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AN ALTERNATIVE PROOF OF A PROPERTY OF THE RADON TRANSFORM ON THE HARDY SPACE H^1

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1. INTRODUCTION

Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and R_ω be the Radon transform relatively to $\omega \in S^{n-1}$, i.e. the integral over almost every hyperplane $\langle x,\omega \rangle = t$.

In this note, we give a non-constructive proof of the following known property :

R_ω is a bounded surjective operator from the Hardy space $H^1(\mathbb{R}^n)$ onto $H^1(\mathbb{R})$.

The boundedness assertion is contained in [1] where it results from an atomic decomposition of H^1 functions. It is also included in [3] where it occurs as a corollary of an identity relating the Radon and Riesz transforms. A constructive proof of the surjectivity of R_ω is contained in [3] too.

The alternative proof we give here is essentially based on duality arguments. In particular, it brings out the fact that the dual operator of $R_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R})$ is $B_\omega : g \to g(\langle .,\omega \rangle) : BMO(\mathbb{R}) \to BMO(\mathbb{R}^n)$.

2. PROOF

By Fubini theorem, we see that

$$(\ast) \quad \int_{\mathbb{R}} g.R_\omega f \, dt = \int_{\mathbb{R}^n} B_\omega g.f \, dx$$

holds for every $f \in \mathcal{S}_o(\mathbb{R}^n)$ and $g \in BMO(\mathbb{R})$, where

$$\mathcal{S}_o(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : \int f \, dx = 0\} \quad \text{and} \quad B_\omega g = g(\langle .,\omega \rangle).$$

On the other hand, we notice the following facts :

a. $R_\omega f \in {}_o(\mathbb{R})$ whenever $f \in \mathcal{S}_o(\mathbb{R}^n)$. This is an easy consequence of the identity relating the Fourier transforms of f and $R_\omega f$.

b. $\mathcal{S}_o(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$, $n \geqslant 1$. This results from lemma 1.5 of [2]. Moreover, $\mathcal{S}_o(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ since it contains the dense subspace $H^1_o(\mathbb{R}^n)$ considered in [4], p 231.

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