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Weighted composition operators between weighted Bloch type spaces

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Abstract

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\phi : \mathbb{D} \to \mathbb{D}$ as well as $\psi : \mathbb{D} \to \mathbb{C}$ be analytic maps. For a holomorphic function f on $\mathbb{D}$ the weighted composition operator $C_{\phi,\psi}$ is defined by $(C_{\phi,\psi}f)(z) = \psi(z)f(\phi(z))$ for every $z \in \mathbb{D}$. We characterize when weighted composition operators acting between weighted Bloch type spaces are bounded resp. compact. Moreover, during these studies we also obtain a characterization of boundedness and compactness of weighted composition operators from weighted Bloch type spaces to weighted Banach spaces of holomorphic functions.

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1 Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and ψ be an analytic map on $\mathbb{D}$. Such maps induce the so-called *weighted composition operator*

$$C_{\phi,\psi} : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \psi(f \circ \phi),$$

where $H(\mathbb{D})$ denotes the set of all holomorphic functions on $\mathbb{D}$. In case that $\psi(z) = 1$ for every $z \in \mathbb{D}$ we simply write C_ϕ and obtain the classical *composition operator*. Such operators have been investigated on various spaces of holomorphic functions and by several authors, see e.g. [4], [5], [6], [9], [10], [13] [16]. In this paper we are interested in operators $C_{\phi,\psi}$ acting between different weighted Bloch type spaces. For a continuous, strictly positive and bounded function (*weight*) on $\mathbb{D}$ we say that the *weighted Bloch type space* B_v is the collection of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{B_v} := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty.$$

Provided, we identify functions that differ by a constant, $\|.\|_{B_v}$ becomes a norm and B_v a Banach space.

In [16] Ohno, Stroethoff and Zhao characterized boundedness and compactness of weighted composition operators in the framework of weighted Bloch type spaces generated by the standard weights, i.e. weights of the form $v_\alpha(z) = (1 - |z|^2)^\alpha$ with $\alpha \in \mathbb{R}$ and $\alpha > 0$.

Using a completely different approach in this article we characterize these properties of $C_{\phi,\psi}$ acting between different more general Bloch type spaces.

2 Notation, auxiliary results and basic facts

First, for the notation as well as a general introduction on the concept of (weighted) composition operators, we refer the reader to the excellent monographs [17] and [7].
In this article we will identify weighted composition operators acting between weighted Bloch type spaces with a sum of operators acting in a different setting which we will explain here. For a weight v we consider

$$\begin{aligned} H_v^\infty &:= \{f \in H(\mathbb{D});\ \|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty\} \text{ and} \\ H_v^0 &:= \{f \in H_v^\infty;\ \forall K \subset\subset \mathbb{D}\ \exists\varepsilon > 0 :\ v(z)|f(z)| < \varepsilon\ \forall z \in \mathbb{D}\backslash K\}. \end{aligned}$$

Endowed with norm $\|.\|_v$ these are Banach spaces. In the sequel we will refer to spaces of type H_v^∞ as *weighted Banach spaces of holomorphic functions*. Such spaces occur naturally in several problems regarding the growth conditions of analytic functions. This means they arise in complex analysis, spectral theory, Fourier analysis, partial differential equations and convolution equations. They were intensively studied e.g. in [3] and [2].
The study of weighted spaces resp. of operators acting on weighted spaces requires the concept of the so-called *associated weight*, which was introduced by Anderson and Duncan in [1] and thoroughly investigated by Bierstedt, Bonet, and Taskinen in [3].
Given a weight v, its associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) := \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \le 1\}} \text{ for every } z \in \mathbb{D}.$$

Bierstedt, Bonet, and Taskinen showed the following very useful facts concerning associated weights:

(a) $0 < v \le \tilde{v}$,

(b) $\tilde{v}$ is continuous,

(c) for every $z \in \mathbb{D}$ there exists a function $f_z \in H_v^\infty$ with $\|f_z\|_v \le 1$ such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.

(d) H_v^∞ and $H_{\tilde{v}}^\infty$ are isometrically isomorphic.

In the following we are mainly interested in *radial* weights, i.e. weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. If such weights satisfy additionally $\lim_{|z|\to 1} v(z) = 0$ they are called *typical*. Since, in general it is quite difficult to compute the associated weight, we need some conditions, when v and $\tilde{v}$ are equivalent, i.e. when there exists a constant $k > 0$ such that

$$v(z) \le \tilde{v}(z) \le kv(z) \text{ for every } z \in \mathbb{D}.$$

We say that a weight v is *essential*, if v and $\tilde{v}$ are equivalent. Bonet, Domański, and Lindström proved that a radial weight which satisfies condition

$$(L1)\ \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0$$

(which is due to Lusky, see [11]) is an essential weight. Examples of such weights include among others the standard weights $v_\alpha(z) = (1-|z|^2)^\alpha$, $\alpha > 0$, as well as the logarithmic weights $w_q(z) = (1 - \log(1 - |z|^2))^q$, $q < 0$. Recall that by [6] weighted composition operators $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ are bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} < \infty.$$

If we assume additionally that v and w are typical weights, then $C_{\phi,\psi} : H_v^\infty \to H_w^\infty$ is compact if and only if $\limsup_{|\phi(z)| \to 1} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = 0$.

Our approach in the study of weighted composition operators involves integration. Thus, let us consider the integration operator

$$I : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto \int_0^z f(u)\, du.$$

We again identify functions that differ by a constant. In [8] Harutyunyan and Lusky studied the boundedness of such operators acting between weighted Banach spaces of holomorphic functions and obtained the following result:

Proposition 1 (Harutyunyan-Lusky [8] Proposition 2.2) *Let v and w be typical weights.*

(a) If H_w^∞ is isomorphic to ℓ^∞ and $\limsup_{r \to 1} \left(-\frac{w^2(r)}{w'(r)v(r)}\right) < \infty$, then the integral operator $I : H_v^\infty \to H_w^\infty$ is bounded.

(b) If $I : H_v^\infty \to H_w^\infty$ is bounded, then $\limsup_{r \to 1} \left(-\frac{w(r)}{v'(r)}\right) < \infty$.

The following characterization is rather technical and not needed in the sequel but for the sake of completeness we will formulate it here. In [12] Lusky showed that H_w^∞ is isomorphic to ℓ^∞ if and only if

$$\forall b_1 > 1\ \exists b_2 > 1\ \exists c > 0\ \forall m, n \geq c :$$
$$|m - n| \geq c \text{ and } \left(\frac{r_m}{r_n}\right)^m \frac{w(r_m)}{w(r_n)} \leq b_1 \Longrightarrow \left(\frac{r_n}{r_m}\right)^n \frac{w(r_n)}{w(r_m)} \leq b_2.$$

Examples of weights with this property include among others the standard weights and the exponential weights $u_\alpha(z) = e^{-\frac{1}{(1-|z|^2)^\alpha}}$, $\alpha > 0$.

The investigation of the compactness of the operator $C_{\phi,\psi} : B_v \to B_w$ requires the following result which can be easily derived from a result of Cowen and MacCluer, see [7] Proposition 3.11.

Proposition 2 (Cowen-MacCluer, [7] Proposition 3.11)) *$C_{\phi,\psi} : B_v \to B_w$ resp. $C_\phi : B_v \to B_w$ is compact if and only if whenever $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in B_v such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $C_{\phi,\psi} f_n \to 0$ resp. $C_\phi f_n \to 0$ in B_w.*

We need the following characterization of the boundedness resp. compactness of operators C_ϕ acting between weighted Bloch type spaces, which in case of standard weights can be easily derived from the boundedness resp. compactness conditions of weighted composition operators acting between weighted Banach spaces of holomorphic functions. We will analyze a more general framework here.

Proposition 3 *Let v and w be weights. Then the composition operator $C_\phi : B_v \to B_w$ is bounded if and only if $\sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))} < \infty$.*

Proof. First, we suppose that $C_\phi : B_v \to B_w$ is bounded. We assume to the contrary that we can find a sequence $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that $\frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))} \geq n$ for every $n \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}$ we can choose $f_n \in B_v$ with $\|f_n\|_{B_v} \leq 1$ and $|f_n'(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$. Then, by the boundedness of the operator $C_\phi : B_v \to B_w$ we can find a constant $c > 0$ such that

$$c \geq w(z_n)|f_n'(\phi(z_n))||\phi'(z_n)| = \frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))} \geq n \text{ for every } n \in \mathbb{N},$$

which is a contradiction.
Conversely, we get that

$$\begin{aligned}\|C_\phi f\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\phi'(z)||f'(\phi(z))| &\leq \sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))}\tilde{v}(\phi(z))|f'(\phi(z))| \\ &\leq \sup_{z\in\mathbb{D}} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))}\|f\|_{B_{\tilde{v}}}\end{aligned}$$

and the claim follows. □

Proposition 4 *Let v and w be typical weights. If $\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}|\phi'(z)| = 0$, then the operator $C_\phi : B_v \to B_w$ is compact. If we assume additionally that $H^\infty_{\frac{v}{v_1}}$ is isomorphic to ℓ^∞ and $\limsup_{r\to 1}\left(-\frac{v(r)}{v'(r)v_1(r)-v_1'(r)v(r)}\right) < \infty$, where $v_1(z) = 1-|z|^2$ for every $z \in \mathbb{D}$, then the converse is also true.*

Proof. Let us first assume that $\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}|\phi'(z)| = 0$. Then, by Proposition 3 the operator C_ϕ must be bounded. The idea now is to use Proposition 2. To do this we take a bounded sequence $(f_n)_n \subset B_v$ with $\|f_n\|_{B_v} \leq 1$ for every $n \in \mathbb{N}$ and $f_n \to 0$ uniformly on compact sets. Given $\varepsilon > 0$ we can find $0 < r_0 < 1$ such that

$$|\phi'(z)|w(z) < \varepsilon\frac{\tilde{v}(\phi(z))}{2} \text{ for every } z \in \mathbb{D} \text{ with } |\phi(z)| > r_0.$$

For n big enough, we have that

$$\sup_{|\phi(z)|\leq r_0} |f_n'(\phi(z))|w(z)|\phi'(z)| < \frac{\varepsilon}{2}$$

and thus

$$\begin{aligned}\|C_\phi(f_n)\|_{B_w} \leq &\sup_{|\phi(z)|\leq r_0} w(z)|\phi'(z)||f_n'(\phi(z))| \\ + &\sup_{|\phi(z)|> r_0} w(z)|\phi'(z)||f_n'(\phi(z))| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\|f_n\|_{B_{\tilde{v}}}.\end{aligned}$$

Hence, by Proposition 2 the operator C_ϕ is compact.

Conversely, we assume that there is a sequence $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ and such that

$$\lim_{n\to\infty} \frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))} = \limsup_{\phi(z)|\to 1} \frac{w(z)|\phi'(z)|}{\tilde{v}(\phi(z))}.$$

By passing to a subsequence we may assume that there is $n_0 \in \mathbb{N}$ such that $|\phi(z_n)|^n \geq \frac{1}{2}$ for every $n \geq n_0$. Moreover, for every $n \in \mathbb{N}$ we can find a function $f_n \in B_v$ with $\|f_n\|_{B_v} \leq 1$ and $|f_n'(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$. Now we choose

$$g_n(z) := z^n \varphi_{\phi(z_n)}(z) f_n(z) \text{ for every } n \in \mathbb{N} \text{ and every } z \in \mathbb{D}.$$

Then

$$g_n'(z) = nz^{n-1} f_n(z)\varphi_{\phi(z_n)}(z) + \varphi'_{\phi(z_n)}(z) z^n f_n(z) + z^n \varphi_{\phi(z_n)}(z) f_n'(z)$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}$. By Proposition 1 the operator $I : H_v^\infty \to H_{\frac{v}{v_1}}^\infty$ is bounded since by assumption we have that

$$\limsup_{r\to 1} \left(-\frac{\left(\frac{v}{v_1}\right)^2 (r)}{\left(\frac{v}{v_1}\right)' (r) v(r)} \right) = \limsup_{r\to 1} \left(-\frac{v(r)}{v'(r)v_1(r) - v_1'(r)v(r)} \right) < \infty.$$

This yields

$$\begin{aligned}
\|g_n\|_{B_v} &\leq \sup_{z\in\mathbb{D}} |\varphi'_{\phi(z_n)}(z)|(1-|z|^2)\frac{v(z)}{(1-|z|^2)}|f_n(z)| + \sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2)\frac{v(z)}{(1-|z|^2)}|f_n(z)| \\
&+ \sup_{z\in\mathbb{D}} v(z)|f_n'(z)| \\
&\leq \sup_{z\in\mathbb{D}} (1-|\varphi_{\phi(z_n)}(z)|^2)\frac{v(z)}{1-|z|^2}|If_n'(z)| + \sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2)\frac{v(z)}{1-|z|^2}|If_n'(z)| + 1 \\
&\leq 1 + \|I\|\|f_n\|_{B_v} + \sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2)\|I\|\|f_n\|_{B_v} \\
&\leq 1 + \|I\| + \sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2)\|I\|.
\end{aligned}$$

An elementary calculation (cf. [14] p. 343) yields

$$\sup_{z\in\mathbb{D}} n|z|^{n-1}(1-|z|^2) = \frac{2n}{n+1}\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}}.$$

Observe that for each $n \geq 2$ the above supremum is a maximum and it is attained at any point on the circle centered at the origin and of radius $r_n = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}$. These maxima form a decreasing sequence which tends to $\frac{2}{e}$. Thus, we consider the sequence $(h_n)_n$ with $h_n(z) = \frac{z^n \varphi_{\phi(z_n)}(z) f_n(z)}{\|z^n\|_{B_v}}$ for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$. This sequence is bounded and converges uniformly to zero on the compact subsets of $\mathbb{D}$. By Proposition 2 we have that

$$\|C_\phi h_n\|_{B_v} \to 0 \text{ if } n \to \infty.$$

Moreover,

$$\begin{aligned}
\|C_\phi h_n\|_{B_v} &\geq w(z_n)|\phi'(z_n)||h_n'(\phi(z_n))| \\
&\geq \frac{1}{2}\frac{w(z_n)|\phi'(z_n)|}{\tilde{v}(\phi(z_n))(1-|\phi(z_n)|^2)}.
\end{aligned}$$

for every $n \in \mathbb{N}$. Thus, the claim follows. □

3 Boundedness

In this section we will investigate when the operator $C_{\phi,\psi} : B_v \to B_w$ is bounded.

Theorem 5 *Let v and w be weights. Then the weighted composition operator $C_{\phi,\psi} : B_v \to B_w$ is bounded if and only if:*

(a) $C_{\phi,\psi'} : B_v \to H_w^\infty$ is bounded, and,

(b) $C_\phi : B_v \to B_{|\psi|w}$ is bounded.

Proof. Obviously, the weighted composition operator $C_{\phi,\psi} : B_v \to B_w$ can be considered as the sum of the operators $C_{\phi,\psi'} : B_v \to H_w^\infty$ and $C_\phi : B_v \to B_{|\psi|w}$ since for every $f \in B_v$ we have that

$$\|C_{\phi,\psi}f\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\psi'(z)f(\phi(z)) + \psi(z)\phi'(z)f'(\phi(z))| \\ \leq \|C_{\phi,\psi'}f\|_w + \|C_\phi f\|_{B_{|\psi|w}}.$$

Hence, if (a) and (b) are fulfilled, the operator $C_{\phi,\psi} : B_v \to B_w$ is bounded.

Conversely, we first prove (b). By Proposition 3 we have to show that $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)||\phi'(z)|}{\tilde{v}(\phi(z))} < \infty$. We fix $a \in \mathbb{D}$ and select $f_a \in B_v$ such that $\|f_a\|_{B_v} \leq 1$ and $|f_a'(\phi(a))| = \frac{1}{\tilde{v}(\phi(a))}$. Next, we put

$$g_a(z) := f_a(z) - f_a(\phi(a)) \text{ for every } z \in \mathbb{D}.$$

Obviously, $g_a \in B_v$ with $\|g_a\|_{B_v} \leq 1$ and $g_a(\phi(a)) = 0$ as well as $|g_a'(\phi(a))| = \frac{1}{\tilde{v}(\phi(a))}$. Since $C_{\phi,\psi} : B_v \to B_w$ is bounded and $a \in \mathbb{D}$ arbitrary, we obtain that $\sup_{a\in\mathbb{D}} \frac{w(a)|\psi(a)||\phi'(a)|}{\tilde{v}(\phi(a))} < \infty$. Hence (b) holds. Since $C_{\phi,\psi} : B_v \to B_w$ is bounded, (a) follows immediately. □

Thus, it remains to analyze when the operator $C_{\phi,\psi'} : B_v \to H_w^\infty$ is bounded. First, observe that this operator can be identified with the operator $C_{\phi,\psi'}I : H_v^\infty \to H_w^\infty$.

Proposition 6 *Let v and w be weights on $\mathbb{D}$. If we can find a weight u such that such that $I : H_v^\infty \to H_u^\infty$ is bounded and such that $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{u(\phi(z))} < \infty$, then the operator $C_{\phi,\psi'} : H_v^\infty \to H_w^\infty$ is bounded. If we assume additionally that v and w are typical and that one of the following conditions holds:*

(a) $\psi' \in H_w^\infty \backslash H_w^0$,

(b) $\psi' \in H_w^0$ with $\liminf_{|z|\to 1} |\psi'(z)| = \alpha > 0$, and $C_\phi : H_w^\infty \to H_w^\infty$,

then the converse is also true.

Proof. Let $f \in H_v^\infty$. Then with u selected as in (b) we get that

$$\|(C_{\phi,\psi'}I)(f)\|_w = \sup_{z\in\mathbb{D}} w(z)|\psi'(z)||(If)(\phi(z))| \\ \leq \sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{u(\phi(z))} u(\phi(z))|(If)(\phi(z))| \leq \sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{u(\phi(z))} \|If\|_u \\ \leq \sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{u(\phi(z))} \|I\|\|f\|_v$$

and the claim follows.
Conversely, since $C_{\phi,\psi'}I : H_v^\infty \to H_w^\infty$ is bounded, we can find a constant $M > 0$ such that

$$\|(C_{\phi,\psi'}I)(f)\|_{B_w} = \sup_{z\in\mathbb{D}} w(z)|\psi'(z)||(If)(\phi(z))| \le M\|f\|_v. \tag{3.1}$$

Now, the aim is to find or construct a weight u such that $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi'(z)|}{u(\phi(z))} < \infty$ and $\lim_{r\to1}\sup_{|\phi(z)|>r} \frac{|\psi'(z)|w(z)}{u(\phi(z))} > 0$. In other words, we have to find a weight u such that the weighted composition operator $C_{\phi,\psi'} : H_u^\infty \to H_w^\infty$ is bounded, but not compact. In case $\psi' \in H_w^\infty \backslash H_w^0$ obviously we can take $u(z) = 1$ for every $z \in \mathbb{D}$. In the second case we may choose $u(z) = w(z)$ for every $z \in \mathbb{D}$. □

Let us give some illustrating examples.

Example 7 (a) Consider the weights $v(z) = (1-|z|^2)^2 = \tilde{v}(z)$ and $w(z) = e^{-\frac{1}{1-|z|^2}}$ as well as $\phi(z) = (z-\frac{1}{2})/(1-\frac{z}{2})$ and $\psi(z) = 1-z$ for every $z \in \mathbb{D}$. Then ϕ is the automorphism of $\mathbb{D}$ which interchanges 0 and $\frac{1}{2}$ and we have that $\phi'(z) = \frac{3}{4}(1-\frac{z}{2})^{-2}$ for every $z \in \mathbb{D}$. Next, a simple calculation shows

$$\begin{aligned}\limsup_{|\phi(z)|\to1} \frac{w(z)|\phi'(z)||\psi(z)|}{\tilde{v}(\phi(z))} &= \limsup_{|\phi(z)|\to1} \frac{3}{4}\left[e^{-\frac{1}{1-|z|^2}}|1-z|\Big/\left(\left(1-\left|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}\right|^2\right)^2\left|1-\frac{z}{2}\right|^2\right)\right] \\ &= 0.\end{aligned}$$

Since $\lim_{r\to1}\left(-\frac{v(r)}{v_1(r)v'(r)-v_1'(r)v(r)}\right) = \lim_{r\to1}\frac{1}{2r} = \frac{1}{2}$, this means that the operator $C_\phi : B_v \to B_{|\psi|w}$ is not only bounded, but even compact. Moreover,

$$\limsup_{r\to1}\left(-\frac{w(r)^2}{w'(r)v(r)}\right) = \limsup_{r\to1}\frac{e^{-\frac{1}{1-r^2}}(1-r^2)}{2r} = 0 < \infty.$$

Thus, the operator $I : H_v^\infty \to H_w^\infty$ is bounded. Furthermore,

$$\limsup_{|\phi(z)|\to1}\frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = \limsup_{|\phi(z)|\to1} e^{-\frac{1}{1-|z|^2}}|1-z|\Big/\left(1-\left|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}\right|^2\right) = 0.$$

Finally, we can conclude that $C_{\phi,\psi} : B_v \to B_w$ is bounded.

(b) Select now $w(z) = 1-|z|$ and $v(z) = e^{-\frac{1}{1-|z|}}$ as well as $\phi(z) = \frac{z-1}{2}$ and $\psi(z) = z$ for every $z \in \mathbb{D}$. Then

$$\sup_{z\in\mathbb{D}}\frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = \sup_{z\in\mathbb{D}}\frac{(1-|z|)|1-z|}{e^{-\frac{1}{1-|\frac{z-1}{2}|}}} = \infty.$$

Hence, $C_\phi : B_v \to B_{|\psi|w}$ is not bounded and thus the same is true for the operator $C_{\phi,\psi} : B_v \to B_w$.

4 Compactness

In this section we will characterize when weighted composition operators $C_{\phi,\psi}$ acting between weighted Bloch type spaces are compact.

Theorem 8 *Let v and w be typical weights such that $H^\infty_{\frac{v}{v_1}}$ is isomorphic to ℓ^∞ and $\limsup_{r\to 1}\left(-\frac{v(r)}{v'(r)v_1(r)-v_1'(r)v(r)}\right) < \infty$. Then $C_{\phi,\psi} : B_v \to B_w$ is compact if and only if*

(a) $C_{\phi,\psi'} : B_v \to H^\infty_w$ is compact, and,

(b) $C_\phi : B_v \to B_{|\psi|w}$ is compact.

Proof. The compactness of $C_{\phi,\psi} : B_v \to B_w$ follows immediately from (a) and (b), if we recall that it can be identified with the sum of $C_{\phi,\psi'} : B_v \to H^\infty_w$ and $C_\phi : B_v \to B_{|\psi|w}$. Conversely, we start with proving (b). As in the proof of Proposition 4 we may assume that there is a sequence $(z_n)_n \subset \mathbb{D}$ with $|(\phi(z_n)| \to 1$ such that

$$\limsup_{|\phi(z)|\to 1} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = \lim_{n\to\infty} \frac{w(z_n)|\psi(z_n)|}{\tilde{v}(\phi(z_n))}.$$

and we consider the same sequence

$$h_n(z) := \frac{z^n f_n(z)\varphi_{\phi(z_n)}(z)}{\|z^n\|_{B_v}} \text{ for every } z \in \mathbb{D},\ n \in \mathbb{N},$$

which is bounded in B_v and tends to zero uniformly on the compact subsets of $\mathbb{D}$. Therefore, by the compactness of the operator $C_{\phi,\psi} : B_v \to B_w$ we have that

$$\|C_{\phi,\psi} h_n\|_{B_w} \to 0.$$

Moreover,

$$\begin{aligned}\|C_{\phi,\psi} h_n\|_{B_w} &\geq w(z_n)|\psi'(z_n)h_n(\phi(z_n)) + \psi(z_n)\phi'(z_n)h_n'(\phi(z_n))| \\ &\geq w(z_n)\frac{|\psi(z_n)\phi'(z_n)|}{\tilde{v}(\phi(z_n))(1-|\phi(z_n)|^2}|\phi(z_n)|^n \\ &\geq \frac{1}{2}\frac{w(z_n)|\psi(z_n)\phi'(z_n)|}{\tilde{v}(\phi(z_n))|(1-|\phi(z_n)|^2)}.\end{aligned}$$

Hence $\limsup_{|\phi(z)|\to 1} \frac{w(z)}{|\psi(z)|}\tilde{v}(\phi(z)) = 0$ and thus the operator $C_\phi : B_v \to B_{|\psi|w}$ is compact. Since, by hypothesis, the operator $C_{\phi,\psi} : B_v \to B_w$ is compact, the compactness of $C_{\phi,\psi'} : B_v \to H^\infty_w$ follows immediately. □

It remains to analyze when $C_{\phi,\psi'} : B_v \to H^\infty_w$ is compact. Again we can (and will) identify the operator $C_{\phi,\psi'} : B_v \to H^\infty_w$ with $C_{\phi,\psi'} I : H^\infty_v \to H^\infty_w$.

Proposition 9 *Let v and w be weights. If one of the following conditions holds*

(a) There is a weight u such that $I : H^\infty_v \to H^\infty_u$ is bounded and $\limsup_{|\phi(z)|\to 1} \frac{w(z)|\psi'(z)|}{u(\phi(z))} = 0$ (i.e. $C_{\phi,\psi'} : H^\infty_u \to H^\infty_w$ is compact),

(b) *There is a weight u such that $I : H_v^\infty \to H_u^\infty$ is compact and* $\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{u(\phi(z))} < \infty$ *(i.e.* $C_{\phi,\psi'} : H_u^\infty \to H_w^\infty$ *is bounded),*

then the operator $C_{\phi,\psi'}I : H_v^\infty \to H_w^\infty$ is compact. If we assume additionally that v and w are typical and one of the following conditions holds,

(c) $\psi' \in H_w^\infty \backslash H_w^0$,

(d) $\psi' \in H_w^0$ *with* $\liminf_{|z|\to 1} |\psi'(z)| = \alpha > 0$, *and* $C_\phi : H_w^\infty \to H_w^\infty$,

then the converse is also true.

Proof. First, we assume that one of (a) and (b) holds. The arguments are standard but for a better understanding we give the proof here. Let $(f_n)_n \subset H_v^\infty$ be a bounded sequence. We first suppose that (a) is valid. Since I, by hypothesis, is bounded, the sequence $(If_n)_n \subset H_u^\infty$ must also be bounded. Hence, the compactness of $C_{\phi,\psi'} : H_u^\infty \to H_w^\infty$ yields, that the sequence $(C_{\phi,\psi'}If_n)_n \subset H_w^\infty$ must have a convergent subsequence, hence the claim follows.
In case that (b) holds, $(If_n)_n \subset H_u^\infty$ must have a convergent subsequence, since $I : H_v^\infty \to H_u^\infty$ is compact. By assumption, $C_{\phi,\psi'} : H_u^\infty \to H_w^\infty$ is continuous, hence the sequence $(C_{\phi,\psi'}If_n)_n \subset H_w^\infty$ must also be convergent. Finally, the operator $C_{\phi,\psi'}I : H_v^\infty \to H_w^\infty$ is compact.

Conversely, we proceed exactly as in the proof of boundedness and choose in case that (c) is valid the weight u to be $u(z) = 1$ for every $z \in \mathbb{D}$ and in case (d) the weight u to be $u(z) = w(z)$ for every $z \in \mathbb{D}$. Thus, in both cases we get an operator $C_{\phi,\psi'} : H_u^\infty \to H_w^\infty$ which is bounded, but not compact. □

Example 10 (a) The operator given in Example 7 (a) is obviously compact.

(b) We choose $v(z) = w(z) = 1 - |z|$ and $\phi(z) = \frac{z-\frac{1}{2}}{1-\frac{z}{2}}$ as well as $\psi(z) = 1 - z$ for every $z \in \mathbb{D}$. Then

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)||\phi'(z)|}{\tilde{v}(\phi(z))} = \sup_{z\in\mathbb{D}} \frac{3}{4} \frac{(1-|z|)|1-z|}{(1-|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}|)|1-\frac{z}{2}|^2} < \infty,$$

but

$$\limsup_{|\phi(z)|\to 1} \frac{3}{4} \frac{(1-|z|)|1-z|}{(1-|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}|)|1-\frac{z}{2}|^2} \neq 0.$$

This means that $C_\phi : B_v \to B_{|\psi|w}$ is bounded, but not compact. Moreover,

$$\limsup_{r\to 1} \left(-\frac{w(r)^2}{w'(r)v(r)}\right) = \limsup_{r\to 1}(1-r) = 0 < \infty.$$

Hence $I : H_v^\infty \to H_w^\infty$ is bounded and

$$\sup_{z\in\mathbb{D}} \frac{w(z)|\psi(z)|}{\tilde{v}(\phi(z))} = \sup_{z\in\mathbb{D}} \frac{(1-|z|)|1-z|}{1-|\frac{z-\frac{1}{2}}{1-\frac{z}{2}}|} < \infty.$$

Thus, $C_{\phi,\psi} : B_v \to B_w$ is bounded, but not compact.

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