Bulletin de la Société Royale des Sciences de Liège, Vol. 67, 1-2, 1998, pp. 69-75

SYMMETRIES OF A COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM

L. REMEZO

Key-words : classical and quantum Hamiltonians, symmetries, algebras.

Mots-clés : Hamiltoniens classique et quantique, symétries, algèbres.

ABSTRACT

We put in evidence the symmetries associated with the *quantum* versions of a completely integrable Hamiltonian recently introduced by Calogero. The corresponding symmetry algebra is seen to be sp(2;IR).

RESUME

Nous déterminons les symétries associées aux versions quantiques d'un Hamiltonien complètement intégrable récemment introduit par Calogero. L'algèbre de symétrie correspondante pour chacune des versions est sp(2;IR).

* Institut de Physique au Sart Tilman, Bâtiment B5, B-4000 LIEGE 1 (Belgique)

présenté par J. Beckers le 14 mai 1998
accepté le 20 mai 1998

I Introduction

Recently, much attention has been devoted to the study of a particular family of hamiltonian systems

$$H = \sum_{j,k=1}^{n} p_j p_k \, f(q_j - q_k) \tag{I.1}$$

where the function f fulfills the conditions

$$f(-x) = f(x) \text{ and } f'(-x) = -f'(x) \ , \tag{I.2}$$

the prime meaning the first derivative with respect to x. The main reason is its possible application in fluids mechanics and non linear partial differential equations [1]. Calogero played an important role in that development. For example he has established the complete integrability [2] and the corresponding quantum versions [3] of the classical Hamiltonian

$$H = \sum_{j,k=1}^{n} p_j p_k \, [\lambda + \mu \, \cos(q_j - q_k)] \tag{I.3}$$

where n is an arbitrary positive integer and λ, μ are arbitrary real constants.

In this paper we want to complete Calogero's considerations by searching for first order symmetry operators associated with two *quantum* versions [3] of (I.3), i.e.

$$\begin{aligned} H^{(1)} &= \lambda P^2 + \mu \sum_{j,k=1}^{n} (\alpha \, \cos(q_j - q_k) p_j p_k + \beta \, p_j p_k \, \cos(q_j - q_k) \\ &+ \, (1 - \alpha - \beta) \, p_j \, \cos(q_j - q_k) \, p_k) \ , \end{aligned} \tag{I.4}$$

and

$$H^{(2)} = \lambda \, P^2 + \mu \sum_{j,k=1}^{n} p_j e^{i(q_j - q_k)} p_k \tag{I.5}$$

where α and β are arbitrary constants and

$$P = \sum_{j=1}^{n} p_j \ . \tag{I.6}$$

Our motivations are the increasing interest in the above Hamiltonians and the fundamental role of symmetries and algebras in quantum physics. We limit ourselves for simplicity to the case $\lambda + \mu = 0$ and $n = 2$. The associated algebra for both (I.4) and (I.5) is shown to be $sp(2; \Re)$.

The second section deals with symmetries of $H^{(1)}$ while those of $H^{(2)}$ are presented in the third one.

II Symmetries of $H^{(1)}$

As hermitian Hamiltonians are physically interesting we consider the case $\alpha = \beta$ in (I.4) and deal with :

$$\begin{aligned} H^{(1)} &= g_1(q_1,q_2)p_1p_2 + g_2(q_1,q_2)p_1 \\ &+ g_3(q_1,q_2)p_2 + g_4(q_1,q_2) \end{aligned} \tag{II.1}$$

where

$$\begin{aligned} g_1 &= 2\lambda\,[1-\cos(q_1-q_2)] \quad , \quad g_2 = i\hbar\lambda\ \sin(q_1-q_2)\ , \\ g_3 &= -i\hbar\lambda\ \sin(q_1-q_2) \quad , \quad g_4 = 2\alpha\hbar^2\lambda\ \cos(q_1-q_2) \end{aligned} \tag{II.2}$$

Next, let us recall [4] that a first order linear differential operator

$$Q = f_1(q_1,q_2)p_1 + f_2(q_1,q_2)p_2 + f_3(q_1,q_2) \tag{II.3}$$

is a symmetry operator of the Schrödinger equation associated with $H^{(1)}$ if it transforms a solution into another one. Here we deal with the time-independent Schrödinger equation and the condition on Q is then [5]

$$\left[H^{(1)}, Q\right] = \gamma Q \tag{II.4}$$

where γ is a constant.

When the operator equation (II.4) is applied to a function one gets the following system :

$$\begin{aligned} g_1\frac{\partial f_1}{\partial q_2} &= 0\ , \\ g_1\frac{\partial f_2}{\partial q_1} &= 0\ , \\ \frac{\hbar}{i}g_1\frac{\partial f_1}{\partial q_1} + \frac{\hbar}{i}g_1\frac{\partial f_2}{\partial q_2} &- \frac{\hbar}{i}f_1\frac{\partial g_1}{\partial q_1} - \frac{\hbar}{i}f_2\frac{\partial g_1}{\partial q_2} = 0\ , \\ -\hbar^2 g_1\frac{\partial^2 f_2}{\partial q_1\partial q_2} + \frac{\hbar}{i}g_1\frac{\partial f_3}{\partial q_1} + \frac{\hbar}{i}g_2\frac{\partial f_2}{\partial q_1} &+ \frac{\hbar}{i}g_3\frac{\partial f_2}{\partial q_2} - \frac{\hbar}{i}f_1\frac{\partial g_3}{\partial q_1} - \frac{\hbar}{i}f_2\frac{\partial g_3}{\partial q_2} = \gamma f_2\ , \\ -\hbar^2 g_1\frac{\partial^2 f_1}{\partial q_1\partial q_2} + \frac{\hbar}{i}g_1\frac{\partial f_3}{\partial q_2} + \frac{\hbar}{i}g_2\frac{\partial f_1}{\partial q_1} &+ \frac{\hbar}{i}g_3\frac{\partial f_1}{\partial q_2} - \frac{\hbar}{i}f_1\frac{\partial g_2}{\partial q_1} - \frac{\hbar}{i}f_2\frac{\partial g_2}{\partial q_2} = \gamma f_1\ , \\ -\hbar^2 g_1\frac{\partial^2 f_3}{\partial q_1\partial q_2} + \frac{\hbar}{i}g_2\frac{\partial f_3}{\partial q_1} + \frac{\hbar}{i}g_3\frac{\partial f_3}{\partial q_2} &- \frac{\hbar}{i}f_1\frac{\partial g_4}{\partial q_1} - \frac{\hbar}{i}f_2\frac{\partial g_4}{\partial q_2} = \gamma f_3 \end{aligned} \tag{II.5}$$

The use of (II.2) reduces the system to :

$$\begin{aligned}
\frac{\partial f_1}{\partial q_2} &= 0 \ , \\
\frac{\partial f_2}{\partial q_1} &= 0 \ , \\
[1-\cos(q_1-q_2)]\left(\frac{\partial f_1}{\partial q_1}+\frac{\partial f_2}{\partial q_2}\right) &- \sin(q_1-q_2)(f_1-f_2)=0 \ , \\
2\lambda\,[1-\cos(q_1-q_2)]\,\frac{\partial f_3}{\partial q_1} &- i\hbar\lambda\ \sin(q_1-q_2)\frac{\partial f_2}{\partial q_2} \\
&+ i\hbar\lambda\ \cos(q_1-q_2)(f_1-f_2)=\frac{i}{\hbar}\gamma f_2 \ , \\
2\lambda\,[1-\cos(q_1-q_2)]\,\frac{\partial f_3}{\partial q_2} &+ i\hbar\lambda\ \sin(q_1-q_2)\frac{\partial f_1}{\partial q_1} \\
&- i\hbar\lambda\ \cos(q_1-q_2)(f_1-f_2)=\frac{i}{\hbar}\gamma f_1 \ , \\
-i\hbar 2\lambda\,[1-\cos(q_1-q_2)]\,\frac{\partial^2 f_3}{\partial q_1\partial q_2} &+ i\hbar\lambda\ \sin(q_1-q_2)\left(\frac{\partial f_3}{\partial q_1}-\frac{\partial f_3}{\partial q_2}\right) \\
&+ 2\alpha\hbar^2\lambda\ \sin(q_1-q_2)(f_1-f_2)=\frac{i}{\hbar}\gamma f_3 \ .
\end{aligned} \tag{II.6}$$

A suitable change of variables to solve (II.6) is

$$\begin{cases} x = q_1 - q_2 \\ y = q_1 + q_2 \end{cases} \tag{II.7}$$

which leads to :

$$\begin{aligned}
\frac{\partial f_1}{\partial y} &= \frac{\partial f_1}{\partial x} \ , \\
\frac{\partial f_2}{\partial y} &= -\frac{\partial f_2}{\partial x} \ , \\
2(1-\cos x)\frac{\partial}{\partial x}(f_1-f_2) &- \sin x(f_1-f_2)=0 \ , \\
2\lambda(1-\cos x)\left(\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial y}\right)+2i\hbar\lambda\ \sin x\frac{\partial f_2}{\partial x} &+ i\hbar\lambda\ \cos x(f_1-f_2)=\frac{i}{\hbar}\gamma f_2 \ , \\
2\lambda(1-\cos x)\left(-\frac{\partial f_3}{\partial x}+\frac{\partial f_3}{\partial y}\right)+2i\hbar\lambda\ \sin x\frac{\partial f_1}{\partial x} &- i\hbar\lambda\ \cos x(f_1-f_2)=\frac{i}{\hbar}\gamma f_1 \ , \\
2i\hbar\lambda(1-\cos x)\left(\frac{\partial^2 f_3}{\partial x^2}-\frac{\partial^2 f_3}{\partial y^2}\right)+2i\hbar\lambda\ \sin x\frac{\partial f_3}{\partial x} &+ 2\alpha\hbar^2\lambda\ \sin x(f_1-f_2)=\frac{i}{\hbar}\gamma f_3 \ .
\end{aligned} \tag{II.8}$$

Solving first the subsystem of the first three equations, we obtain :

$$\begin{aligned} f_1 &= \frac{-1}{2\sqrt{1-\cos x}}\left[be^{\frac{i}{2}y}(e^{ix}-1)+ce^{-\frac{i}{2}y}(e^{-ix}-1)\right]+a \ , \\ f_2 &= \frac{1}{2\sqrt{1-\cos x}}\left[be^{\frac{i}{2}y}(e^{-ix}-1)+ce^{-\frac{i}{2}y}(e^{ix}-1)\right]+a \ , \end{aligned} \tag{II.9}$$

where a, b and c are arbitrary constants.

For the remaining last three equations, we have a solution only in the case

$$\gamma = 0 \ \text{ and } \ \alpha = \frac{1}{4} \tag{II.10}$$

which is

$$f_3 = \frac{-i\hbar\sin x}{2\sqrt{1-\cos x}}\left(be^{\frac{i}{2}y}+ce^{-\frac{i}{2}y}\right)+d \tag{II.11}$$

d being an arbitrary constant.
Then the solution of the whole system is given by equations (II.9) and (II.11).

Setting one for each of the above constants (a, b, c and d) and zero for the remaining ones, we obtain four symmetry operators associated with $H^{(1)}$. They correspond to the following expressions (II.3) :

$$\begin{aligned} Q_1 &= p_1+p_2 \ , \\ Q_2 &= \frac{e^{\frac{i}{2}y}}{2\sqrt{1-\cos x}}\left[-(e^{ix}-1)p_1+(e^{-ix}-1)p_2-i\hbar\ \sin x\right] \ , \\ Q_3 &= \frac{e^{-\frac{i}{2}y}}{2\sqrt{1-\cos x}}\left[-(e^{-ix}-1)p_1+(e^{ix}-1)p_2-i\hbar\ \sin x\right] \ , \\ Q_4 &= I \end{aligned} \tag{II.12}$$

where I denotes the identity operator.

The next step is to derive the corresponding algebra. From commutation relations we deduce that the operators Q_1, Q_2 and Q_3 form the algebra $sp(2,\Re) \cong so(2,1)$ as ($\hbar = 1$) :

$$[Q_2,Q_3] = -Q_1 \ , \ [Q_1,Q_2] = Q_2 \ \text{ and } \ [Q_1,Q_3] = -Q_3 \ . \tag{II.13}$$

They evidently commute with Q_4.

III Symmetries of $H^{(2)}$

We first extend the expression (I.5) of $H^{(2)}$ in powers of p_1 and p_2 like (II.1) for $H^{(1)}$:

$$H^{(2)} = h_1(q_1,q_2)p_1p_2 + h_2(q_1,q_2)p_1 + h_3(q_1,q_2)p_2 + h_4(q_1,q_2) \tag{III.1}$$

where

$$h_1 = 2\lambda\left[1-\cos(q_1-q_2)\right] \quad , \quad h_2 = -\hbar\lambda e^{-i(q_1-q_2)} \ ,$$
$$h_3 = -\hbar\lambda e^{i(q_1-q_2)} \quad , \quad h_4 = 0 \ . \tag{III.2}$$

Next, we follow step by step the procedure initiated in section II to put in evidence symmetry operators, the difference being the replacement of (II.2) by (III.2) and its consequences. Then, it turns out that symmetry operators associated with $H^{(2)}$ are given by :

$$\begin{aligned} K_1 &= p_1 + p_2 \ , \\ K_2 &= \frac{e^{\frac{i}{2}y}}{2\sqrt{1-\cos x}}\left[-(e^{ix}-1)p_1 + (e^{-ix}-1)p_2 - 2i\hbar\sin x\right] \ , \\ K_3 &= \frac{e^{-\frac{i}{2}y}}{2\sqrt{1-\cos x}}\left[-(e^{-ix}-1)p_1 + (e^{ix}-1)p_2\right] \ , \\ K_4 &= I \ . \end{aligned} \tag{III.3}$$

Commuting the above operators one gets the following central extension ($\hbar = 1$) :

$$[K_1, K_2] = K_2 \quad , \quad [K_1, K_3] = -K_3 \ ,$$
$$[K_2, K_3] = -(K_1 + K_4) \quad , \quad [K_i, K_4] = 0 \ ; \ i = 1, 2, 3 \tag{III.4}$$

which is isomorphic with $sp(2;\Re)$. Then we can say that $H^{(2)}$ is also associated with $sp(2;\Re)$.

Let us remark that the third quantum version of H [3] proposed by Calogero cannot be extended in integer powers of p_1 and p_2 with the coefficients depending only on q_1 and q_2. Our method cannot be applied to it.

References

[1] R. Camassa and D.D. Holm, , Phys.Rev.Lett. **71** (1993) 1661.

[2] F. Calogero, Phys.Lett. **A201** (1995) 306.

[3] F. Calogero and J.F. Van Diejen, J.Math.Phys **37**(9) (1996) 4243.

[4] W.I. Fushchich and A.G. Nikitin, Symmetries of equations of quantum mechanics, Allerton Press, Inc., New York (1994).

[5] P.J. Olver, Applications of Lie groups to differential equations, Springer, New York, (1986).