor norms and operator ideals;* North Holland Math. Studies 176, 1993.

[3] FLORET, K.: *Trace duality and accessible operator ideals;* 44° Sem. Bras. Análise (1996) 119–151.

[4] GILBERT, J.E. – T. LEIH: *Factorization, tensor products and bilinear forms in Banach space theory;* Notes in Banach spaces, Univ. Texas Press (1980) 182–305.

[5] GROTHENDIECK, A.: *Résumé de la théorie métrique des produits tensoriels topologiques;* Bol. Soc. Mat. São Paulo 8 (1956) 1–79.

[6] LOPEZ MOLINA, J.A. – E.A. SÁNCHEZ PÉREZ: *Tensor norms factoring through interpolation spaces;* preprint 1997.

[7] PIETSCH, A.: *Operator ideals;* Deutscher Verlag der Wiss., 1978 and North-Holland 1980.

[8] REISNER, S.: *On Banach spaces having the property G.L.;* Pacific J. Math. 83 (1979) 505–521.

[9] SAPHAR, P.: *Produits tensoriels d'espaces de Banach et classes d'applications linéaires;* Studia Math. 38 (1970) 71–100.

[10] MATTER-CHO, U.: *Absolutely continuous operators and super-reflexivity;* Dissertation University Zürich, 1985.

[11] DEFANT, A. – M. JUNGE: *On some matrix inequalities in Banach spaces;* Rev. R. Acad. Cien. (Madrid) 90 (1996) 133-140.

Klaus Floret
Fachbereich Mathematik
Carl von Ossietzky Universität Oldenburg
26111 Oldenburg, Germany
email: floret@mathematik.uni-oldenburg.de