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CONSTANTS OF MOTION OF CHARGED MESONS WITH CLASSICAL INTERACTIONS *

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RESUME

Nous étudions les formalismes hamiltonien et lagrangien décrivant des mésons scalaires (et vectoriels) plongés dans un champ électromagnétique extérieur. L'ensemble complet des constantes du mouvement est obtenu en utilisant la description relativiste de Sakata-Taketani.

1. INTRODUCTION

We recently discussed (Beckers and Hussin, 1984) the full set of constants of motion characterizing relativistic or nonrelativistic electrons interacting with (external) classical electromagnetic fields. Based on Hamiltonian and Lagrangian formalisms, such a study deals with recent gauge theoretical developments (Jackiw and Manton, 1980) and symmetries of potentials (Beckers and Hussin, 1983). Through group extensions (Bargmann, 1954) and associated extended Lie algebras, we completed the fundamental and standard approach of Johnson and Lippmann (1949). In particular, such developments exploited the U(1)-gauge theory as well as the minimal electromagnetic coupling principle.

Relativistic or nonrelativistic electrons are respectively described by the Dirac or Schrödinger equations. These are time first order descriptions whose associated Hamiltonians and Lagrangians are very well known. If the considered charged particles are (scalar or vector) mesons, the corresponding elements are not so simple : from Bhabba's equations (Bhabba, 1939) or Kemmer's formulation (Kemmer, 1939) Hamiltonian formalisms have been obtained but with extra conditions (eliminating the redundant components). The only net formalism for such mesons is the one presented by Sakata and Taketani (1940) (see also Baym, 1969) : in particular, for scalar mesons, it corresponds to a Hamiltonian version of the time second order description associated with the Klein-Gordon equation.

Here we apply our recent developments (Beckers and Hussin, 1984) to the case of scalar mesons by using the Sakata-Taketani formulation with electromagnetic interactions. We get the full set of constants of motion and manage all the difficul-

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ties inherent to this spin-0 formalism. First we just recall the sufficient elements of Sakata-Taketani's formulation (§ 2). Then the constants of motion are pointed out and discussed through the Hamiltonian (§ 3) and Lagrangian (§ 4) formalisms. The case of vector mesons is briefly considered (§ 5).

2. RELATIVISTIC SCALAR MESONS AND SAKATA-TAKETANI'S FORMULATION

The motion of free relativistic scalar mesons is described by the Klein-Gordon equation :

$$(\Box + m^2)\psi = 0 \quad , \quad \Box = - p^\mu p_\mu \quad , \quad p^\mu = i\partial^\mu \tag{2.1}$$

where $\psi(x)$ is a scalar wave function. This equation can also be written as a time "first order" equation :

$$i\partial_t \phi = \frac{1}{2m} \vec{p}^2 (\sigma^3 + i\sigma^2)\phi + m\sigma^3 \phi \tag{2.2}$$

in terms of the wave function ϕ given by

$$\phi = \frac{1}{2m}\begin{pmatrix} m\psi + \psi_0 \\ m\psi - \psi_0 \end{pmatrix} \quad , \quad \psi_0 = i\partial_t \quad . \tag{2.3}$$

This corresponds to the well known Sakata-Taketani equation (Sakata-Taketani,1940) and its Hamiltonian operator :

$$H^0_{S.T.} = \frac{1}{2m} \vec{p}^2 (\sigma^3 + i\sigma^2) + m\sigma^3 \tag{2.4}$$

where the $\vec{\sigma}$-matrices refer to the usual (2×2) Pauli matrices.

Such an equation (2.2) can be obtained from the Lagrangian density :

$$L_{0,S.T.} = \frac{1}{2m} \vec{\nabla}\bar{\phi}(\sigma^3 + i\sigma^2)\cdot\vec{\nabla}\phi + \frac{i}{2}[(\partial_t\bar{\phi})\phi - \bar{\phi}\partial_t\phi] + m\bar{\phi}\sigma^3\phi \tag{2.5}$$

where $\bar{\phi}$ is the "adjoint" wave function $\bar{\phi} = \phi^\dagger\sigma^3$. This Lagrangian density is strictly invariant under infinitesimal Poincaré transformations :

$$x \to x' : x'^\mu = x^\mu + \omega^\mu{}_\nu x^\nu + a^\mu = x^\mu - \xi^\mu \quad , \tag{2.6}$$

if the wave functions ϕ and $\bar{\phi}$ admit the following transformation laws :

$$\begin{aligned} \phi'(x') &= \phi(x) - \frac{\vec{v}\cdot\vec{p}}{2m} (\sigma^3 + i\sigma^2)\phi(x) = \phi(x) + \Delta\phi(x) \quad , \\ \bar{\phi}'(x') &= \bar{\phi}(x) + \frac{\vec{v}\cdot\vec{p}}{2m} \bar{\phi}(x)(\sigma^3 + i\sigma^2) = \bar{\phi}(x) + \overline{\Delta\phi}(x) \quad , \end{aligned} \tag{2.7}$$

where $\vec{v}$ corresponds as usual to the three infinitesimal parameters of the Lorentz boosts.

When the scalar mesons interact with an arbitrary electromagnetic field F , the Klein-Gordon equation takes the form

$$(\Pi^\mu \Pi_\mu - m^2)\psi = 0 \tag{2.8}$$

with

$$\{\Pi^\mu\} = \{\Pi^0, \vec{\Pi}\} \quad , \quad \Pi^0 = p^0 + eV = i\partial_t + eV \quad , \quad \vec{\Pi} = \vec{p} + e\vec{A} \quad , \tag{2.9}$$

when $A = \{A^\mu\} = \{A^0, \vec{A}\}$ is a fourpotential associated with the field F . From

Sakata-Taketani's developments applied to the interacting case, we get (Baym,1969)

$$i\partial_t\phi = H_{S.T.}\phi \quad , \tag{2.10}$$

where the wave function ϕ is given by (2.3) but with

$$\psi_0 = (i\partial_t + eV)\psi \tag{2.11}$$

and where the Hamiltonian takes the form :

$$H_{S.T.} = \frac{1}{2m}\,\vec{\Pi}^2(\sigma^3 + i\sigma^2) + m\sigma^3 - eV\mathbb{1} \quad . \tag{2.12}$$

In this context, the total Lagrangian density reads

$$L = L_{S.T.} + L_M \tag{2.13}$$

with

$$L_{S.T.} = \frac{1}{2m}(\vec{\Pi}^*\bar{\phi})(\sigma^3 + i\sigma^2)\vec{\Pi}\phi + \frac{i}{2}[(\partial_t\bar{\phi})\phi - \bar{\phi}\partial_t\phi] + m\bar{\phi}\sigma^3\phi - eV\bar{\phi}\phi \tag{2.14}$$

and

$$L_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad . \tag{2.15}$$

The invariance of the theory under Poincaré transformations is realized if ϕ , $\bar{\phi}$ and A transform according to :

$$\left.\begin{aligned} \phi'(x') &= \phi(x) - \frac{1}{2m}\,\vec{v}\cdot\vec{\Pi}(\sigma^3 + i\sigma^2)\phi(x) \quad , \\ \bar{\phi}'(x') &= \bar{\phi}(x) + \frac{1}{2m}\,(\vec{v}\cdot\vec{\Pi})\bar{\phi}(x)(\sigma^3 + i\sigma^2) \end{aligned}\right. \tag{2.16}$$

and

$$A'_\mu(x') = A_\mu(x) - \xi^\nu\partial_\nu A_\mu(x) \quad . \tag{2.17}$$

3. CONSTANTS OF MOTION FROM THE HAMILTONIAN FORMALISM

Through the Hamiltonian formalism and within the free and interacting cases we search for complete sets of constants of motion, i.e. for operators C satisfying the equation

$$\dot{C} = \frac{d}{dt}C = \frac{\partial C}{\partial t} + i[H,C] = 0 \quad . \tag{3.1}$$

In the free case i.e. with the Hamitonian (2.4), such a set is easily obtained. It contains the momentum $\vec{p}$, the angular momentum $\vec{L} = \vec{r}\wedge\vec{p}$ and the quantities $\vec{K}$ given by

$$\vec{K} = t\vec{p} - \vec{r}H^0_{S.T.} + \frac{i\vec{p}}{2m}(\sigma^3 + i\sigma^2) \quad . \tag{3.2}$$

Then, the operators $P^0 = H^0_{S.T.}$, $\vec{P} = \vec{p}$, $\vec{L} = \vec{r}\wedge\vec{p}$ and $\vec{K} \equiv (3.2)$ are evidently the ten generators of the Poincaré Lie algebra. It is a consequence of the invariance of the Klein-Gordon equation under the Poincaré group. The realization of the generators is the expected one but we notice that in $\vec{K} \equiv (3.2)$, we have an additional term due to the Sakata-Taketani formulation. This extra term comes in due to the fact that although the Klein-Gordon wave function $\psi(x)$ has to be scalar, the function $\phi(x)$ given by (2.3) is not (see (2.7)).

Let us presently discuss the case of charged scalar mesons interacting with a particular constant electromagnetic field $F = (\vec{E},\vec{B})$ chosen as the parallel field (Bacry, Combe and Richard, 1970) :

$$F_{/\!/} \equiv \{\vec{E} = (0,0,E), \vec{B} = (0,0,B)\} \tag{3.3}$$

and where the four potential included in $H_{S.T.} \equiv (2.12)$ is the associated gauge symmetrical potential

$$V = -\frac{1}{2}Ez \quad , \quad \vec{A} = \frac{1}{2}(-By,Bx,-Et) \ . \tag{3.4}$$

From the following identities

$$\dot{\vec{p}} = -\frac{e}{2}(B\Pi^2(\sigma^3 + i\sigma^2), -B\Pi^1(\sigma^3 + i\sigma^2), E\Pi) \quad , \quad \dot{\vec{r}} = \frac{1}{m}\vec{\Pi}(\sigma^3 + i\sigma^2) \tag{3.5}$$

and

$$\dot{H}_{S.T.} = -\frac{eE}{2m}\Pi^3(\sigma^3 + i\sigma^2) \quad , \tag{3.6}$$

we obtain the constants of motion

$$\begin{gathered} \vec{\Pi} = \vec{p} - e\vec{A} \quad , \quad \Pi^0 = H_{S.T.} - eV \quad , \\ J^3 = xp^2 - yp^1 \quad , \quad K^3 = tp^3 - zH_{S.T.} + \frac{i\Pi^3}{2m}(\sigma^3 + i\sigma^2) \quad . \end{gathered} \tag{3.7}$$

These constants are thus associated with the generators of an extension by R of the $F_{/\!/}$-kinematical algebra $\mathbb{G}_{F_{/\!/}} = \{J^3,K^3,P^\mu\}$ (Bacry, Combe and Richard, 1970). As a last comment, let us notice that the expressions of the constants (3.7) explicitly depend on the form of the potential A associated with the field $F_{/\!/}$. It can be easily shown that the generators of $G_{F_{/\!/}}$ which are symmetries of a potential give rise to constants of motion unchanged with respect to the free case (Beckers and Hussin, 1983, 1984).

4. CONSTANTS OF MOTION FROM THE LAGRANGIAN FORMALISM

Let us briefly consider the Lagrangian formalism in order to find the constants of motion through Noether's theorem (Noether, 1918; Hill, 1951). This theorem essentially says that the invariance (strict or up to a fourdivergence) of a Lagrangian density under infinitesimal transformations implies the existence of a conserved current J ($\partial_\mu J^\mu = 0$) and of a constant of the motion

$$C = \int d\vec{r} J^0 \ . \tag{4.1}$$

In our particular case, the Poincaré invariance of the free Lagrangian density (2.5) is a consequence of the transformation laws (2.7) on ϕ and $\bar{\phi}$. Then, the associated constant of motion can be written in the form

$$C_{0,S.T.} = \int d\vec{r}\ [\frac{i}{2}(\Delta\bar{\phi}\phi - \bar{\phi}\Delta\phi) - L_{0,S.T.}\xi^0]$$

or, with suitable limit conditions,

$$C_{0,S.T.} = -\frac{i}{2}\int d\vec{r}\;\bar{\phi}\Delta\phi + h.c. \quad . \tag{4.2}$$

These results are the expected ones if we remember that in the Sakata-Taketani formulation the scalar product reads

$$\langle\phi,\phi\rangle = \int d\vec{r}\;\phi^{\dagger}\sigma^{3}\phi = \int d\vec{r}\;\bar{\phi}\phi \quad . \tag{4.3}$$

In the interacting case (with an arbitrary electromagnetic field), the total Lagrangian density (2.13) is evidently invariant under Poincaré transformations (2.16) and (2.17) on ϕ , $\bar{\phi}$ and A and, moreover, under usual local gauge transformations on the corresponding wave functions and potential.

In order to obtain the equivalence between such elements and the preceding results, we have to combine coordinate and gauge transformations (Jackiw-Manton, 1980) in such a way that the electromagnetic potential will be an invariant quantity $(A'(x) = A(x))$, so that the Noether theorem still applies if we have the following transformation laws :

$$\begin{aligned} \phi'(x') &= \phi(x) - \frac{\vec{v}_F\cdot\vec{\Pi}}{2m}(\sigma^3 + i\sigma^2)\phi(x) - ieW_F\phi(x) \quad , \\ \bar{\phi}'(x') &= \bar{\phi}(x) + \frac{\vec{v}_F\cdot\vec{\Pi}}{2m}\bar{\phi}(x)(\sigma^3 + i\sigma^2) + ieW_F\bar{\phi}(x) \end{aligned} \tag{4.4}$$

and

$$A'_{\mu}(x') = A_{\mu}(x) + (\partial_{\mu}\xi^{\nu}_{F})A_{\nu}(x) - \partial_{\mu}W_F(x) \quad , \tag{4.5}$$

where ξ_F refers to the Poincaré transformations (2.6) leaving the field F invariant and $W_F(x)$ to compensating gauge transformations for A (Janner and Janssen, 1971). The associated constant of motion is given by

$$C_F = -\frac{i}{2}\int d\vec{r}\;\bar{\phi}(\Delta_F - ieW_F)\phi + h.c. \quad , \tag{4.6}$$

which can be compared with $C_{0,S.T.} \equiv (4.2)$ when the Poincaré transformations associated with the symmetries of the field F are considered.

Finally, the identification with the results obtained in the Hamiltonian approach is easily realized if we consider the particular $F_{/\!/}$ field and the potential $A^{(0)} \equiv (3.4)$. We effectively obtain the constants of motion

$$\begin{aligned} \langle\pi^{\mu}\rangle = \int d\vec{r}\;\bar{\phi}\pi^{\mu}\phi \quad , \quad \langle J^3\rangle = \int d\vec{r}\;\bar{\phi}J^3\phi \quad , \\ \langle K^3\rangle = \int d\vec{r}\;\bar{\phi}K^3\phi \quad . \end{aligned} \tag{4.7}$$

5. RELATIVISTIC VECTOR MESONS IN THE SAKATA-TAKETANI FORMULATION

In the Sakata-Taketani formulation, the description of free vector mesons is given by the equation

$$i\partial_t\psi = H^0\psi \tag{5.1}$$

with the Hamiltonian operator

$$H^0 = \frac{1}{2m}\vec{p}^2(\sigma^3 + i\sigma^2) - i\sigma^2\frac{(\vec{S}\cdot\vec{p})^2}{m} + m\sigma^3 \qquad (5.2)$$

where $\vec{S} = (S^1,S^2,S^3)$ are the 3×3 spin-one matrices commuting with the σ^i's.

In this case and with the Hamiltonian (5.2), the constants of motion are associated with the usual momentum operator $\vec{p}$, the total angular momentum $\vec{J} = \vec{L} + \vec{S}$ and the operator $\vec{K}$ given by

$$\vec{K} = t\vec{p} - \vec{r}H^0 + \frac{i\vec{p}}{2m}(\sigma^3 + i\sigma^2) + \frac{\sigma^2}{2m}[(\vec{S}\cdot\vec{p})\vec{S} + \vec{S}(\vec{S}\cdot\vec{p})] \ . \qquad (5.3)$$

The result (5.3) differs from (3.2) by a supplementary contribution due to the spin term in the Hamiltonian (5.2). This explicit calculation has to take into account the $\vec{S}$-matrix properties of Kemmer algebras such as

$$S_iS_jS_k + S_kS_jS_i = \delta_{ij}S_k + \delta_{kj}S_i \qquad (5.4)$$

and

$$(\vec{S}\cdot\vec{p})^3 = \vec{p}^2(\vec{S}\cdot\vec{p}) \quad . \qquad (5.5)$$

Finally, if we consider the interaction of such vector mesons with a specific external electromagnetic field, the constants of motion are still associated with the generators of an extended algebra $\bar{\mathbb{C}}$ by R of the symmetry algebra $\mathbb{C}_F$ of the field under consideration. They can be easily obtained using the interacting Hamiltonian :

$$H = \frac{1}{2m}\vec{\Pi}^2(\sigma^3 + i\sigma^2) - i\sigma^2\frac{(\vec{S}\cdot\vec{\Pi})^2}{m} - eV\mathbb{1} + m\sigma^3 \ . \qquad (5.6)$$

For example, if the case (3.3)-(3.4) is considered, we can get the six constants of motion corresponding to Eqs.(3.7). We leave the search for these constants as an exercise for the reader.

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