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A NOTE ON A CLASS OF ANALYTIC FUNCTIONS IN THE UNIT DISK II *

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ABSTRACT. Let $A(\alpha)$ be the class of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk U and satisfy

$$|f(z)/z - 1| < \alpha \qquad (z \in U)$$

for some α $(0 < \alpha \leqq 1)$. The object of the present paper is to show some distortion theorems for the fractional calculus of f(z) belonging to the class $A(\alpha)$.

I. INTRODUCTION

Many essentially equivalent definitions of the fractional calculus (that is, the fractional integrals and the fractional derivatives) have been given in the literature (cf., e.g., [2], [4], [5], [8], [9], and [10]). We find it convenient to recall here the following definitions which were used by Owa [6].

DEFINITION 1. The fractional integral of order λ is defined by

$$(1.1) \qquad D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta ,$$

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where $\lambda > 0$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z - \zeta)^{\lambda-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

DEFINITION 2. The fractional derivative of order λ is defined by

$$(1.2) \qquad D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} \, d\zeta ,$$

where $0 < \lambda < 1$, $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z - \zeta)^{-\lambda}$ is removed by requiring $\log(z - \zeta)$ to be real when $(z - \zeta) > 0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n + \lambda)$ is defined by

$$(1.3) \qquad D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) ,$$

where $0 < \lambda < 1$ and $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$.

Let $\mathcal{A}(\alpha)$ denote the class of functions of the form

$$(1.4) \qquad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the unit disk $\mathcal{U} = \{z: |z| < 1\}$ and the satisfy the following condition

$$(1.5) \qquad \left| \frac{f(z)}{z} - 1 \right| < \alpha \qquad (z \in \mathcal{U})$$

for some α $(0 < \alpha \leq 1)$.

This class $\mathcal{A}(\alpha)$ was studied by Padmanabhan [7] and Chandra and Singh [1].

Now, we need the following lemma by Nehari [3].

LEMMA. *Let $\phi(z)$ be analytic in the unit disk $\mathcal{U}$ and satisfy $|\phi(z)| \leqq 1$ for $z \in \mathcal{U}$. Then*

$$(1.6) \qquad |\phi'(z)| \leqq \frac{1 - |\phi(z)|^2}{1 - |z|^2} \qquad (z \in \mathcal{U}).$$

2. DISTORTION THEOREMS

THEOREM I. *Let the function $f(z)$ defined by (1.4) belong to the class $\mathcal{A}(\alpha)$. Then we have*

$$(2.1) \qquad |D_z^{1+\lambda} f(z)| \geqq \operatorname{Max}\left\{0, \frac{1}{\Gamma(2-\lambda)|z|^\lambda}\left\{(1-\lambda) - (2-\lambda)\alpha|z| - \frac{\alpha|z|^2}{1-|z|^2}\right\}\right\}$$

and

$$(2.2) \qquad |D_z^{1+\lambda} f(z)| \leqq \frac{1}{\Gamma(2-\lambda)|z|^\lambda}\left\{(1-\lambda) + (2-\lambda)\alpha|z| + \frac{\alpha|z|^2}{1-|z|^2}\right\}$$

for $0 < \alpha \leqq 1$, $0 < \lambda < 1$ and $z \in \mathcal{U} - \{0\}$.

PROOF. Let the function $g(z)$ be defined by

$$(2.3) \qquad g(z) = \frac{\Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)}{z} - 1,$$

where $0 < \lambda < 1$. Then $g(z)$ is analytic in the unit disk $\mathbb{U}$ and has simple zero at the origin. Consequently we can write that

$$(2.4) \qquad g(z) = \frac{\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)}{z} - 1 = z\phi(z) ,$$

where $\phi(z)$ is an analytic function in the unit disk $\mathbb{U}$ and satisfies $|\phi(z)| < \alpha$ for $z \in \mathbb{U}$. Further we know that

$$(2.5) \qquad \left| \frac{\phi'(z)}{\alpha} \right| \leqq \frac{1 - |\phi(z)|^2/\alpha^2}{1 - |z|^2}$$

for $0 < \alpha \leqq 1$ and $z \in \mathbb{U}$ by means of Lemma.

Differentiating both sides of (2.4), we can show that

$$(2.6) \qquad \Gamma(2-\lambda)z^{1+\lambda}D_z^{1+\lambda}f(z) = (1-\lambda)\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z) + z^2\phi(z) + z^3\phi'(z)$$

$$= z(1-\lambda)(1 + z\phi(z)) + z^2\phi(z) + z^3\phi'(z)$$

$$= z(1-\lambda) + (2-\lambda)z^2\phi(z) + z^3\phi'(z) ,$$

hence further,

$$(2.7) \qquad D_z^{1+\lambda}f(z) = \frac{1}{\Gamma(2-\lambda)z^{\lambda}} \{(1-\lambda) + (2-\lambda)z\phi(z) + z^2\phi'(z)\}.$$

Thus we obtain two estimates which we require, because, with the aid of (2.5), we have

$$(2.8) \qquad |\phi'(z)| \leqq \frac{\alpha\{1 - |\phi(z)|^2/\alpha^2\}}{1 - |z|^2}$$

$$\leqq \frac{\alpha}{1 - |z|^2}$$

for $z \in \mathbb{U}$. This completes the proof of the theorem.

THEOREM 2. *Let the function* f(z) *defined by* (1.4) *belong to the class* $\mathcal{A}(\alpha)$. *Then we have*

$$(2.9) \qquad |D_z^{1-\lambda} f(z)| \geqq \operatorname{Max}\left\{ 0,\ \frac{|z|^{\lambda}}{\Gamma(2+\lambda)} \left\{ (1+\lambda) - (2+\lambda)\alpha|z| - \frac{\alpha|z|^2}{1-|z|^2} \right\} \right\}$$

and

$$(2.10) \qquad |D_z^{1-\lambda} f(z)| \leqq \frac{|z|^{\lambda}}{\Gamma(2+\lambda)} \left\{ (1+\lambda) + (2+\lambda)\alpha|z| + \frac{\alpha|z|^2}{1-|z|^2} \right\}$$

for $0 < \alpha \leqq 1$, $\lambda > 0$ *and* $z \in \mathcal{U}$.

PROOF. Let the function h(z) be defined by

$$(2.11) \qquad h(z) = \frac{\Gamma(2+\lambda) z^{-\lambda} D_z^{-\lambda} f(z)}{z} - 1$$

for $\lambda > 0$. Then we can show two estimates of the theorem in the same way as in the proof of Theorem 1.

3. FUNCTIONS WITH INITIAL ZERO COEFFICIENTS

In this section, we show two distortion theorems for functions $f(z) \in \mathcal{A}(\alpha)$ with initial zero coefficients.

THEOREM 3. *Let the function* f(z) *defined by*

$$(3.1) \qquad f(z) = z + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (p \in \mathbb{N} = \{1,2,3,\cdots\})$$

belong to the class $\mathcal{A}(\alpha)$. *Then we have*

$$(3.2) \qquad |D_z^{1+\lambda}f(z)| \geqq \mathrm{Max}\left\{0, \frac{1}{\Gamma(2-\lambda)|z|^{\lambda}}\left\{(1-\lambda) - (p+1-\lambda)\alpha|z|^{p} - \frac{\alpha|z|^{p+1}}{1-|z|^{2}}\right\}\right\}$$

and

$$(3.3) \qquad |D_z^{1+\lambda}f(z)| \leqq \frac{1}{\Gamma(2-\lambda)|z|^{\lambda}}\left\{(1-\lambda) + (p+1-\lambda)\alpha|z|^{p} + \frac{\alpha|z|^{p+1}}{1-|z|^{2}}\right\}$$

for $0 < \alpha \leqq 1$, $0 < \lambda < 1$ *and* $z \in \mathcal{U} - \{0\}$.

PROOF. Let the function G(z) be defined by

$$(3.4) \qquad G(z) = \frac{\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)}{z} - 1,$$

where $0 < \lambda < 1$. Then G(z) is analytic in the unit disk $\mathcal{U}$ and has p zeros at the origin. Hence we can write that

$$(3.5) \qquad G(z) = \frac{\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)}{z} - 1 = z^{p}\phi(z),$$

where $\phi(z)$ is an analytic function in the unit disk $\mathcal{U}$ and satisfies $|\phi(z)| < \alpha$ for $z \in \mathcal{U}$. Consequently we obtain that

$$(3.6) \qquad D_z^{1+\lambda}f(z) = \frac{1}{\Gamma(2-\lambda)z^{\lambda}}\{(1-\lambda) + (p+1-\lambda)z^{p}\phi(z) + z^{p+1}\phi'(z)\},$$

which gives two estimates of the theorem with (2.5). Thus we have the theorem.

REMARK 1. Let $p = 1$ in Theorem 3, then we have Theorem 1.

THEOREM 4. Let the function $f(z)$ defined by (3.1) belong to the class $\mathcal{A}(\alpha)$. Then we have

$$|D_z^{1-\lambda} f(z)| \geqq \operatorname{Max}\left\{0,\ \frac{|z|^{\lambda}}{\Gamma(2+\lambda)}\left\{(1+\lambda) - (p+1+\lambda)\alpha|z|^p - \frac{\alpha|z|^{p+1}}{1-|z|^2}\right\}\right\} \tag{3.7}$$

and

$$|D_z^{1-\lambda} f(z)| \leqq \frac{|z|^{\lambda}}{\Gamma(2+\lambda)}\left\{(1+\lambda) + (p+1+\lambda)\alpha|z|^p + \frac{\alpha|z|^{p+1}}{1-|z|^2}\right\} \tag{3.8}$$

for $0 < \alpha \leqq 1$, $\lambda > 0$ and $z \in \mathcal{U}$.

The proof of Theorem 4 is obtained by using the same technique as in the proof of Theorem 3.

REMARK 2. Let $p = 1$ in Theorem 4, then we have Theorem 2.

REMARK 3. We have not been able to obtain sharp estimates for $|D_z^{1+\lambda} f(z)|$ and $|D_z^{1-\lambda} f(z)|$ in our theorems.

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