RELATIVE PARACOMPACTNESS
AS TAUTNESS CONDITION IN SHEAF THEORY

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RESUME: Nous introduisons la paracompacité relative. Cette notion nous permet d’obtenir un critère de faîteur qui unifie et généralise les résultats classiques de [2].

INTRODUCTION

Let $X$ be a topological space, $S$ a subset of $X$, $\mathcal{F}$ a family of supports in $X$ and $\Gamma_S$ the set of the open neighborhoods of $S$ in $X$, ordered by the relation $\supset$. In this paper, we consider only sheaves of abelian groups. We say that $S$ is $\mathcal{F}$-taut in $X$ if the canonical morphism

$$
(\tau_S : \lim_{\mathcal{V} \in \mathcal{F}} H^*_\mathcal{F}(\mathcal{V}, F|_\mathcal{V}) \longrightarrow H^*_\mathcal{F}(S, F|_S))
$$

is an isomorphism whenever $F$ is a sheaf on $X$. In [2] G.E. Bredon proves that it is equivalent to say that the canonical morphism

$$
(\tau_{SX} : \Gamma_\mathcal{F}(X, F) \longrightarrow \Gamma_\mathcal{F}(S, F|_S))
$$

is onto and that $F|_S$ is $\mathcal{F}\cap S$-acyclic whenever $F$ is a flabby sheaf on $X$. The tautness appears in the hypothesis of many important theorems of sheaf theory. So, for practical use, we need criteria stating that $S$ is $\mathcal{F}$-taut in $X$ under more explicit topological assumptions on $S$ and $\mathcal{F}$. For example, it is trivial to see that an open subset of $X$ is $\mathcal{F}$-taut. In [2] it is proved that $S$ is $\mathcal{F}$-taut in $X$ if one of the following conditions is satisfied:

a) $\mathcal{F}$ is paracompactifying for the pair $(X, S)$

b) $\Phi$ is paracompactifying, $X$ is completely paracompact, $S$ is arbitrary.

c) $\Phi$ is paracompactifying, $S$ is closed in $X$.

d) $\Phi$ is maximum, $S$ is compact and relatively Hausdorff in $X$.

The purpose of this paper is to prove a tautness criterion which unifies and generalizes the preceding ones. For this reason, we introduce in definition 1 the notion of relative paracompactness of $S$ in $X$. We say that $\Phi$ is $S$-paracompactifying if every element of $\Phi$ has a neighborhood belonging to $\Phi$ and if $S \cap F$ is relatively paracompact in $F$ whenever $F$ belongs to $\Phi$. Our main result states that $S$ is $\Phi$-taut in $X$ if $\Phi$ is $S$-paracompactifying.

RELATIVE PARACOMPACTNESS

In order to avoid confusions, let us recall the following definitions.

An open covering of $S$ in $X$ is a set $U$ of open subsets of $X$, such that $U \supseteq S$. For a set $U$ of subsets of $X$ we write $U(S) = \{ U : U \in U, U \cap S \neq \emptyset \}$. We say then that:

a) $U$ is punctually finite on $S$ if $U(\{s\})$ is finite for every element $s$ of $S$.

b) $U$ is locally finite on $S$ if each element of $S$ has a neighborhood $V$ such that $U(V)$ is finite.

c) A $S$-refinement of an open covering $U$ of $S$ in $X$ is an open covering $V$ of $S$ in $X$ such that every element of $V$ is contained in some element of $U$.

Now let us introduce the following

DEFINITION 1. The subset $S$ of $X$ is

a) relatively Hausdorff in $X$ if two distinct points of $S$ have disjoint neighborhoods in $X$.

b) relatively normal in $X$ if two disjoint closed subsets of $S$ have disjoint neighborhoods in $X$.

c) relatively paracompact in $X$ if every covering of $S$ in $X$ has a $S$-refinement which is locally finite on $S$ and if moreover $S$ is relatively Hausdorff in $X$. 

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REMARK 2. It is clear that $X$ is relatively Hausdorff (resp. normal; paracompact) in $X$ if and only if $X$ is Hausdorff (resp. normal; paracompact).

Slight modifications of classical proofs give the following three results.

PROPOSITION 3. If $S$ is relatively normal in $X$ and if $U$ is an open covering of $S$ in $X$ which is punctually finite on $S$ then there exists a family $(V_U)_{U \in S}$ of open subsets of $X$, covering $S$ and such that $V_U$ is contained in $U$ for every $U$ belonging to $U$.

PROPOSITION 4. If $F$ is a closed subset of $S$ and if $S$ is relatively paracompact in $X$ then every open covering of $F$ in $X$ has a $T$-refinement which is locally finite on $S$. In particular $F$ is relatively paracompact in $X$.

PROPOSITION 5. If $S$ is relatively paracompact in $X$ then $S$ is relatively normal in $X$.

The following easy results are also useful.

PROPOSITION 6. If $S$ is relatively paracompact in $X$ and if $Y$ is a subset of $X$ containing $S$ then $S$ is relatively paracompact in $Y$. In particular $S$ is paracompact.

Proof: Let $U$ be an open covering of $S$ in $Y$. It is clear that there exists an open covering $V$ of $S$ in $X$ such that $V \cap Y = U$. Thus there exists a $S$-refinement $W$ of $V$ which is locally finite on $S$. We see directly that $W \cap Y$ is a $S$-refinement of $U$ in $Y$ which is locally finite on $S$. To conclude, we just have to note that $S$ is relatively Hausdorff in $Y$. //

PROPOSITION 7. If $S$ has a fundamental system of paracompact neighborhoods in $X$ then $S$ is relatively paracompact in $X$.

Proof: Let $U$ be an open covering of $S$ in $X$. Let us choose a paracompact neighborhood $V$ of $S$ in $X$ contained in $U$. Since $U \cup V$ is an open covering of $V$ in $V$, there exists a $V$-refinement $V'$ of $U \cup V$ in $V$ which is locally finite on $V$. Thus $V \cap V$ is a $S$-refinement of $U$ in $X$ which is locally finite on $S$. To conclude, it remains to prove that $S$ is relatively Hausdorff in $X$. Let $x, y$ be two distincts elements of $S$ and $W$ a paracompact neighborhood of $S$ in $X$. Since $W$ is a Hausdorff space, there exist neighborhoods $V_x, V_y$ of $x$ and $y$. //
and $y$ in $W$, such that $\forall x. y \neq y$. But $W$ is a neighborhood of $x$ (resp. $y$) in $X$. Then $x$ and $y$ have disjoint neighborhoods in $X$. 

COROLLARY 8.

a) A subset $S$ of a completely paracompact space (e.g. a metric space) $X$ is relatively paracompact in $X$.

b) A closed subset $S$ of a paracompact space is relatively paracompact in $X$.

Proof: a) Since $X$ is completely paracompact, every open subset of $X$ is paracompact and we may apply proposition 7.

b) Since $X$ is paracompact we know that $X$ is normal and the closed neighborhoods of $S$ in $X$ form a fundamental system of paracompact neighborhoods of $S$ in $X$. So we may apply proposition 7.

PROPOSITION 9. If $S$ is compact and relatively Hausdorff in $X$ then $S$ is relatively paracompact in $X$.

Proof: Let $U$ be an open covering of $S$ in $X$. Since $U \cap S$ is an open covering of $S$, there exists a finite subset $V$ of $U \cap S$ which covers $S$. Let us choose a finite subset $W$ of $U$ such that $W \cap S = V$. Clearly $W$ is a $S$-refinement of $U$ which is locally finite on $S$. Since $S$ is relatively Hausdorff in $X$, the proof is complete.

A TAUTNESS CRITERION

PROPOSITION 10. If $S$ is relatively paracompact in $X$ then the canonical morphism

$$
(r_S : \lim_{U \in V_S} \Gamma(U, F | U) \to \Gamma(S, F | S))
$$

is an isomorphism for every sheaf $F$ on $X$.

Proof: It is clear that $r_S$ is injective, thus we just have to prove that it is onto. Let $s$ be a section of $F$ over $S$. For every $x \in S$, let us choose a neighborhood $U_x$ of $x$ in $X$ and a section $s_x$ of $F$ over $U_x$ such that $s_x | U_x \cap S = s | U_x \cap S$. 

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We know that $S$ is relatively paracompact in $X$ and that $U = \{U_x : x \in S\}$ is an open covering of $S$ in $X$, thus there exists a $S$-refinement $V$ of $U$ which is locally finite on $S$. For every $V$ belonging to $V$, let us choose a an element $x_V$ of $S$, such that $V \subseteq U_{x_V}$. Let $s^V_U$ denote the section $s^V_U | V$. Since $S$ is relatively normal in $X$, the proposition 3 gives us a family $(V_U)^{-1} \subseteq V$ of open subsets of $X$ covering $S$ and such that $\overline{V_U} \subseteq V$ whenever $V \subseteq V$. For every $V$ belonging to $V$ we denote by $s^{V_U}_V$ the section $s^V_U | \overline{V_U}$. Let us set

$$B = \bigcap_{U, V \in V} \{y : (y \in \overline{U_U} \cap \overline{V_U}) \Rightarrow (s^V_U(y) = s^U_V(y))\}.$$ 

We shall establish that $B$ is a neighborhood of $S$. Let $x$ be an element of $S$. Since $V$ is locally finite on $S$ there exists an open neighborhood $\omega$ of $x$ in $X$ such that $V(\omega)$ is finite. Let us set

$$\omega = \omega \setminus \left( \bigcup_{V \in V(\omega)} \overline{U_U} \right)_{x \in \overline{V_U}}.$$ 

Clearly $\omega$ is still a neighborhood of $x$ in $X$ and $x \in \overline{V_U}$ if $\overline{V_U} \cap \omega \neq \emptyset$. Let us set

$$\omega'' = \omega \cap \bigcap_{\overline{V_U} \cap \omega \neq \emptyset} \{y : y \in V \cap U, s^V_U(y) = s^V_U(y)\}.$$ 

We see immediately that $\omega''$ is an open neighborhood of $x$ in $X$ and that

$$(y \in \omega'' \cap \overline{V_U} \cap \overline{V_U}) \Rightarrow (s^V_U(y) = s^V_U(y))$$

if $U, V$ belong to $V$. Thus $\omega''$ is contained in $B$ and $B$ is a neighborhood of $x$ in $X$. Since $x$ is an arbitrary point of $S$, this proves that $S$ is contained in $B$. Now, since $V$ is locally finite on $S$, we know that $(\overline{V_U} : V \subseteq V)$ is locally finite on an open neighborhood $\Omega$ of $S$ in $X$. Let $\Omega'$ be the set $\Omega \cap (\bigcup \overline{V_U}) \cap B$. Clearly, $\Omega'$ is an open neighborhood of $S$ in $X$. For every $V \subseteq V$ let $F_V$ be the set $\overline{V_U} \cap \Omega'$ and let $s''_V$ be the section $s^V_U | F_V$. The family $(F_V)^{-1} \subseteq V$ defines
nes a closed locally finite covering of \( \Omega' \) and \( s''_{V}|_{F'' \cap F'} \) equals
\( s''_{U}|_{F'' \cap F'} \) if \( U, V \in V \). Thus there exists a section \( s \) of \( F \) over \( \Omega' \)
such that \( s|_{F'} = s''_{V} \) if \( V \in V \). This shows that \( s|_{S} = 0 \). Since \( \sigma \) is
an arbitrary section of \( F \) over \( S \), we have proved that \( r_{S} \) is onto.///

**DEFINITION 11.** The family of supports \( \Phi \) is \( A \)-paracompactifying
if every element of \( \Phi \) has a neighborhood which belongs to \( \Phi \) and if
\( F \cap \Omega \) is relatively paracompact in \( F \) for every \( F \) belonging to \( \Phi \).

**PROPOSITION 12.** If \( \Phi \) is \( S \)-paracompactifying and if \( F \) is a clo-
csed subset of \( S \) then \( \Phi \) is \( F \)-paracompactifying.

**Proof:** Let \( F' \) be an element of \( \Phi \). We know that \( F' \cap S \) is rel-
atively paracompact in \( F' \) and that \( F \cap F' \) is closed in \( F' \cap S \),
thus, by proposition 4, \( F \cap F' \) is relatively paracompact in \( F' \).///

**PROPOSITION 13.** If \( \Phi \) is \( S \)-paracompactifying, then
a) \( \Phi \cap S \) is paracompactifying in \( S \),
b) the canonical morphism,

\[
(r_{S} : \lim_{U \in V} \Gamma_{\Phi \cap U}(U, F|_{U}) \longrightarrow \Gamma_{\Phi \cap S}(S, F|_{S}))
\]

is an isomorphism for every sheaf \( F \) on \( X \),
c) \( F|_{S} \) is \( \Phi \cap S \)-soft for every flabby sheaf \( F \) on \( X \).

**Proof:** a) If \( F \in \Phi \), \( F \cap S \) relatively paracompact in \( F \) and
proposition 6 shows that \( F \cap S \) is paracompact.

b) Let \( F \) be a sheaf on \( X \). It is clear that \( r_{S} \) is injective, so we
just have to prove that it is onto. Let \( \sigma \) be a section of \( F \) over \( S \)
with support belonging to \( \Phi \cap S \). Let us choose an element \( F' \) of \( \Phi \)
such that \( \text{supp}(\sigma) = F \cap S \) and a neighborhood \( F' \) of \( F \) belonging to
\( \Phi \). Since \( F' \cap S \) is relatively paracompact in \( F' \), proposition 10
shows that the section \( \sigma \) extends to a section \( \sigma' \) of \( F \) over a neigh-
borhood \( V \) of \( S \cap F' \) in \( F' \). Let \( G \) be the set \( \text{supp}(\sigma') \). By construction,
we know that \( G \cap S \subset F' \cap S \) and that \( F' \cap S \subset V \). Thus \( S \setminus \\text{supp}(\sigma') \)
is contained in \( S \setminus G \). This proves that \( S \) is contained in the open set
\((X \setminus G) \cup \setminus V \). Let us denote by \( \Omega \) this open set and by \( \sigma'' \) the
section of \( F \) over \( \Omega \) which is equal to \( 0 \) on \( X \setminus G \) and to \( \sigma'|_{S} \) on \( V \). We see immediately that \( \sigma|_{S} = \sigma \) and that \( \text{supp}(\sigma'') \subset G \). Since \( G \subset F' \),

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σ" is an element of $\Gamma_{\mathfrak{S}_2}(\mathbb{C}, F|_S)$, such that $\tau_{\mathbb{S}_S}(\sigma") = \sigma$. Since $\sigma$ is an arbitrary element of $\Gamma_{\mathfrak{S}_2}(\mathbb{C}, F|_S)$ we have proved that $\tau_{\mathbb{S}_S}$ is onto.

c) Let $F$ be a flabby sheaf on $X$, $B$, $B'$ two elements of $\mathfrak{F} \cap S$ such that $B \subseteq B'$ and $\sigma$ a section of $F$ over $B$. Since $B$ is closed in $S$, we know, by proposition 12, that $\sigma$ is $\mathfrak{F}$-paracompactifying and what is proved above shows that there exists an open neighborhood $\mathcal{U}$ of $B$ in $X$ and an element $\sigma'$ of $\Gamma_{\mathfrak{S}_2}(\mathbb{C}, F|_{\mathcal{U}})$ such that $\sigma'|_{B} = \sigma$. Since $F$ is flabby, there exists a section $\sigma''$ of $F$ over $X$, such that $\sigma''|_{\mathcal{U}} = \sigma'$. Let us denote by $\sigma''$ the section $\sigma'|_{B'}$. It is clear that $\sigma''|_{B} = \sigma$. Since $\sigma$ is an arbitrary section of $F$ over $B$, we have proved that $F|_S$ is $\mathfrak{F} \cap S$ - soft.///

**CRITERION 14.** If $\phi$ is $\mathfrak{F}$-paracompactifying then $S$ is $\phi$-taut.

Proof: It is an easy consequence of the preceding proposition if we remember that an open subset of $X$ is $\phi$-taut and that a $\mathfrak{F}$-soft sheaf is $\phi$ - acyclic if $\phi$ is paracompactifying.///

**REMARK 15.** If $S$ satisfies the condition a) (resp. b); c); d) then proposition 7 (resp. 7; 7; 9) shows that $\phi$ is $\mathfrak{F}$-paracompactifying and the preceding result shows that $S$ is $\phi$-taut.///

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**REFERENCES**


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