AN ALTERNATIVE PROOF OF A PROPERTY OF THE RADON TRANSFORM
ON THE HARDY SPACE $H^1$

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1. INTRODUCTION

Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and $R_\omega$ be the Radon transform relatively to $\omega \in S^{n-1}$, i.e. the integral over almost every hyperplane $\langle x, \omega \rangle = t$.

In this note, we give a non-constructive proof of the following known property:

$R_\omega$ is a bounded surjective operator from the Hardy space $H^1(\mathbb{R}^n)$ onto $H^1(\mathbb{R})$.

The boundedness assertion is contained in [1] where it results from an atomic decomposition of $H^1$ functions. It is also included in [3] where it occurs as a corollary of an identity relating the Radon and Riesz transforms. A constructive proof of the surjectivity of $R_\omega$ is contained in [3] too.

The alternative proof we give here is essentially based on duality arguments. In particular, it brings out the fact that the dual operator of $R_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R})$ is $B_\omega : g \mapsto g(\langle ., \omega \rangle) : \text{BMO}(\mathbb{R}) \to \text{BMO}(\mathbb{R}^n)$.

2. PROOF

By Fubini theorem, we see that

$$\left( \int \mathbb{R} \, g \, R_\omega f \, dx \right) \mathbb{R} \, g \mathbb{R} \, f \, dx$$

holds for every $f \in \mathcal{G}_0(\mathbb{R}^n)$ and $g \in \text{BMO}(\mathbb{R})$, where

$$\mathcal{G}_0(\mathbb{R}^n) = \{ f \in \mathcal{G}(\mathbb{R}^n) : \int f \, dx = 0 \}$$

and $B_\omega g = g(\langle ., \omega \rangle)$.

On the other hand, we notice the following facts:

a. $R_\omega f \in \mathcal{G}_0(\mathbb{R}^n)$ whenever $f \in \mathcal{G}(\mathbb{R}^n)$. This is an easy consequence of the identity relating the Fourier transforms of $f$ and $R_\omega f$.

b. $\mathcal{G}_0(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$, $n \geq 1$. This results from lemma 1.5 of [2].

Moreover, $\mathcal{G}_0(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ since it contains the dense subspace $H^1(\mathbb{R}^n)$ considered in [4], p 231.

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