AN ALTERNATIVE PROOF OF A PROPERTY OF THE RADON TRANSFORM 
ON THE HARDY SPACE $H^1$ 

C. CARTON-LEBRUN

1. INTRODUCTION

Let $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and $R_\omega$ be the Radon transform relatively to $\omega \in S^{n-1}$, i.e. the integral over almost every hyperplane $\langle x, \omega \rangle = t$.

In this note, we give a non-constructive proof of the following known property:

$R_\omega$ is a bounded surjective operator from the Hardy space $H^1(\mathbb{R}^n)$ onto $H^1(\mathbb{R})$.

The boundedness assertion is contained in [1] where it results from an atomic decomposition of $H^1$ functions. It is also included in [3] where it occurs as a corollary of an identity relating the Radon andiesz transforms. A constructive proof of the surjectivity of $R_\omega$ is contained in [3] too.

The alternative proof we give here is essentially based on duality arguments. In particular, it brings out the fact that the dual operator of $R_\omega : H^1(\mathbb{R}^n) \to H^1(\mathbb{R})$ is $B_\omega : g \to g(\langle ., \omega \rangle) : \text{BMO}(\mathbb{R}) \to \text{BMO}(\mathbb{R}^n)$.

2. PROOF

By Fubini theorem, we see that

\[(\star) \quad g, R_\omega f \, dx = \int_{\mathbb{R}^n} B_\omega g, f \, dx \]

holds for every $f \in \mathscr{V}_c(\mathbb{R}^n)$ and $g \in \text{BMO}(\mathbb{R})$, where

\[\mathscr{V}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : \text{f dx} = 0\} \quad \text{and} \quad B_\omega g = g(\langle ., \omega \rangle).\]

On the other hand, we notice the following facts:

a. $R_\omega f \in \mathcal{S}(\mathbb{R})$ whenever $f \in \mathcal{V}(\mathbb{R}^n)$. This is an easy consequence of the identity relating the Fourier transforms of $f$ and $R_\omega f$.

b. $\mathcal{V}_c(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$, $n \geq 1$. This results from lemma 1.5 of [2].

Moreover, $\mathcal{V}_c(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$ since it contains the dense subspace $H^1_c(\mathbb{R}^n)$ considered in [4], p 231.

Présenté par H. Garnier, le 26 avril 1984.
c. \( R_\omega \) is a one-to-one bicontinuous map from \( BMO(\mathbb{R}) \) into \( BMO(\mathbb{R}^n) \). Indeed, if \( m_\mu = \int_Q g \, dx \), a straightforward comparison of the \( BMO \) norms of \( g \) and \( g(\langle \cdot, \omega \rangle) \), defined as \( \sup_Q (m_\mu |g - m_\mu|) \) over all cubes \( Q \subset \mathbb{R}^n \), with \( n = 1 \) and \( n > 1 \) respectively, yields the required assertion.

From the above remarks, we first deduce that

\[
\sup_{\omega} \frac{\| f \|_{H^1(\mathbb{R})}}{\| f \|_{H^1(\mathbb{R}^n)}} \leq C \| f \|_{H^1(\mathbb{R}^n)}, \quad \forall f \in L^1(\mathbb{R}^n),
\]

which implies that \( R_\omega \) has a bounded extension on the whole of \( H^1(\mathbb{R}) \).

Moreover, owing to the boundedness of \( R_\omega \) as an operator from \( L^1(\mathbb{R}^n) \) into \( L^1(\mathbb{R}) \), this extended operator coincides with the usual Radon transform defined as an integral.

From (2), we next conclude that \( R_\omega \) is the dual operator of \( E_\nu \). The surjectivity thesis is then a consequence of the Banach closed range theorem ([5], corollary 1, p 206).

**Remark.** If \( N(R_\omega) \) denotes the null-space of \( R_\omega : H^1(\mathbb{R}^n) \rightarrow H^1(\mathbb{R}) \), we notice that

\[ N(R_\omega) = \{ f \in H^1(\mathbb{R}^n) : \langle g(\langle \cdot, \omega \rangle), f \rangle = 0, \quad \forall g \in BMO(\mathbb{R}) \} \]

\[ N(R_\omega) = \{ g(\langle \cdot, \omega \rangle) : g \in BMO(\mathbb{R}) \}. \]

3. REFERENCES


Université de l’Etat à Mons
Service de Mathématique
Avenue du Champ de Mars, 24
B - 7000 - MONS
BELGIQUE

128