Bulletin de la Société Royale des Sciences de Liège, 53e année, 3-4, 1984, pp. 143-157

ON A CLASS OF ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENT II

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ABSTRACT. Sarangi and Uralegaddi studied the class $\tilde{C}(\alpha)$ consisting of functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

satisfying $\text{Re}\{f'(z)\} > \alpha$ $(0 \leqq \alpha < 1)$. We introduce the class $\tilde{C}(\alpha,p)$ $(0 \leqq \alpha < 1,\ 0 \leqq p \leqq 1)$ of functions $f(z) \in \tilde{C}(\alpha)$ with fixed second coefficient. The object of the present paper is to show coefficient inequalities, distortion theorems and closure theorem for functions $f(z)$ in $\tilde{C}(\alpha,p)$, and to determine the radii of starlikeness and convexity for $\tilde{C}(\alpha,p)$. Further we consider the modified Hadamard product of functions $f(z)$ belonging to the class $\tilde{C}(\alpha,p)$.

I. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$. Further let $C(\alpha)$ denote the subclass of A consisting of functions satisfying

$$\text{Re}\{f'(z)\} > \alpha \qquad (z \in U) \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$.

In particular, the class $C(0)$ was studied by MacGregor [1].

Présenté par H. Garnir, le 4 mai 1984.

Let $\tilde{A}$ denote the subclass of A whose members have the form

$$(1.3) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geqq 0).$$

We denote by $\tilde{C}(\alpha)$ the class obtained by taking intersection of $C(\alpha)$ with $\tilde{A}$, that is, $\tilde{C}(\alpha) = C(\alpha) \cap \tilde{A}$.

The class $\tilde{C}(\alpha)$ was studied by Sarangi and Uralegaddi [3], and Owa and Uralegaddi [2].

In [3], Sarangi and Uralegaddi gave the following lemma.

LEMMA 1. Let the function $f(z)$ be defined by (1.3). Then $f(z)$ is in the class $\tilde{C}(\alpha)$ if and only if

$$(1.4) \qquad \sum_{n=2}^{\infty} na_n \leqq 1 - \alpha.$$

By virtue of Lemma 1, we introduce the following class of analytic functions with fixed second coefficient.

DEFINITION. Let $\tilde{C}(\alpha,p)$ be the class of functions of the form

$$(1.5) \qquad f(z) = z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

such that $f(z) \in \tilde{C}(\alpha)$, where $0 \leqq \alpha < 1$ and $0 \leqq p \leqq 1$.

2. COEFFICIENT INEQUALITIES

THEOREM 1. Let the function $f(z)$ be defined by (1.5). Then $f(z)$ is in the class $\tilde{C}(\alpha,p)$ if and only if

$$(2.1) \qquad \sum_{n=3}^{\infty} na_n \leqq (1-p)(1-\alpha).$$

The result is sharp.

PROOF. Putting $a_2 = p(1 - \alpha)/2$ in Lemma 1, we have

$$(2.2) \qquad p(1 - \alpha) + \sum_{n=3}^{\infty} n a_n \leqq 1 - \alpha$$

which gives (2.1). Further we can observe that the result is sharp for the function given by

$$(2.3) \qquad f(z) = z - \frac{p(1 - \alpha)}{2} z^2 - \frac{(1 - p)(1 - \alpha)}{n} z^n$$

for $n \geqq 3$.

COROLLARY 1. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then

$$(2.4) \qquad a_n \leqq \frac{(1 - p)(1 - \alpha)}{n}$$

for $n \geqq 3$. Equality is attained for the function f(z) given by (2.3).

COROLLARY 2. Let $0 \leqq \alpha_1 \leqq \alpha_2 < 1$ and $0 \leqq p_1 \leqq p_2 \leqq 1$. Then

$$(2.5) \qquad \tilde{C}(\alpha_1,p_1) \supset \tilde{C}(\alpha_2,p_2).$$

THEOREM 2. Let

$$(2.6) \qquad f_2(z) = z - \frac{p(1 - \alpha)}{2} z^2$$

and

$$(2.7) \qquad f_n(z) = z - \frac{p(1 - \alpha)}{2} z^2 - \frac{(1 - p)(1 - \alpha)}{n} z^n$$

for $n \geqq 3$, where $0 \leqq \alpha < 1$ and $0 \leqq p \leqq 1$. Then f(z) is in the class $\tilde{C}(\alpha,p)$ if and only if it can be expressed in the form

$$(2.8) \qquad f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) ,$$

where $\lambda_n \geqq 0$ $(n \geqq 2)$ and

$$(2.9) \qquad \sum_{n=2}^{\infty} \lambda_n = 1.$$

PROOF. Suppose that

$$(2.10) \quad f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z)$$

$$= z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} \frac{(1-p)(1-\alpha)}{n} \lambda_n z^n$$

$$= z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} a_n z^n ,$$

where

$$(2.11) \qquad a_n = \frac{(1-p)(1-\alpha)}{n} \lambda_n \geqq 0 \qquad (n \geqq 3).$$

Then we know that

$$(2.12) \qquad \sum_{n=3}^{\infty} n a_n = (1-p)(1-\alpha) \sum_{n=3}^{\infty} \lambda_n$$

$$\leqq (1-p)(1-\alpha)$$

which implies that $f(z) \in \tilde{C}(\alpha,p)$ by means of Theorem 1.

Conversely, we suppose that

$$(2.13) \quad f(z) = z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} a_n z^n \qquad (a_n \geqq 0)$$

is in the class $\tilde{C}(\alpha,p)$. Then we have (2.4) for $n \geqq 3$. Taking

$$\lambda_n = \frac{na_n}{(1-p)(1-\alpha)} \qquad (n \geq 3) \tag{2.14}$$

and

$$\lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n , \tag{2.15}$$

we obtain the representation (2.8). This completes the proof of the theorem.

3. Distortion theorems

We need the following lemmas in order to get the distortion inequalities for functions $f(z)$ belonging to the class $\tilde{C}(\alpha,p)$.

LEMMA 2. Let $0 \leq \alpha < 1$,

$$p_0 = \frac{-(10+\alpha) + \sqrt{132 - 12\alpha + \alpha^2}}{2(1-\alpha)} , \tag{3.1}$$

$$r_0 = \frac{-4(1-p) + \sqrt{16(1-p)^2 + 3p^2(1-p)(1-\alpha)}}{p(1-p)(1-\alpha)} , \tag{3.2}$$

and $f_3(z)$ be defined as in Theorem 2. Then

$$|f_3(re^{i\theta})| \geq r - \frac{p(1-\alpha)}{2} r^2 - \frac{(1-p)(1-\alpha)}{3} r^3 \tag{3.3}$$

for $0 \leq p \leq 1$ and $0 \leq r < 1$. Equality is attained for $\theta = 0$. For either $0 \leq p < p_0$ and $0 \leq r \leq r_0$ or $p_0 \leq p \leq 1$,

$$|f_3(re^{i\theta})| \leq r + \frac{p(1-\alpha)}{2} r^2 - \frac{(1-p)(1-\alpha)}{3} r^3 \tag{3.4}$$

with equality for $\theta = \pi$. Further, for $0 \leq p < p_0$ and $r_0 \leq r < 1$,

$$|f_3(re^{i\theta})| \leq \frac{r}{2} \{3p^2(1-\alpha) + 16(1-p)\}^{1/2} \tag{3.5}$$

$$\times \left\{ \frac{1}{4(1-p)} + \frac{1-\alpha}{6} r^2 + \frac{(1-p)(1-\alpha)^2}{36} r^4 \right\}^{1/2}$$

with equality for $\theta = \theta_0$, where

$$\theta_0 = \cos^{-1}\left(\frac{p(1-p)(1-\alpha)r^2 - 3p}{8(1-\alpha)r} \right) . \tag{3.6}$$

PROOF. A simple computation gives that

$$\frac{\partial}{\partial\theta} |f_3(re^{i\theta})|^2 \tag{3.7}$$

$$= \frac{1}{3}(1-\alpha)r^3 \sin\theta \{3p + 8(1-p)r\cos\theta - p(1-p)(1-\alpha)r^2\}.$$

Hence $\partial|f_3(re^{i\theta})|^2/\partial\theta = 0$ for $\theta_1 = 0$, $\theta_2 = \pi$ and $\theta_3 = \theta_0$. Further, since θ_3 is a valid root only when $|\cos\theta_0| \leqq 1$, we have a third root if and only if $r_0 \leqq r < 1$ and $0 \leqq p < p_0$. Consequently we can prove the lemma by comparing the extremal values $|f_3(re^{i\theta_k})|$ $(k = 1, 2, 3)$ on the appropriate intervals.

LEMMA 3. Let the function $f_n(z)$ be defined by (2.7) and $n \geqq 4$. Then

$$|f_n(re^{i\theta})| \leqq |f_4(-r)| \tag{3.8}$$

for $0 \leqq r < 1$.

PROOF. Since r^n/n is a decreasing function of n $(n \geqq 4)$, we can see that

$$|f_n(re^{i\theta})| \leqq r + \frac{p(1-\alpha)}{2} r^2 + \frac{(1-p)(1-\alpha)}{n} r^n \tag{3.9}$$

$$\leq r + \frac{p(1-\alpha)}{2} r^2 + \frac{(1-p)(1-\alpha)}{4} r^4$$

$$= -f_4(-r)$$

which implies (3.8). Thus the lemma is completed.

THEOREM 3. Let the function $f(z)$ defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then, for $0 \leq r < 1$,

(3.10) $$|f(re^{i\theta})| \geq r - \frac{p(1-\alpha)}{2} r^2 - \frac{(1-p)(1-\alpha)}{3} r^3$$

with equality for the function $f_3(z)$ at $z = r$. Further, for $0 \leq r < 1$,

(3.11) $$|f(re^{i\theta})| \leq \operatorname{Max}\{\operatorname*{Max}_{\theta}|f_3(re^{i\theta})|, -f_4(-r)\},$$

where $\operatorname*{Max}_{\theta}|f_3(re^{i\theta})|$ is given by Lemma 2.

We can prove the theorem by comparing the bounds of Lemma 2 and Lemma 3.

COROLLARY 3. Let the function $f(z)$ defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then the unit disk $U = \{z: |z| < 1\}$ is mapped on a domain that contains the disk $|w| < (4 + 2\alpha + p\alpha - p)/6$.

LEMMA 4. Let $0 \leq \alpha < 1$,

(3.12) $$p_1 = \frac{-(4+\alpha) + \sqrt{32 - 8\alpha + \alpha^2}}{2(1-\alpha)},$$

(3.13) $$r_1 = \frac{-2(1-p) + \sqrt{4(1-p)^2 + p^2(1-p)(1-\alpha)}}{p(1-p)(1-\alpha)},$$

and $f_3(z)$ be defined as in Theorem 2. Then, for $0 \leq p \leq 1$ and

$0 \leqq r < 1,$

$$|f_3'(re^{i\theta})| \geqq 1 - p(1 - \alpha)r - (1 - p)(1 - \alpha)r^2 \tag{3.14}$$

with equality for $\theta = 0$. For either $0 \leqq p < p_1$ and $0 \leqq r \leqq r_1$ or $p_1 \leqq p \leqq 1$,

$$|f_3'(re^{i\theta})| \leqq 1 + p(1 - \alpha)r - (1 - p)(1 - \alpha)r^2 \tag{3.15}$$

with equality for $\theta = \pi$. Further, for $0 \leqq p < p_1$ and $r_1 \leqq r < 1$,

$$|f_3'(re^{i\theta})| \leqq \{4(1 - p) + p^2(1 - \alpha)\}^{1/2} \times \left\{\frac{1}{4(1 - p)} + \frac{1 - \alpha}{2} r + \frac{(1 - p)(1 - \alpha)^2}{4} r^4\right\}^{1/2} \tag{3.16}$$

with equality for $\theta = \theta_0$, where

$$\theta_0 = \cos^{-1}\left(\frac{p(1 - p)(1 - \alpha)r^2 - p}{4(1 - p)r}\right). \tag{3.17}$$

PROOF. Since

$$\frac{\partial}{\partial\theta} |f_3'(re^{i\theta})|^2 \tag{3.18}$$

$$= 2(1 - \alpha)r\sin\theta\,\{p + 4(1 - p)r\cos\theta - p(1 - p)(1 - \alpha)r^2\},$$

$\partial|f_3'(re^{i\theta})|^2/\partial\theta = 0$ gives that $\theta_1 = 0$, $\theta_2 = \pi$ and $\theta_3 = \theta_0$. Hence, in the same way as in the proof of Lemma 2, we have the lemma.

LEMMA 5. Let the function $f_n(z)$ be defined by (2.7) and $n \geqq 4$. Then

$$|f_n'(re^{i\theta})| \leqq |f_4'(-r)| \tag{3.19}$$

for $0 \leqq r < 1$.

PROOF. Note that r^{n-1} is decreasing in n ($n \geqq 4$). This implies that

$$|f_n'(re^{i\theta})| \leqq 1 + p(1-\alpha)r + (1-p)(1-\alpha)r^{n-1} \tag{3.20}$$
$$\leqq 1 + p(1-\alpha)r + (1-p)(1-\alpha)r^3$$
$$= f_4'(-r).$$

This completes the proof of the lemma.

THEOREM 4. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then, for $0 \leqq r < 1$,

$$|f'(re^{i\theta})| \geqq 1 - p(1-\alpha)r - (1-p)(1-\alpha)r^2 \tag{3.21}$$

with equality for the function $f_3(z)$ at $z = r$. Further, for $0 \leqq r < 1$,

$$|f'(re^{i\theta})| \leqq \mathrm{Max}\{\, \underset{\theta}{\mathrm{Max}}|f_3'(re^{i\theta})|,\ f_4'(-r)\}, \tag{3.22}$$

where $\underset{\theta}{\mathrm{Max}}|f_3'(re^{i\theta})|$ is given by Lemma 4.

The proof the the theorem is obtained by comparing the bounds of Lemma 4 and Lemma 5.

COROLLARY 4. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then f'(z) includes a disk with its center at the origin and radius α.

4. CLOSURE THEOREM

THEOREM 5. Let the functions

$$f_i(z) = z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geqq 0) \tag{4.1}$$

be in the class $\tilde{C}(\alpha,p)$ for every $i = 1, 2, 3, \cdots, m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geqq 0) \tag{4.2}$$

is also in the same class $\tilde{C}(\alpha,p)$, where

$$\sum_{i=1}^{m} c_i = 1. \tag{4.3}$$

PROOF. By the definition of $h(z)$, we have the following expression

$$h(z) = z - \frac{p(1-\alpha)}{2} z^2 - \sum_{n=3}^{\infty} \left(\sum_{i=1}^{m} c_i a_{n,i} \right) z^n . \tag{4.4}$$

Since $f_i(z) \in \tilde{C}(\alpha,p)$, in view of Theorem 1, we obtain that

$$\sum_{n=3}^{\infty} n a_{n,i} \leqq (1-\alpha)(1-p) \tag{4.5}$$

for $i = 1, 2, 3, \cdots, m$. Hence we can show that

$$\sum_{n=3}^{\infty} n \left(\sum_{i=1}^{m} c_i a_{n,i} \right) = \sum_{i=1}^{m} c_i \left(\sum_{n=3}^{\infty} n a_{n,i} \right) \tag{4.6}$$

$$\leqq \left(\sum_{i=1}^{m} c_i \right) (1-\alpha)(1-p) = (1-\alpha)(1-p)$$

which gives that $h(z) \in \tilde{C}(\alpha,p)$ with the aid of Theorem 1.

5. MODIFIED HADAMARD PRODUCT

Let $f_i(z)$ $(i = 1, 2)$ be defined by

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geqq 0). \tag{5.1}$$

Then we denote by $f_1 * f_2(z)$ the modified Hadamard product of $f_1(z)$

and $f_2(z)$, that is,

$$(5.2) \qquad f_1{*}f_2(z) = z - \sum_{n=2}^{\infty} a_{n,1}a_{n,2}z^n .$$

THEOREM 6. Let the function f(z) defined by (1.5) be in the class $\mathcal{C}(\alpha,p)$. Then the modified Hadamard product $f{*}f(z)$ belongs to the class $\mathcal{C}(\alpha(2-\alpha),p^2/2)$.

PROOF. The definition of modified Hadamard product gives that

$$(5.3) \qquad f{*}f(z) = z - \frac{p^2(1-\alpha)^2}{4} z^2 - \sum_{n=3}^{\infty} a_n^2 z^n$$

$$= z - \frac{p^2}{4}\{1-\alpha(2-\alpha)\}z^2 - \sum_{n=3}^{\infty} a_n^2 z^n .$$

Since $0 \leqq \alpha(2-\alpha) < 1$ and $0 \leqq p^2/2 \leqq 1/2$, it suffices to prove that

$$(5.4) \qquad \sum_{n=3}^{\infty} n a_n^2 \leqq \{1-\alpha(2-\alpha)\}\left(1 - \frac{p^2}{2}\right)$$

$$= (1-\alpha)^2\left(1 - \frac{p^2}{2}\right)$$

by means of Theorem 1. But, in view of Theorem 1, we can see that

$$(5.5) \qquad \sum_{n=3}^{\infty} n a_n^2 \leqq \frac{(1-\alpha)(1-p)}{2} \sum_{n=3}^{\infty} n a_n \leqq \frac{(1-\alpha)^2(1-p)^2}{2}$$

$$\leqq (1-\alpha)^2\left(1 - \frac{p^2}{2}\right) .$$

Thus we obtain that $f{*}f(z) \in \mathcal{C}(\alpha(2-\alpha),p^2/2)$.

6. Radii of starlikeness and convexity

With Noshiro-Warschawski theorem, we know that the function $f(z)$ in $\tilde{\mathcal{C}}(\alpha,p)$ is univalent in the unit disk $\mathcal{U}$. Then we determine the radii of starlikeness and convexity for $\tilde{\mathcal{C}}(\alpha,p)$.

A function $f(z)$ of $\mathcal{A}$ is said to be starlike of order β if $f(z)$ satisfies

$$\text{(6.1)} \qquad \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \qquad (z \in \mathcal{U})$$

for some β $(0 \leqq \beta < 1)$. Further a function $f(z)$ of $\mathcal{A}$ is said to be convex of order β if $f(z)$ satisfies

$$\text{(6.2)} \qquad \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta \qquad (z \in \mathcal{U})$$

for some β $(0 \leqq \beta < 1)$.

THEOREM 7. Let the function $f(z)$ defined by (1.5) be in the class $\tilde{\mathcal{C}}(\alpha,p)$. Then $f(z)$ is starlike of order β $(0 \leqq \beta < 1)$ in the disk $|z| < r_1(\alpha,\beta,p)$, where $r_1(\alpha,\beta,p)$ is the largest value for which

$$\text{(6.3)} \qquad \frac{p(1-\alpha)(1-\beta)}{2} r + \frac{(1-p)(1-\alpha)(n-\beta)}{n} r^{n-1} \leqq 1 - \beta$$

for $n \geqq 3$. The result is sharp.

PROOF. It is easy that

$$\text{(6.4)} \qquad \left|\frac{zf'(z)}{f(z)} - 1\right| \leqq \frac{\frac{p(1-\alpha)}{2} r + \sum_{n=3}^{\infty}(n-1)a_n r^{n-1}}{1 - \frac{p(1-\alpha)}{2} r - \sum_{n=3}^{\infty} a_n r^{n-1}}$$

$$\leqq 1 - \beta$$

for $|z| \leqq r < 1$ if and only if

$$(6.5) \qquad \frac{p(1-\alpha)(2-\beta)}{2} r + \sum_{n=3}^{\infty} (n-\beta) a_n r^{n-1} \leqq 1-\beta.$$

Since $f(z) \in \tilde{\mathcal{C}}(\alpha,p)$, by means of Theorem 1, we may set

$$(6.6) \qquad a_n = \frac{(1-p)(1-\alpha)}{n} \lambda_n \qquad (n \geqq 3),$$

where $\lambda_n \geqq 0$ $(n \geqq 3)$ and

$$(6.7) \qquad \sum_{n=3}^{\infty} \lambda_n \leqq 1.$$

For each fixed r, choose the integer $n_0 = n(r)$ for which $(n-\beta)r^{n-1}/n$ is maximal. Then it follows that

$$(6.8) \qquad \sum_{n=3}^{\infty} (n-\beta) a_n r^{n-1} \leqq \frac{(1-p)(1-\alpha)(n_0-\beta)}{n_0} r^{n_0-1} .$$

Consequently f(z) is starlike of order β in $|z| \leqq r_1(\alpha,\beta,p)$ provided that

$$(6.9) \qquad \frac{p(1-\alpha)(2-\beta)}{2} r + \frac{(1-p)(1-\alpha)(n_0-\beta)}{n_0} r^{n_0-1}$$

$$\leqq 1-\beta.$$

Now, find the value $r_0 = r_0(\alpha,\beta,p)$ and corresponding $n_0(r_0)$ so that

$$(6.10) \qquad \frac{p(1-\alpha)(2-\beta)}{2} r_0 + \frac{(1-p)(1-\alpha)(n_0-\beta)}{n_0} r_0^{n_0-1}$$

$$= 1-\beta.$$

It is this value r_0 that is the radius of starlikeness of order β of $\tilde{\mathcal{C}}(\alpha,p)$.

Finally we can see that the result of the theorem is sharp for the function f(z) given by (2.3).

COROLLARY 5. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then f(z) is starlike in the disk $|z| < r_2(\alpha,p)$, where $r_2(\alpha,p)$ is the largest value for which

$$\frac{p(1-\alpha)}{2} r + (1-p)(1-\alpha)r^{n-1} \leq 1 \qquad (n \geq 3). \tag{6.11}$$

The result is sharp.

PROOF. Putting $\beta = 0$ in Theorem 7, we have the corollary.

THEOREM 8. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then f(z) is convex of order β $(0 \leq \beta < 1)$ in the disk $|z| < r_3(\alpha,\beta,p)$, where $r_3(\alpha,\beta,p)$ is the largest value for which

$$p(1-\alpha)(2-\beta)r + (1-p)(1-\alpha)(n-\beta)r^{n-1} \leq 1-\beta \qquad (n \geq 3). \tag{6.12}$$

The result is sharp.

PROOF. The function f(z) defined by (1.5) will be convex of order β in the disk $|z| \leq r$ for which

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{p(1-\alpha)r + \sum_{n=3}^{\infty} n(n-1)a_n r^{n-1}}{1 - p(1-\alpha)r - \sum_{n=3}^{\infty} na_n r^{n-1}} \leq 1-\beta, \tag{6.13}$$

that is,

$$(6.14) \qquad p(1-\alpha)(2-\beta)r + \sum_{n=3}^{\infty} n(n-\beta)a_n r^{n-1} \leqq 1-\beta.$$

In the same way as in the proof of Theorem 7, we can show the theorem. Further the result of the theorem is sharp for the function f(z) given by (2.3).

Finally putting $\beta = 0$ in Theorem 8, we have the following corollary.

COROLLARY 6. Let the function f(z) defined by (1.5) be in the class $\tilde{C}(\alpha,p)$. Then f(z) is convex in the disk $|z| < r_4(\alpha,p)$, where $r_4(\alpha,p)$ is the largest value for which

$$(6.15) \qquad 2p(1-\alpha)r + n(1-p)(1-\alpha)r^{n-1} \leqq 1 \qquad (n \geqq 3).$$

The result is sharp.

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