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SOME RESULTS ON TRANSFORMING h-TRIPLE

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INTRODUCTION

$(S, \mathfrak{A}, \varphi)$ is said to be a h-triple if a fixed non-empty set S with elements s, a fixed non-empty family $\mathfrak{A}$ of subsets of S and a fixed non-negative real-valued set function φ on $\mathfrak{A}$ satisfy the conditions :

(i) $A_1, A_2 \in \mathfrak{A}$ imply $A_1 \cap A_2 \in \mathfrak{A}$,

(ii) $A_1, A_2 \in \mathfrak{A}$ imply that there is a finite, pairwise disjoint subfamily $\{B_i\}_{i=1}^n$ of $\mathfrak{A}$ such that $A_1 - A_2 = \bigcup_{i=1}^n B_i$,

(iii) S is the union of a sequence of sets from $\mathfrak{A}$,

(iv) φ is superadditive in the sense that $\varphi(A) \geqq \sum_{i=1}^n \varphi(A_i)$ if $\{A_i\}_{i=1}^n$ is a finite, pairwise disjoint subfamily of $\mathfrak{A}$ with $\bigcup_{i=1}^n A_i = A \in \mathfrak{A}$.

Suppose that we are given a set $S_1 \neq \varnothing$, a h-triple $(S_2, \mathfrak{A}_2, \varphi_2)$ and a mappping $T: S_1 \to S_2$ with the property $TT^{-1}A_2 \in \mathfrak{A}_2$ for each $A_2 \in \mathfrak{A}_2$. Obviously, the class $\mathfrak{A}_1 \equiv \{T^{-1}A_2 : A_2 \in \mathfrak{A}_2\}$ satisfies the above conditions (i)-(iii) and a non-negative real-valued set function φ_1 on $\mathfrak{A}_1$ can be introduced by setting $\varphi_1(A_1) = \varphi_2(TA_1)$ for each $A_1 \in \mathfrak{A}_1$. Moreover, φ_1 can be proved to be superadditive. Hence we may obtain Hayes integrals, measurable sets, measurable functions on S_1 and on S_2 respectively. The purpose of the present paper is to investigate how they are related.

In the sequel, all definitions and notations are found in [1] (numbers in brackets refer to the bibliography at the end of this paper).

I. HAYES INTEGRALS

In this section, we shall prove that if $f: S_2 \to R$, then $\Phi_1(f \circ T, P_1) = \Phi_2(f, TP_1)$ for every $P_1 \in \mathfrak{P}(S_1)$ and $\Phi_2(f, P_2) = \Phi_1(f \circ T, T^{-1}P_2)$ for every $P_2 \in \mathfrak{P}(S_2)$ with $P_2 \subset TS_1$.

Lemma I.1. — If $f: S_2 \to R$ is non-negative and bounded, then so is $f \circ T: S_1 \to R$, $\varphi_1(f \circ T, A_1) = \varphi_2(f, TA_1)$ for every $A_1 \in \mathfrak{A}_1$ and $\varphi_1(f \circ T, T^{-1}A_2) \leqq \varphi_2(f, A_2)$ for every $A_2 \in \mathfrak{A}_2$.

Proof. — The first equality follows directly from the definitions for $\varphi_1(f \circ T, \;)$ and $\varphi_2(f, \;)$. Owing to $TT^{-1}A_2 \subset A_2$, the second inequality follows easily from the first equality and monotonity of $\varphi_2(f, \;)$.

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Theorem I.2. — If $f: S_2 \to R$ is non-negative, then so is $f \circ T: S_1 \to R$ and $\varphi_1^*(f \circ T, P_1) = \varphi_2^*(f, TP_1)$ for every $P_1 \in \mathfrak{P}(S_1)$. In particular, $\varphi_1^*(P_1) = \varphi_2^*(TP_1)$.

Proof. — We need only prove this for the case when $f: S_2 \to R$ is bounded, for the general case follows from the bounded case and $(f \circ T)^{(n)} = f^{(n)} \circ T$, where $(f \circ T)^{(n)} = (f \circ T) \wedge n$, $f^{(n)} = f \wedge n$.

Let $\{A_{1i}\}_i \subset \mathfrak{A}_1$ be any covering for P_1. Thus $\{TA_{1i}\}_i \subset \mathfrak{A}_2$ is a covering for TP_1, and by lemma I.1, we obtain

$$\sum_i \varphi_1(f \circ T, A_{1i}) = \sum_i \varphi_2(f, TA_{1i}) \geqq \varphi_2^*(f, TP_1).$$

This implies $\varphi_1^*(f \circ T, P_1) \geqq \varphi_2^*(f, TP_1)$. By applying the inequality of lemma I.1, $\varphi_1^*(f \circ T, P_1) \leqq \varphi_2^*(f, TP_1)$ can be obtained similarly. Hence $\varphi_1^*(f \circ T, P_1) = \varphi_2^*(f, TP_1)$.

Corollary I.3. — If $f: S_2 \to R$, then $\Phi_1(f \circ T, P_1) = \Phi_2(f, TP_1)$ for every $P_1 \in \mathfrak{P}(S_1)$ in the sense that one of them is defined, so is the other and they possess the same value.

Corollary I.4. — If $f: S_2 \to R$ is non-negative, then $\varphi_1^*(f \circ T, T^{-1}P_2) \leqq \varphi_2^*(f, P_2)$ for every $P_2 \in \mathfrak{P}(S_2)$.

It should be remarked that $\varphi_1^*(f \circ T, T^{-1}P_2) = \varphi_2^*(f, P_2)$ for $P_2 \in \mathfrak{P}(S_2)$ with $P_2 \subset TS_1$. From this we have the following.

Corollary I.5. — If $T: S_1 \to S_2$ is onto and $f: S_2 \to R$, then $\Phi_2(f, P_2) = \Phi_1(f \circ T, T^{-1}P_2)$ for every $P_2 \in \mathfrak{P}(S_2)$ in the sense that one of them is defined, so is the other and they possess the same value.

II. MEASURABLE SETS AND MEASURABLE FUNCTIONS

We shall consider the classes $\mathfrak{M}(\varphi_1, f \circ T)$, $\mathfrak{M}(\varphi_2, f)$ for every non-negative function $f: S_2 \to R$; $\mathscr{C}(\varphi_1, P_1)$, $\mathscr{C}(\varphi_2, TP_1)$ for every $P_1 \in \mathfrak{P}(S_1)$; and $\mathscr{C}(\varphi_1, T^{-1}P_2)$, $\mathscr{C}(\varphi_2, P_2)$ for every $P_2 \in \mathfrak{P}(S_2)$.

Therorem II.1. — If $f: S_2 \to R$ is non-negative, then $T\mathfrak{M}(\varphi_1, f \circ T) \subset \mathfrak{M}(\varphi_2, f)$ and $T^{-1}\mathfrak{M}(\varphi_2, f) \subset \mathfrak{M}(\varphi_1, f \circ T)$. In particular, $T\mathfrak{M}(\varphi_1^*) \subset \mathfrak{M}(\varphi_2^*)$ and $T^{-1}\mathfrak{M}(\varphi_2^*) \subset \mathfrak{M}(\varphi_1^*)$.

Proof. — We prove the first inclusion only, since the second inclusion can be obtained analogously. Let E_1 be any set of $\mathfrak{M}(\varphi_1, f \circ T)$, then there are $H_1 \in (\mathfrak{A}_1)_\sigma$ and $N_1 \in \mathfrak{P}(S_1)$ such that $\varphi_1^*(f \circ T, N_1) = 0$ and $E_1 = H_1 - N_1$. Clearly, $TH_1 \in (\mathfrak{A}_2)_\sigma \subset \mathfrak{M}(\varphi_2, f)$. $TE_1 \subset TH_1 - TN_1$ and $TE_1 \subset TH_1$ imply $TE_1 = (TH_1 - TN_1) \cup (TE_1 - (TH_1 - TN_1))$ and $TE_1 - (TH_1 - TN_1) \subset TH_1 - (TH_1 - TN_1) \subset TN_1$ respectively. By theorem I.2, we have

$$\varphi_2^*(f, TE_1 - (TH_1 - TN_1)) \leqq \varphi_2^*(f, TN_1) = \varphi_1^*(f \circ T, N_1) = 0.$$

This implies $TE_1 - (TH_1 - TN_1) \in \mathfrak{M}(\varphi_2, f)$ and $TN_1 \in \mathfrak{M}(\varphi_2, f)$. Hence $TE_1 \in \mathfrak{M}(\varphi_2, f)$.

Corollary II.2. — If $T: S_1 \to S_2$ is onto and $f: S_2 \to R$ is non-negative, then $T\,\mathfrak{M}(\varphi_1, f \circ T) = \mathfrak{M}(\varphi_2, f)$.

Lemma II.3. — If $f: S_2 \to R$, then $\Omega(f \circ T, P_1) = \Omega(f, TP_1)$ for every $P_1 \in \mathfrak{P}(S_1)$ and $\Omega(f \circ T, T^{-1}P_2) \leqq \Omega(f, P_2)$ for every $P_2 \in \mathfrak{P}(S_2)$.

This lemma follows from definition of Ω.

Theorem II.4. — Let $P_1 \in \mathfrak{P}(S_1)$ and $f: S_2 \to R$. A necessary and sufficient condition for $f \circ T \in \mathscr{C}(\varphi_1, P_1)$ is $f \in \mathscr{C}(\varphi_2, TP_1)$.

Proof. — Necessity is straightforward, and we shall prove sufficiency only. Let $f \in \mathscr{C}(\varphi_2, TP_1)$. For $\varepsilon > 0$, there is a sequence $\{A_{2i}\}_i \subset \mathfrak{A}_2$ such that $\bigcup_i A_{2i} \supset TP_1$ and

$$\sum \{\varphi_2(A_{2i}) : \Omega(f, A_{2i}) \geqq \varepsilon\} < \varepsilon.$$

Thus $\{T^{-1}A_{2i}\}_i \subset \mathfrak{A}_1$ and $\cup T^{-1}A_{2i} \supset P_1$. By lemmas I.1 and II.3, we have

$$\begin{aligned} &\sum \{\varphi_1(T^{-1}A_{2i}) : \Omega(f \circ T, T_{-1}A_{2i}) \geqq \varepsilon\} \\ &\leqq \sum \{\varphi_2(A_{2i}) : \Omega(f \circ T, T^{-1}A_{2i}) \geqq \varepsilon\} \\ &\leqq \sum \{\varphi_2(A_{2i}) : \Omega(f, A_{2i}) \geqq \varepsilon\} < \varepsilon. \end{aligned}$$

Hence $f \circ T \in \mathscr{C}(\varphi_1, P_1)$.

Corollary II.5. — Let $P_2 \in \mathfrak{P}(S_2)$ and $f: S_2 \to R$. If $f \in \mathscr{C}(\varphi_2, P_2)$, then $f \circ T \in \mathscr{C}(\varphi_1, T^{-1}P_2)$.

Proof. — $TT^{-1}P_2 \subset P_2$ implies $\mathscr{C}(\varphi_2, P_2) \subset \mathscr{C}(\varphi_2, TT^{-1}P_2)$. Thus the conclusion follows from theorem II.4.

It should be noted that the converse of corollary II.5 need not be true. For example, let $S_1 = (0, 1]$, $S_2 = R$, $\mathfrak{A}_2 = \{(a, b] : a \leqq b\}$, $\varphi_2((a, b]) = b - a$ and $T: S_1 \to S_2$ be the inclusion mapping. If $f = \chi_{R_a \cap [2, 3]}$, where R_a is the set of rational numbers, then $f \circ T = 0 \in C(\varphi_1, S_1)$. However $f \notin \mathscr{C}(\varphi_2, S_2)$.

III. CONCLUDING REMARK

Finally, we examine the case when $(S_2, \mathfrak{A}_2, \varphi_2)$ satisfies a further condition (v), i.e., if $H_2 \in (\mathfrak{A}_2)_\sigma$, then $\Phi_2(\chi_{H2}, S_2) = \varphi_2^*(H_2)$.

Theorem III.1. — If $(S_2, \mathfrak{A}_2, \varphi_2)$ satisfies (v), then so does $(S_1, \mathfrak{A}_1, \varphi_1)$.

Proof. — Let $H_1 \in (\mathfrak{A}_1)_\sigma$, then $TH_1 \in (\mathfrak{A}_2)_\sigma$. By hypothesis and theorem I.2, $\Phi_2(\chi_{TH_1}, S_2) = \varphi_2^*(TH_1) = \varphi_1^*(H_1)$. Also, we have $\chi_{H_1} \leqq \chi_{TH_1} \circ T$. Hence

$$\begin{aligned} \Phi_1(\chi_{H_1}, S_1) &\leqq \Phi_1(\chi_{TH_1} \circ T, S_1) = \varphi_1^*(\chi_{TH_1} \circ T, S_1) \\ &= \varphi_2^*(\chi_{TH_1}, TS_1) \leqq \varphi_2^*(\chi_{TH_1}, S_2) = \Phi_2(\chi_{TH_1}, S_2) = \varphi_1^*(H_1). \end{aligned}$$

We may assume $H_1 = \bigcup_{i=1}^{\infty} A_{1i}$, $\{A_{1i}\}_i \subset \mathfrak{A}_1$ and $A_{1i} \cap A_{1j} = \emptyset$ for $i \neq j$ ([1], lemma 1.6). Thus

$$\begin{aligned} \Phi_1(\chi_{H_1}, S_1) &= \varphi_1^*(\chi_{H_1}, S_1) \geqq \varphi_1^*(\chi_{H_1}, H_1) = \sum_i \varphi_1^*(\chi_{H_1}, A_{1i}) \\ &= \sum_i \varphi_1^*(A_{1i}) = \varphi_1^*(H_1) \quad \text{([1], theorem 3.5).} \end{aligned}$$

Hence $\Phi_1(\chi_{H_1}, S_1) = \varphi_1^*(H_1)$.

By virtue of [2], for every $E_2 \in \mathfrak{M}(\varphi_2^*)$, the class $\mathscr{C}(\varphi_2, E_2)$ coincides with the class of φ_2^*-measurable functions on E_2, and for $E_1 \in \mathfrak{M}(\varphi_1^*)$, the class $\mathscr{C}(\varphi_1, E_1)$ coincides with the class of φ_1^*-measurable functions on E_1. From this and [2], we obtain the following well-known results ([3], pp. 182-183) as direct consequences of corollaries I.5 and II.5 respectively :

(a) if f is φ_2^*-measurable on $E_2 \in \mathfrak{M}(\varphi_2^*)$, then $f \circ T$ is φ_1^*-measurable on $T^{-1}E_2$,

(b) for every φ_2^*-measurable function f on $E_2 \in \mathfrak{M}(\varphi_2^*)$,

$$\int_{E_2} f d\varphi_2^* = \int_{T^{-1}E_2} f \circ T \, d\varphi_1^*$$

in the sense that one of them is defined, so is the other and they possess the same value, if $T: S_1 \to S_2$ is onto.

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