Bulletin de la Société Royale des Sciences de Liège, Vol. 80, 2011, p. 549 - 554

Solving light curves of WR+O binaries: the regularization approach

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Abstract: Extended semi-transparent atmospheres of WR stars in eclipsing WR+O binaries present some difficulties for analysis of their light curves. We present an approach to the problem based on solving the most general form of integral equations describing a light curve of a WR+O binary: Fredholm's equations of the first kind. The unknown functions are the brightness and opacity distributions across the disk of the WR component. The equations represent an ill-posed problem. To get a stable, unique solution one needs to impose some *a priori* constraints on the solution. We review various physically justified sets of constraints and, using artificially simulated light curves with known solutions, show how these constraints and the so-called regularization technique work to retrieve the functions of interest. The influence of errors in the input light curve on the solution is shown and discussed. The algorithms and the corresponding computer programs are open to the scientific community.

1 Introduction

Eclipsing binaries containing a WR component provide a potential possibility to directly probe WR winds and to get important clues about their structure. However, strong semi-transparent WR winds make parametric modeling in binaries rather problematic. In the most general case the light loss during an eclipse in a binary is described by the equation

$$1 - l(\Delta) = \iint\limits_{S(\Delta)} I_c(\vec{\rho}) I_a(\vec{r})\, d\sigma \tag{1}$$

where l is the normalized flux (maximal flux outside of eclipses is equal to 1), $I_c(\vec{\rho})$ is the brightness distribution across the disk of the eclipsed component, $I_a(\vec{r})$ is the opacity distribution across the disk of the component in front, S is the eclipsed area, Δ is the projected distance between the centers of the stellar disks (normalized by the orbital separation), i is the orbital inclination angle. For simplicity, in the rest of the paper we assume a circular orbit, although the method outlined below can be easily generalized to elliptical orbits. For a circular orbit, $\Delta = \sqrt{\cos^2 i + \sin^2 i \sin^2 \theta}$, where θ is the orbital phase. In case of spherical stars with thin atmospheres the function I_a for a given component is simply

$$I_a(\vec{r}) = \begin{cases} 1, & |\vec{r}| < r_* \\ 0, & |\vec{r}| \geq r_* \end{cases}, \tag{2}$$

while I_c may be parametrized e.g., by the linear darkening law

$$I_c(s) = I_0\left(1 - x + x\sqrt{1 - \frac{s^2}{r_*^2}}\right) \tag{3}$$

With these assumptions, equation (1) becomes a parametric problem which can be solved by e.g. minimizing χ^2 error between the model and observed light curves. Many minimization techniques have been proposed in the literature (see e.g. the semi-analytical approach by Russell & Merrill (1952) for classical binaries or the purely numerical method of Wilson (1979) for close binaries with tidally distorted components). In case of WR+O binaries the functions I_a and I_c for the WR component cannot be easily represented by any parametric expressions. In the present paper we implement a method for directly solving (1) without making any strong assumptions about I_a and I_c for the WR component.

In Section 2 we describe the method itself. In Section 3 we show how the method works on artificially simulated light curves of a WR+O binary and discuss the influence of errors in the input data on the solution. We discuss the results and potential application of the method to other problems in Section 4.

2 Method: model assumptions and basic equations

The method outlined below follows the approach first suggested by Cherepashchuk and his co-authors (see e.g., Goncharsky, Cherepashchuk & Yagola 1985, Antokhin & Cherepashchuk 2001 and references therein). To simplify the problem, we make the following assumptions: (i)the binary components are spherical, the functions I_a and I_c are axially symmetrical; (ii)the O component is a "normal" main-sequence star so its I_a and I_c functions can be represented by the equations (2) and (3) respectively. With these assumptions, equation (1) written for both eclipses, becomes

$$1 - l_1(\Delta) = \int_0^{R_{\mathrm{w}}} K_1(s,\Delta) I_0 I_a(s)\, ds \tag{4a}$$

$$1 - l_2(\Delta) = \int_0^{R_{\mathrm{w}}} K_2(s,\Delta) I_c(s)\, ds \tag{4b}$$

$$L_{\mathrm{O}} + L_{\mathrm{WR}} = I_0 \pi r_{\mathrm{O}}^2 (1 - \frac{x}{3}) + 2\pi \int_0^{R_{\mathrm{w}}} I_c(s) s\, ds = 1, \tag{4c}$$

where indexes 1 and 2 refer to the primary (WR star in front) and secondary (O star in front) eclipses respectively, K_1 and K_2 are the equation kernels describing the geometry of the eclipsed areas, I_0 is the brightness of the O component in the center of its disk, I_a and I_c are opacity and brightness functions for the WR component, R_{w} is the radius of the WR disk (wind), r_{O} is the radius of the O star, L_{O} and L_{WR} are the luminosities of the O and WR stars respectively. The third equation is the normalization condition on the luminosities of the binary components. Expressions for K_1 and K_2 will be given in the detailed forthcoming paper (Antokhin 2011).

The unknown quantities in (4) are the I_a and I_c functions, the orbital inclination angle i and the radius of the O star r_{O}. At any given pair of i and r_{O}, (4) can be solved as follows: (i)solve (4b) and obtain I_c; (ii)substitute I_c to (4c) and obtain I_0; (iii)substitute I_0 to (4a) and solve it to obtain I_a. The equations(4a,b) have the form of the well known Fredholm's integral equation of the first kind

$$Az = \int_a^b K(x,s)z(s)ds = u(x), x \in [c,d] \tag{5}$$

Such equations present a so-called ill-posed problem, that is, infinitely small perturbations in the input data may result in arbitrarily large fluctuations in the solution. Obtaining a unique stable solution requires some *a-priori* knowledge about the unknown function. According to Tikhonov et al. (1995), there are two possible approaches to the problem: (i)to solve Fredholm's equation on a so-called compact set of unknown functions; (ii)to use some sort of regularization technique. We consider these two approaches below.

2.1 Solving (4) on a compact set

A set Y of a metric space Z is said to be compact if from any infinite sequence of its elements one can extract a sub-sequence converging to some element $y \in Y$. Considering our particular problem, the examples of the compact sets are (i)non-negative monotonically non-increasing functions; (ii)non-negative convex functions; (iii)non-negative convex-concave functions. It can be demonstrated (see, e.g. Tikhonov et al. 1995) that Fredholm's equation of the first kind has a unique solution if the latter is searched for on a compact set. The above examples of compact sets seem to be quite reasonable assumptions about the unknown brightness and opacity distributions across the disk of the WR component. They are loose enough to not restrict the functions by any parametric form like the linear limb darkening. Solving (4a), (4b) consists in minimizing the residual (the norm) squared $||Az - u||^2$, $z \in S$ (see (5)). Here u is the observed light curve $(1 - l_{1,2}(\Delta))$, Az is the model light curve (the elements of the matrix A are constructed in such a way that the product Az is the numerical approximation of the integral in (5)). A particular expression for the norm depends on the metric used to measure the distance between the model and the input light curve. This approach was used in Antokhin & Cherepashchuk (2001) and other previous papers.

2.2 Regularization approach

One problem with solving (4) on a compact set is that it does not require the solution to be smooth. As we will see below, this leads to unrealistic solutions. One can stabilize the solution and require it to be smooth by using the regularization technique developed by Tikhonov (Tikhonov et al. 1995). The basic idea is very simple: instead of minimizing the residual, one has to minimize the function $||Az - u||^2 + \alpha||z||^2$. The second term in this expression is the so-called stabilizing term. It is small when z is smooth and large when z is oscillating. Thus, minimizing this function one can minimize the residuals while keeping z smooth. The regularization parameter α controls the relative weight of the stabilising term.

The central question in this technique is how to choose α. Tikhonov et al. (1995) showed that there exists a way of choosing α based on the uncertainty of the input data δ such that the resulting approximate solution converges to the true solution as long as $\delta \to 0$. In case the A operator is known exactly, α must be chosen in such a way that $||Az^\alpha_\delta - u_\delta|| = \delta$, where u_δ is the input data set containing some noise, and z^α_δ is the approximate solution obtained with this data set.

In its original form the regularization technique requires the solution to be smooth and does not impose other constraints. The technique can, however, be combined with some *a priori* constrains on the unknown functions like those listed in the previous subsection.

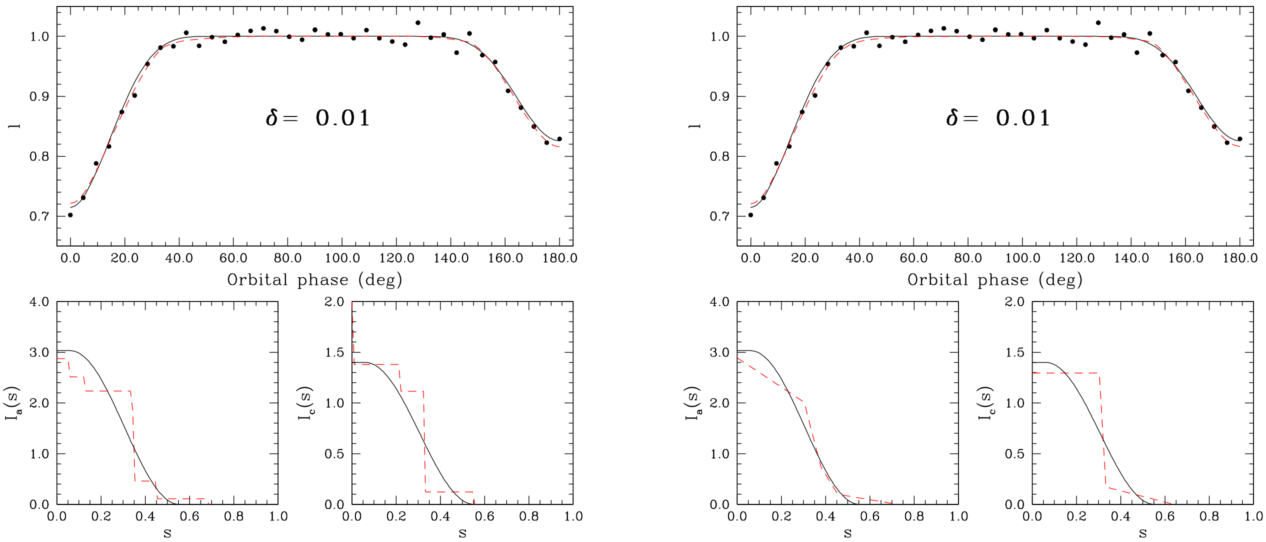

Figure 1: Solution of (4) on compact sets. Left: assumption (i) of section 2.1. Right: assumption (iii). Solid lines represent the exact I_a and I_c (bottom plots) and the exact light curve (top plots). Dots on the top plots show the simulated light curve with added noise. Dashed lines represent approximate solutions (bottom plots) and model light curves (top plots).

3 Simulated light curves

To demonstrate how various approaches to solving (4) work, we created a simulated light curve of a WR+O binary using smooth convex-concave functions I_a and I_c. Gaussian noise with various standard deviations was then added to the exact light curve to produce several simulated light curves used as input in (4). The values of i and $r_{\rm O}$ were set to $i = 78°$, $r_{\rm O} = 0.2$.

In Fig.1 two examples of solving (4) on compact sets are shown. The approximate solutions (dashed lines) demonstrate some characteristic features: stair-like structures in the left hand plot, broken lines in the right hand plot. The reason for these structures is that in an attempt to minimize the residual (recall that input data contain noise), the algorithm always uses as much flexibility as it is allowed to. In model (i) the solution must be non-increasing. This means that it *may* be non-decreasing in some parts. Similarly, in the convex (concave) part of the model (iii) the second derivative of the solution must be non-positive (non-negative); this means that it *may* be equal to zero. It is important to note that the above structures will *always* be present in approximate solutions of this kind, as long as the data contain some noise. Clearly, such brightness and opacity distributions are not very meaningful.

In Fig.2 the regularization approach is shown. In the left hand plot, the input light curve is the same as in Fig.1. The unknown functions are assumed to be non-negative, convex-concave, smooth (the functions themselves and their first derivatives continuous). The right-hand plot shows how the approximate solution approaches the true solution if the error of the input data decreases.

4 Discussion

Provided that an input light curve has sufficiently good accuracy ($\delta = 0.001$ seems to be a reasonable expectation from modern photometry), the regularization technique allows one to obtain empirical distributions of brightness and opacity across the disk of the WR component in an eclipsing WR+O binary. I_c can be used to estimate, e.g., the brightness temperature of the WR star. More inter-

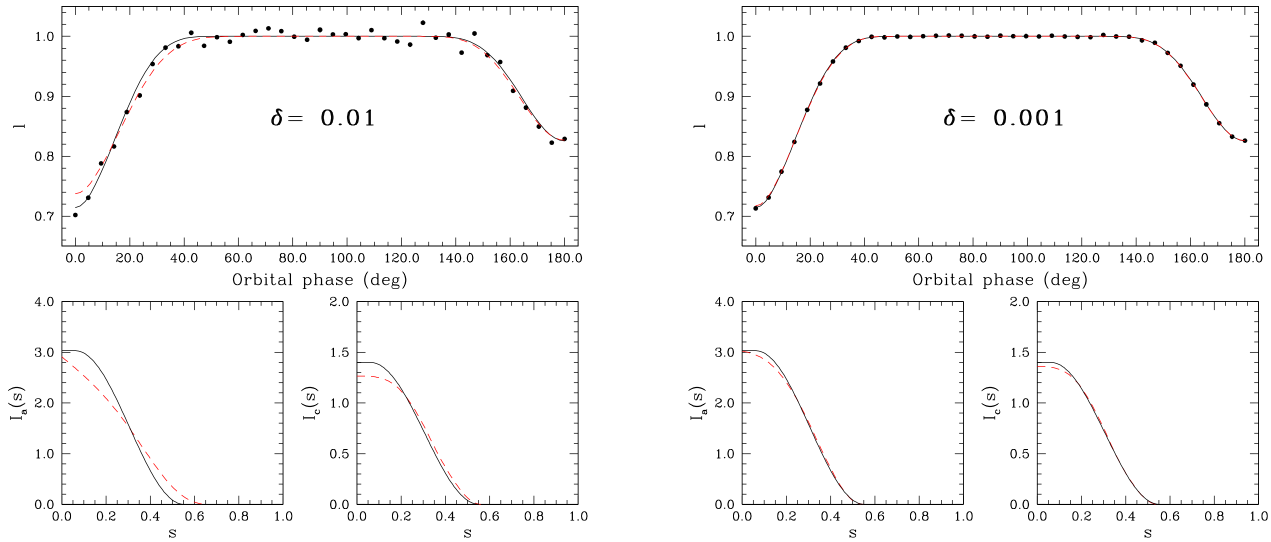

Figure 2: Regularization approach. I_a and I_c are assumed to be convex-concave.

estingly, I_a allows one to get an empirical distribution of the velocity in the WR wind. Indeed, $I_a(s) = 1 - e^{-\tau(s)}$, where τ is the optical depth of the WR wind along the line of sight at the impact distance s. In turn,

$$\tau(s) = 2 \int_s^{\infty} \frac{\epsilon(r) r dr}{\sqrt{r^2 - s^2}}, \tag{6}$$

where ϵ is the linear absorption coefficient. Recalling that the main absorption agent in the WR wind in the optical continuum is electronic scattering and using the continuity equation, we get $\epsilon(r) = \sigma_T n_e = \frac{k \sigma_T \dot{M}}{4\pi m_p v(r) r^2}$, where $k \simeq 0.5$ if helium is fully ionized, $k \simeq 0.25$ in the He II zone[1], σ_T is the scattering cross section, $\dot{M}$ is the mass loss rate, m_p is the proton mass, $v(r)$ is the velocity law. Thus, from the empirical $I_a(s)$ one can obtain $\tau(s)$, and, solving (6), called Abel's equation, obtain $\epsilon(r)$ and hence $v(r)$. Such an empirical velocity law can be used as a constraint in any self-consistent theory of WR winds. Solving Abel's equation as well as application of the technique presented in the current paper, to real objects, will be a subject of forthcoming papers.

Potential applications of the regularization technique are much wider than the particular problem discussed in the present paper. This technique can be used whenever a problem can be described by Fredholm's equation of the first kind (CoRoT observations of exoplanets, lunar occultations of stars, correction of observational data for the response function of the receiver being a few examples of its use). The computer code for solving light curves of WR+O binaries and the underlying libraries for solving Fredholm's equation using non-trivial *a priori* constraints on the solution are freely available to all interested sides on request.

Acknowledgements

This research is supported by the Russian Foundation for Basic Research through the grants No 08-02-01220 and 11-02-00940.

[1]Helium is the most abundant element in WR winds.

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Discussion

G. Rauw: You mentioned the impact of the errors on the data on the stability of your solution. One can of course reduce these errors by using e.g. CoRoT, but what about the intrinsic variability of the WR star that will also appear in the light curve?

I. Antokhin: As long as you use a regularization technique, the stability of the solution is not a problem. You need good accuracy of the data to get an accurate estimate of the "exact" brightness and opacity distributions. As for the intrinsic irregular variability, one should obtain as much data (long runs) as possible so these variations will be smoothed in the mean light curve. Filtering based on e.g. Fourier decomposition is also possible.