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THE TENSOR ALGEBRA OF THE SPACE s

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Dedicated to the memory of Pascal Laubin

In view of recent work on noncommutative geometry involving various Fréchet spaces (resp. algebras), isomorphic to the space s of rapidly decreasing sequences, and their tensor algebras (see [2]) it appears to be of interest to determine the isomorphy type of the tensor algebra $T(s)$ of s. We show that $T(s) \cong s$ and that, moreover, for any complemented subspace E of s with $\dim E \geq 2$ we have $T(E) \cong s$. This, in particular, includes all nuclear power series spaces of infinite type as e.g. the space of entire holomorphic functions in any finite dimension.

It is interesting to compare the situation with the case of the symmetric tensor algebra $S(E)$. The isomorphy type of $S(E)$ for certain power series spaces E has been determined in [1] There we also have $S(s) \cong s$, however $S(E) \not\cong s$ in general, even for power series spaces. Of course, the proof of Theorem 5 owes much to the methods developed in [1].

For any Fréchet space we set

$$E^{\otimes n} := E \otimes \cdots \otimes E$$

the n-fold complete π-tensor product, and by $T(E)$ we denote the completion of

$$\bigoplus_{n=1}^{\infty} E^{\otimes n}$$

with respect to the Fréchet space topology given by the seminorms

$$x = x_1 \oplus \cdots \oplus x_n \mapsto \sum_n p^{\otimes n}(x_n)$$

where p runs through all continuous seminorms on E and $p^{\otimes n}$ denotes the n-fold π-tensorproduct of p. $T(E)$ is called the tensor algebra of E.

By

$$x_1 \otimes \cdots \otimes x_n \times y_1 \otimes \cdots \otimes y_m := x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m$$

it becomes a Fréchet m-algebra. In a natural way $E \subset T(E)$ and every continuous linear map $E \longrightarrow A$, where A is an Fréchet m-algebra extends to a uniquely determined continuous algebra homomorphism $T(E) \longrightarrow A$. If E and F are Fréchet spaces then every continuous linear map $\varphi : E \longrightarrow F$ extends into a continuous algebra homomorphism $T(\varphi) : T(E) \longrightarrow T(F)$.

If $\| \ \|_1 \leq \| \ \|_2 \leq \ldots$ is a fundamental system of seminorms on the Fréchet space E then $T(E)$ may be represented in the following form

$$(1) \quad T(E) = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \prod_{n=1}^{\infty} E^{\otimes n} : \|x\|_k = \sum_{n=1}^{\infty} e^{kn} \|x_n\|_k^{\otimes n} < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

For $E = \mathbb{C}$ we obtain clearly $T(E) = H(\mathbb{C})$ the space of entire functions in one complex variable. A less trivial situation appears already for $\dim E = 2$.

We define the space s of rapidly decreasing sequences:

$$s = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |x|_k = \sum_{n=1}^{\infty} |x_n| n^k < +\infty \text{ for all } k \in \mathbb{N} \right\}$$

equipped with the norms $| \ |_k$, $k \in \mathbb{N}$, and we obtain:

Lemma 1 $T(\mathbb{C}^2) \cong s$.

Proof: We think of $\mathbb{C}^2$ as equipped with the ℓ_1-norm. Then $(\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n}$ again equipped with the ℓ_1-norm (see [3, Chap. I, §2, n°2, Cor. 4, p. 61]). So we obtain counting the natural basis of $\mathbb{C}^{2^n}$ from 2^{n-1} to $2^n - 1$:

$$T(\mathbb{C}^2) = \left\{ x = (x_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_{n=1}^{\infty} 2^{kn} \left(\sum_{j=2^{n-1}}^{2^n - 1} |x_j| \right) \text{ for all } k \right\}.$$

For $2^{n-1} \leq j < 2^n$ we have $2^{-k} \cdot 2^{kn} \leq j^k < 2^{kn}$. Therefore $T(\mathbb{C}^2) \cong s$. □

Corollary 2 *For every Fréchet space E with $\dim E \geq 2$ the Fréchet space $T(E)$ contains s as a complemented subspace.*

Proof: Choose a 2-dimensional subspace $F \subset E$ and a projection P onto F. Then $T(P)$ is a projection onto $T(F) \cong T(\mathbb{C}^2) \cong s$. □

For any unbounded sequence $0 \leq \alpha_1 \leq \alpha_2 \ldots$ we set

$$\Lambda_\infty(\alpha) = \left\{ x = (x_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |x|_k = \sum_{n=1}^{\infty} e^{k\alpha_n} |x_n| < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

$\Lambda_\infty(\alpha)$ equipped with the norms $(|\ |_k)_{k \in \mathbb{N}}$ is a Fréchet space. It is called power series space of infinite type. Notice that for $\alpha_j = \log j$ we obtain the space s.

We obtain, again using [3, Chap. I, §2, n°2, Cor. 4, p. 61]

$$\Lambda_\infty^{\otimes n}(\alpha) = \left\{ x = (x_j)_{j \in \mathbb{N}^n} : |x|_k^{\otimes n} = \sum_{j \in \mathbb{N}^n} |x_j| e^{k(\alpha_{j_1} + \cdots + \alpha_{j_n})} < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

We put $J = \dot{\bigcup}_{n=1}^{\infty} \mathbb{N}^n$ and for $j = (j_1, \ldots, j_n) \in J$ we set $w(j) = n + \alpha_{j_1} + \cdots + \alpha_{j_n}$. Then (1) takes the form

$$T(\Lambda_\infty(\alpha)) = \left\{ x = (x_j)_{j \in J} : \|x\|_k = \sum_{j \in J} |x_j| e^{kw(j)} < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

Let $j \mapsto l(j)$ be an enumeration of J with increasing $w(j)$ and put $\beta_l = w(j)$ for $l = l(j)$. Then

$$T(\Lambda_\infty(\alpha)) = \left\{ x = (x_l)_{l \in \mathbb{N}} : \|x\|_k = \sum_{l=1}^{\infty} |x_l| e^{k\beta_l} < +\infty \text{ for all } k \in \mathbb{N} \right\}.$$

Lemma 3 *With β as defined before, we have $T(\Lambda_\infty(\alpha)) = \Lambda_\infty(\beta)$.*

Lemma 4 *For any α we have $\beta_l \leq a \log l + b$ with suitable a and b .*

Proof: This follows from Lemma 1. Notice that, choosing $F = \text{span}\{e_1, e_2\}$ in its proof, then we have $T(F) \cong s$ represented as $\Lambda_\infty(\beta)$ restricted to M, where M is a suitable subset of $\mathbb{N}$. e_k denote the canonical basis vectors. □

We are now ready to prove our main theorem.

Theorem 5 $T(s) \cong s$.

Proof: We have to estimate β_l. By Lemma 4 we have $\beta_l \leq a \log l + b$. To get a reverse estimate we put for $r \geq 0$

(2)
$$m_n(r) = \#\{j \in \mathbb{N}^n : \sum_{\nu=1}^{n} \log j_\nu \leq r\}.$$

To any $j \in \mathbb{N}^n$ with $\sum_{\nu=1}^{n} \log j_\nu \leq r$ we assign the cube

$$Q_n(j) = \{x \in \mathbb{R}^n : j_\nu - 1 < x_\nu < j_\nu,\ \nu = 1, \ldots, n\}$$

and put

$$W_n(r) = \{x \in \mathbb{R}^n : x_\nu > 0 \text{ for all } \nu,\ \sum_{\nu=1}^{n} \log_+ x_\nu \leq r\}$$

where $\log_+ x = \max(0, \log x)$. Then we have

$$\bigcup_{j \in \mathbb{N}^n \cap W_n(r)} Q_n(j) \subset W_n(r)$$

and therefore

$$m_n(r) \leq \mu W_n(r) =: \mu_n(r)$$

where μ denotes the Lebesgue measure. We determine $\mu_n(r)$ inductively over n. Obviously $\mu_1(r) = e^r$ and

$$\begin{aligned} \mu_{n+1}(r) &= \int_0^{e^r} \mu_n(r - \log_+ t)dt \\ &= \int_0^r \mu_n(r-s)e^s ds + \mu_n(r). \end{aligned}$$

By induction we obtain

$$\mu_n(r) = e^r \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{r^k}{k!}$$

for all $r \geq 0$.

We use the notations of our preliminary analysis and put

$$\begin{aligned} m(r) &:= \{j \in J : w(j) \leq r\} \\ &= \bigcup_{n \in \mathbb{N}} \{j \in \mathbb{N}^n : \sum_{\mu=1}^{n} \log j_\mu \leq r - n\} \\ &= \sum_{n=1}^{[r]} m_n(r-n). \end{aligned}$$

Then we obtain

$$\begin{aligned} m(r) &\leq \sum_{n=1}^{[r]} \mu_n(r-n) \\ &= \sum_{n=1}^{[r]} e^{r-n} \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(r-n)^k}{k!} \\ &\leq \sum_{n=1}^{[r]} e^{2(r-n)} 2^{n-1} \\ &\leq e^{2r}. \end{aligned}$$

If for $j \in J$ we have $w(j) = r$, then obviously $l(j) \leq m(r)$, hence with $l = l(j)$ and $\beta_l = w(j)$

$$l \leq e^{2r} = e^{2\beta_l}.$$

This proves

$$\beta_l \geq \frac{1}{2} \log l.$$

and completes the proof of the theorem. □

As an immediate consequence of Lemma 1 and Theorem 5 we obtain:

Theorem 6 *If E is a complemented subspace of s and* $\dim E \geq 2$ *then* $T(E) \cong s$.

Proof: s is isomorphic to a complemented subspace of $T(E)$ by Lemma 1, If P is a projection in s onto a subspace isomorphic to E the $T(P)$ is a projection in $T(s) \cong s$ onto a subspace isomorphic to $T(E)$. This implies the result, see e.g. [4, Lemma 31.2]. □

In view of the theory developed in [2] let us close with a final remark. We use the notation of [2, 2.2]. If A is a Fréchet m-algebra then id : $A \longrightarrow A$ generates a continuous algebra homomorphism $\alpha : T(A) \longrightarrow A$, namely the one which sends $x_1 \otimes \cdots \otimes x_n$ into $x_1 x_2 \ldots x_n$. We set $J(A) = \ker \alpha$ and obtain the so called universal extension

$$0 \longrightarrow J(A) \longrightarrow T(A) \xrightarrow{\alpha} A \longrightarrow 0.$$

Now, $J(A)$ contains $T(A)$ as a complemented subspace. To see this pick an $a \neq 0$. If $a^2 = 0$ then $u \mapsto a \otimes a \otimes u$ imbeds $T(A)$ as a complemented subspace into $J(A)$. If $a^2 \neq 0$ the map $u = \sum_{n=1}^{\infty} u_n \mapsto \sum_{n=2}^{\infty} v_n$, $(u_n, v_n \in A^{\otimes n})$ does the same where for $n \in \mathbb{N}$ we set $v_{2k} = -a^2 \otimes a \otimes \cdots \otimes a \otimes u_k$ and $v_{2k+1} = a \otimes a \otimes a \otimes \cdots \otimes a \otimes u_k$. Therefore we get from Theorem 6 using again [4, Lemma 31.2]

Corollary 7 *If A is isomorphic to a complemented subspace of s and* $\dim A \geq 2$ *then* $J(A) \cong s$.

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