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Orthosymplectic supersymmetries of modules of differential operators

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Abstract Let $\mathfrak{F}_\lambda$ be the space of tensor densities of degree λ on the supercircle $S^{1|1}$. We consider the space $\mathfrak{D}^k_{\lambda,\mu}$ of k-th order linear differential operators from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ as $\mathfrak{osp}(1|2)$-module and we determine the superalgebra $\mathfrak{I}^k_{\lambda,\mu}$ of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the $\mathfrak{osp}(1|2)$-action.

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1 Introduction

In the classical literature, thanks to the Gelfand-Fuchs cohomology (see [5]), the differential operators invariant with respect of the diffeomorphism groups can be interpreted in terms of the Lie algebras of vector fields. Usually, one considers differential or bidifferential operators acting on various spaces of tensor fields on smooth manifolds. Many classification results of such operators are available now. In [6], the authors considered the $\mathrm{Diff}(S^1)$-module $D_{\lambda,\mu}$ of linear differential operators $A : \mathcal{F}_\lambda \to \mathcal{F}_\mu$, where $\mathcal{F}_\lambda$ and $\mathcal{F}_\mu$ are the spaces of tensor densities on S^1 of degree λ and μ respectively. The module $D_{\lambda,\mu}$ has a natural filtration:

$$\mathrm{D}^0_{\lambda,\mu} \subset \mathrm{D}^1_{\lambda,\mu} \subset \cdots \mathrm{D}^k_{\lambda,\mu} \subset \cdots .$$

In [6] the algebra of symmetries $\mathcal{I}^k_{\lambda,\mu}$ of linear maps $T : D^k_{\lambda,\mu} \to D^k_{\lambda,\mu}$ commuting with the $\mathrm{Diff}(S^1)$-action was computed.

In this paper we are interested in the study of the analogue super structures. Namely we consider the superspace $\mathfrak{D}_{\lambda,\mu}$ of differential linear operators $A : \mathfrak{F}_\lambda \to \mathfrak{F}_\mu$, where $\mathfrak{F}_\lambda$ and $\mathfrak{F}_\mu$ are the spaces of tensor densities on the supercircle $S^{1|1}$ of degree λ and μ respectively. Naturally the Lie superalgebra $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$ acts on $\mathfrak{D}_{\lambda,\mu}$, but, we restrict ourselves to the superalgebra $\mathfrak{osp}(1|2)$ which can be realized as a subalgebra of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$. The $\mathfrak{osp}(1|2)$-module $\mathfrak{D}_{\lambda,\mu}$ is also filtered [7]:

$$\mathfrak{D}^0_{\lambda,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^1_{\lambda,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu} \subset \cdots \subset \mathfrak{D}^{k-\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^k_{\lambda,\mu} \subset \cdots .$$

In this context, we compute the space $\mathfrak{I}^k_{\lambda,\mu}$ of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the $\mathfrak{osp}(1|2)$-action which we call the superalgebra of the orthosymplectic supersymmetries of the modules $\mathfrak{D}^k_{\lambda,\mu}$. We give explicitly the dimension of this space. Some examples of elements in $\mathfrak{I}^k_{\lambda,\mu}$ for some particular values of k, λ, μ are investigated. We hope to be able, in the immediate future, to determine the algebra of linear maps on $\mathfrak{D}^k_{\lambda,\mu}$ commuting with the action of the superalgebra $\mathcal{K}(1)$ of contact vector fields. A work in this framework is currently being.

2 Preliminaries

The supercircle $S^{1|1}$ is the simplest supermanifold of dimension 1|1 generalizing S^1. It can be defined in terms of its superalgebra of functions, denoted by $C^\infty_{\mathbb{C}}(S^{1|1})$ and consisting of elements of the form:

$$F: \ (x, \theta) \ \longmapsto \ f_0(x) + f_1(x)\theta,$$

where x is an arbitrary parameter on S^1 (the even variable), θ is the odd variable ($\theta^2 = 0$) and f_0, f_1 are C^∞ complex valued functions. We denote by F' the derivative of F with respect to x, i.e, $F' : (x, \theta) \mapsto f_0'(x) + f_1'(x)\theta$. Let $\mathrm{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\mathrm{Vect}_{\mathbb{C}}(S^{1|1}) = \left\{ F_0\partial_x + F_1\partial_\theta \mid \ F_i \in C^\infty_{\mathbb{C}}(S^{1|1}) \right\},$$

where ∂_θ stands for $\frac{\partial}{\partial\theta}$ and ∂_x stands for $\frac{\partial}{\partial x}$. The space $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$ is a Lie superalgebra. Consider the vector fields D and $\overline{D}$ defined by (see [8] for the interpretation of these fields):

$$D = \partial_\theta + \theta\partial_x \quad \text{and} \quad \overline{D} = \partial_\theta - \theta\partial_x,$$

one can easily checks that

$$D^{2j} = -\overline{D}^{2j} = \partial_x^j, \forall j \in \mathbb{N}.$$

The distribution generated by $\overline{D}$ defines a codimension 1 non-integrable distribution on $S^{1|1}$ called the standard contact structure on $S^{1|1}$ which is equivalently the kernel of differential 1-form

$$\alpha = dx + \theta d\theta.$$

A vector field X is said to be contact if it preserves the contact distribution, i.e.,

$$[X, \overline{D}] = F_X\overline{D},$$

where $F_X \in C^\infty_{\mathbb{C}}(S^{1|1})$ is a function depending on X. We denote by $\mathcal{K}(1)$ the Lie superalgebra of contact vector fields on $S^{1|1}$. An element in $\mathcal{K}(1)$ can be expressed for any $f \in C^\infty_{\mathbb{C}}(S^{1|1})$ as (see [7]):

$$X_f = -f\overline{D}^2 + \frac{1}{2}D(f)\overline{D}. \tag{2.1}$$

The contact bracket is defined by

$$[X_f, X_g] = X_{\{f,g\}}$$

and the space $C^\infty_{\mathbb{C}}(S^{1|1})$ is thus equipped with a Lie superalgebra structure (isomorphic to $\mathcal{K}(1)$) thanks to the bracket:

$$\{f, g\} = fg' - f'g + \frac{1}{2}(-1)^{|f|(|g|+1)}D(f)D(g),$$

where $|\,|$ stands for the parity function. The action of $\mathcal{K}(1)$ on $C^\infty_{\mathbb{C}}(S^{1|1})$ is defined by:

$$\mathfrak{L}_{X_f}(g) = fg' + \frac{1}{2}(-1)^{|f|+1}\overline{D}(f)\cdot\overline{D}(g).$$

We consider the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$, which is the smallest simple Lie superalgebra. It can be defined as the real algebra with basis (H, X, Y, A, B), the elements H, X and Y are even (with parity 0) and the elements A, B are odd (with parity 1), the bracket is graded skewsymmetric, it satisfies the graded Jacobi identity

$$(-1)^{|U||W|}[[U,V],W] + (-1)^{|V||U|}[[V,W],U] + (-1)^{|W||V|}[[W,U],V] = 0.$$

The commutation relations are:

$$\begin{array}{lll} [H,X]=X, & [H,Y]=-Y, & [X,Y]=2H, \\ [H,A]=\frac{1}{2}A, & [X,A]=0, & [Y,A]=-B, \\ [H,B]=-\frac{1}{2}B, & [X,B]=-A, & [Y,B]=0, \\ [A,A]=2X, & [A,B]=2H, & [B,B]=-2Y. \end{array}$$

The even subalgebra $(\mathfrak{osp}(1|2))_0$ of $\mathfrak{osp}(1|2)$ is of course the simple Lie algebra $\mathfrak{sl}(2)$, with basis $\{X,\ Y,\ H\}$. From the relations, it is clear that, as a Lie superalgebra, $\mathfrak{osp}(1|2)$ is generated by its odd part $(\mathfrak{osp}(1|2))_1 = \mathrm{Span}(A, B)$.
We can realize the superalgebra $\mathfrak{osp}(1|2)$ as a subalgebra of $\mathcal{K}(1)$ (and evidently of $\mathrm{Vect}_{\mathbb{C}}(S^{1|1})$) by setting using (2.1)

$$(-X_x, X_1, -X_{x^2}, 2X_\theta, X_{x\theta}) = (H, X, Y, A, B).$$

It is well known that if we identify S^1 with $\mathbb{RP}^1$ with homogeneous coordinates $(x_1 : x_2)$ and choose the affine coordinate $x = x_1/x_2$, the vector fields $\frac{d}{dx}$, $x\frac{d}{dx}$, $x^2\frac{d}{dx}$ are globally defined and correspond to the standard projective structure on $\mathbb{RP}^1$. In this adapted coordinate the action of the algebra $\mathfrak{sl}(2)$ viewed as the subalgebra $\mathrm{Span}\left(\frac{d}{dx}, x\frac{d}{dx}, x^2\frac{d}{dx}\right)$ of the Lie algebra $\mathrm{Vect}(S^1)$ is well defined.
As in the S^1-case, there exist adapted coordinates (x, θ) for which the $\mathfrak{osp}(1|2)$-action is well defined (see [7] for more details).
We denote by $\mathfrak{F}_\lambda$ the space of all weighted densities on $S^{1|1}$ of weight λ:

$$\mathfrak{F}_\lambda = \left\{F\alpha^\lambda \;\mid\; F \in C^\infty_{\mathbb{C}}(S^{1|1})\right\} \qquad (\alpha = dx + \theta d\theta).$$

The action of $\mathfrak{osp}(1|2)$ on $\mathfrak{F}_\lambda$ is given by

$$\mathfrak{L}^\lambda_{X_G}(F\alpha^\lambda) = ((G\partial_x + \frac{1}{2}D(G)\overline{D})(F) + \lambda G'F)\alpha^\lambda.$$

Any differential operator A on $S^{1|1}$ defines a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any λ and any μ by: $A : F\alpha^\lambda \mapsto A(F)\alpha^\mu$, $\mu \in \mathbb{R}$, thus, the space of differential operators form a family of $\mathfrak{osp}(1|2)$-modules denoted $\mathfrak{D}_{\lambda,\mu}$, for the natural action:

$$\mathfrak{L}^{\lambda,\mu}_{X_G}(A) = \mathfrak{L}^\mu_{X_G} \circ A - (-1)^{|A||G|}A \circ \mathfrak{L}^\lambda_{X_G}.$$

For more details see, for instance [1, 2, 7].

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3 The algebra $\mathfrak{I}^k_{\lambda,\mu}$ of orthosymplectic supersymmetries.

Lemma 3.1. *[7] Every differential operator $A \in \mathfrak{D}_{\lambda,\mu}$ can be expressed in the form*

$$A(f\alpha^{\lambda}) = \sum_{i=0}^{\ell} a_i(x,\theta)\overline{D}^i(f)\alpha^{\mu}, \;\; f \in C^{\infty}_{\mathbb{C}}(S^{1|1}),$$

where the coefficients $a_i(x,\theta)$ are arbitrary functions and $\ell \in \mathbb{N}$. Moreover, the differential operators A is of contact order $\frac{\ell}{2}$, we denote by $\mathfrak{D}^k_{\lambda,\mu}$ the subspace of differential operators of contact order $\leq k$. For short, we will write the operator A as:

$$A = \sum_{i=0}^{2k} a_i\overline{D}^i. \tag{3.1}$$

From Lemma 3.1, we win the fine filtration (stable under the $\mathfrak{osp}(1|2)$-action) on the space of differential operators $\mathfrak{D}_{\lambda,\mu}$:

$$\mathfrak{D}^0_{\lambda,\mu} \subset \mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^1_{\lambda,\mu} \subset \mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu} \subset \cdots \subset \mathfrak{D}^{k-\frac{1}{2}}_{\lambda,\mu} \subset \mathfrak{D}^k_{\lambda,\mu} \subset \cdots .$$

Remark 3.2. *Since*

$$\overline{D} = D - 2\theta D^2, \tag{3.2}$$

every operator A in $\mathfrak{D}_{\lambda,\mu}$ can be expressed using D instead of $\overline{D}$ (see [7]).

Let us now define the main object of our study. Recall that a linear operator $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ is said to be local if it preserves the supports of its arguments: $\mathrm{Supp}T(A) \subset \mathrm{Supp}A$, for all $A \in \mathfrak{D}^k_{\lambda,\mu}$. That is forall open subset $U \subset S^{1|1}$, we have

$$A_{|U} = 0 \Rightarrow T(A)_{|U} = 0; \forall A \in \mathfrak{D}^k_{\lambda,\mu}.$$

Definition 3.3. *If $k \in \frac{1}{2}\mathbb{N}$, the supercommutant $\mathfrak{I}^k_{\lambda,\mu}$ is the space of linear local operators:*

$$T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$$

commuting with the $\mathfrak{osp}(1|2)$-action:

$$[\mathfrak{L}^{\lambda,\mu}_{X_F}, T] := \mathfrak{L}^{\lambda,\mu}_{X_F} \circ T - (-1)^{|T||F|} T \circ \mathfrak{L}^{\lambda,\mu}_{X_F} = 0, \quad X_F \in \mathfrak{osp}(1|2).$$

The space $\mathfrak{I}^k_{\lambda,\mu}$ is an associative superalgebra called the superalgebra of orthosymplectic supersymmetries of the modules $\mathfrak{D}^k_{\lambda,\mu}$.

Lemma 3.4. *Let $k \in \frac{1}{2}\mathbb{N}$ and $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ a linear local operator. Then there exists an integer m and some functions $T^{\ell}_{ij} \in C^{\infty}_{\mathbb{C}}(S^{1|1})$ such that*

$$T(a\overline{D}^{\ell}) = \sum_{j=0}^{\ell}\sum_{i=0}^{m} T^{\ell}_{i,j} D^i(a)\overline{D}^j.$$

Proof. Firstly, for all ℓ and for all j, the operator that associates with the function a the component with respect to $\overline{D}^j$ of $T(a\overline{D}^{\ell})$ is a local operator acting between superfunctions. Secondly, by the theorem of Batchelor, there exists a fiber bundle E of rank 1 over S^1 such that the space of superfunctions $\mathbb{C}(S^{1|1})$ is isomorphic to the space of sections of the bundle $\bigwedge^* E \to S^1$. Thus, thanks to the renowned Peetre theorem in classical differential geometry, the coefficients of the operator $T(a\overline{D}^{\ell})$ appear as derivations of the function a. $\square$

4 Examples of orthosymplectic supersymmetries

4.1 The conjugation $*$

Let us denote by $\mathcal{B}$ the Berezin integral $\mathcal{B} : \mathfrak{F}_{\frac{1}{2}} \to \mathbb{C}$ given, for any $f = f_0(x) + \theta f_1(x)$, by the formula

$$\mathcal{B}(f\alpha^{\frac{1}{2}}) = \int_{S^1} f_1(x)dx.$$

So, the product of densities composed with $\mathcal{B}$ yields a bilinear $\mathcal{K}(1)$-invariant form:

$$< . \, , \, . >: \mathfrak{F}_\lambda \otimes \mathfrak{F}_{\frac{1}{2}-\lambda} \to \mathbb{C}, \;\; \lambda \in \mathbb{C}.$$

Thus we get the $\mathcal{K}(1)$-invariant conjugation map $* : \mathfrak{D}_{\lambda,\mu} \to \mathfrak{D}_{\frac{1}{2}-\mu,\frac{1}{2}-\lambda}$ defined by

$$< A\phi, \psi >= (-1)^{A\phi} < \phi, A^*\psi >$$

for any $A \in \mathfrak{D}_{\lambda,\mu}$, $\phi \in \mathfrak{F}_\lambda$ and $\psi \in \mathfrak{F}_{\frac{1}{2}-\mu}$.

Accordingly, in the particular case $\lambda+\mu = \dfrac{1}{2}$, the conjugation map leads an element in $\mathfrak{I}^k_{\lambda,\frac{1}{2}-\lambda}$, for any $\lambda \in \mathbb{C}$. An explicit formula for $*$ was given in [7], for any $k \in \frac{1}{2}\mathbb{N}$, by:

$$\left(\sum_i a_i \overline{D}^i\right)^* = \sum_i (-1)^{[\frac{i+1}{2}]+ia_i} \overline{D}^i \circ a_i$$

where for a real number x, we denote by $[x]$ its integer part.

4.2 The projection P_0

The space of zeroth-order operator $\mathfrak{D}^0_{\lambda,\mu}$ is isomorphic to $\mathfrak{F}_{\mu-\lambda}$, it consists of the operator that are equal to the multiplication by some $(\mu - \lambda)$-density. Moreover, when $\lambda = 0$, we get the projection map

$$P_0 : \mathfrak{D}^k_{0,\mu} \to \mathfrak{F}_\mu \subset \mathfrak{D}^k_{0,\mu}, \;\; A = \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \mapsto a_0\alpha^\mu$$

For short we write: $P_0(A) = a_0$.

4.3 An element of $\mathfrak{I}^k_{\lambda,\frac{1}{2}}$, $k \in \frac{1}{2}\mathbb{N}$ and an element of $\mathfrak{I}^{k+\frac{1}{2}}_{-\frac{k}{2},\mu}$, $k \in \mathbb{N}$.

For $k \in \frac{1}{2}\mathbb{N}$, we consider the map:

$$\Upsilon_k : \mathfrak{D}^k_{\lambda,\frac{1}{2}} \longrightarrow \mathfrak{D}^k_{\lambda,\frac{1}{2}}, \;\; \sum_{\ell=0}^{2k} a_\ell \overline{D}^\ell \longmapsto \sum_{\ell=0}^{2k} D^\ell(a_\ell)$$

and for $k \in \mathbb{N}$, we consider the map:

$$\Lambda_k : \mathfrak{D}^{k+\frac{1}{2}}_{-\frac{k}{2},\mu} \longrightarrow \mathfrak{D}^{k+\frac{1}{2}}_{-\frac{k}{2},\mu}, \;\; \sum_{\ell=0}^{2k+1} a_\ell \overline{D}^\ell \longmapsto a_{2k+1}\overline{D}^{2k+1}.$$

Then we can easily check that $\Upsilon_k \in \mathfrak{I}^k_{\lambda,\frac{1}{2}}$ and $\Lambda_k \in \mathfrak{I}^{k+\frac{1}{2}}_{-\frac{k}{2},\mu}$.

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5 Aff-supersymmetries

We consider the affine subalgebra of the orthosymplectic superalgebra $\mathfrak{osp}(1|2)$

$$\mathrm{Aff} := \mathrm{Span}\{X_1, X_x, X_\theta\}.$$

Obviously, the affine algebra acts on the module $\mathfrak{D}^k_{\lambda,\mu}$. Let us first characterize the algebra of Aff-supersymmetries, i.e, the space of linear local operators $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\lambda,\mu}$ commuting with the Aff-action on $\mathfrak{D}^k_{\lambda,\mu}$.

Proposition 5.1. *A linear operator $T : \mathfrak{D}^k_{\lambda,\mu} \longrightarrow \mathfrak{D}^k_{\lambda,\mu}$ commutes with the action of the vector fields X_1 and X_x if and only if for all ℓ in $\{0, 1, \cdots, 2k\}$, there exist constant coefficients c^ℓ_i and d^ℓ_i such that*

$$T\left(a\overline{D}^\ell\right) = \sum_{i=0}^{\ell} c^\ell_i D^i(a)\overline{D}^{\ell-i} + \theta \sum_{i=1}^{\ell+1} d^\ell_i D^i(a)\overline{D}^{\ell+1-i}. \tag{5.1}$$

Proof. Evidently, by the linearity of $[T, L_X]$, it suffices to check the invariance of T on homogenous operators of the form $a\overline{D}^\ell$ for some function a. Indeed, we have

$$[T, L_X](\sum_{\ell=1}^{k} a_\ell\overline{D}^\ell) = 0, \forall a_0, ..., a_\ell \in \mathbb{C}(S^{1|1}) \Longleftrightarrow [T, L_X](a_\ell\overline{D}^\ell) = 0, \forall \ell = 1, ..., k, \forall a_\ell \in \mathbb{C}(S^{1|1}).$$

Moreover, the invariance with respect to the vector field $X_1 = \partial_x$ yields that the coefficients T^ℓ_{ij} given in Lemma (3.4) do not depend on the even variable x. Thus, we put

$$T(a\overline{D}^\ell) = \sum_{j=0}^{\ell}\sum_{i=0}^{m} T^\ell_{i,j} D^i(a)\overline{D}^j$$

where $T^\ell_{i,j} = c^\ell_{i,j} + \theta d^\ell_{i,j}$ ($c^\ell_{i,j}, d^\ell_{i,j} \in \mathbb{C}$) . The action of X_x on the operator $A = a\overline{D}^\ell$ is explicitly given by the rule

$$\mathfrak{L}^{\lambda,\mu}_{X_x}(A) = \left(xa' + \tfrac{1}{2}\theta\overline{D}(a) + \left(\mu - \lambda - \tfrac{\ell}{2}\right) a\right) \overline{D}^\ell \ .$$

Since the operator X_x is even, the invariance property of T with respect of the vector fields X_x leads to the following condition:

$$\begin{array}{rcl} 0 &=& x\partial_x\left(T(A)\right) - \tfrac{1}{2}\overline{D}(x)\overline{D}(T(A)) + \mu T(A) - T(A) \circ (xD^2 + \tfrac{1}{2}\theta\overline{D} + \lambda) \\ && -T((xa' + \tfrac{1}{2}\theta\overline{D}(a) + (\mu - \lambda - \tfrac{\ell}{2})a)\overline{D}^\ell). \end{array}$$

If we look for the terms in $\overline{D}^{2j}$ for $0 \le j \le \left[\frac{\ell}{2}\right]$ and for the terms in $\overline{D}^{2j+1}$ for $0 \le j \le \left[\frac{\ell-1}{2}\right]$ in this equation, we get

$$\begin{cases} 0 = \left(\frac{i}{2} + j - \left[\frac{\ell}{2}\right]\right) T^\ell_{i,2j} D^i(a) - \frac{1}{2}\theta\overline{D}(T^\ell_{i,2j})D^i(a), & 0 \le i \le m, \ j \le \left[\frac{\ell}{2}\right] \\ 0 = \left(\frac{i}{2} + j - \frac{\ell}{2} + \frac{1}{2}\right) T^\ell_{i,2j+1} D^i(a) - \frac{1}{2}\theta\overline{D}(T^\ell_{i,2j+1})D^i(a), & 0 \le i \le m, \ j \le \left[\frac{\ell-1}{2}\right] \ . \end{cases}$$

By substituting respectively $c^\ell_{i,2j} + \theta d^\ell_{i,2j}$ and $c^\ell_{i,2j+1} + \theta d^\ell_{i,2j+1}$ to $T^\ell_{i,2j}$ and $T^\ell_{i,2j+1}$ we get

$$\begin{cases} \left(\frac{i}{2} + j - \left[\frac{\ell}{2}\right]\right) c^\ell_{i,2j} + \theta\left(\frac{i}{2} + j - \left[\frac{\ell}{2}\right] - \frac{1}{2}\right) d^\ell_{i,2j} = 0, & 0 \le i \le m, \ j \le \left[\frac{\ell}{2}\right] \\ \left(\frac{i}{2} + j - \frac{\ell}{2} + \frac{1}{2}\right) c^\ell_{i,2j+1} + \theta\left(\frac{i}{2} + j - \frac{\ell}{2}\right) d^\ell_{i,2j+1} = 0, & 0 \le i \le m, \ j \le \left[\frac{\ell-1}{2}\right] \end{cases}$$

which proves that $c^{\ell}_{i,j}=0$ for $0\le i\le m, j\neq \ell-i$ and $d^{\ell}_{i,j}=0$ for $1\le i\le m, j\neq \ell+1-i$. Then one has

$$T(a\overline{D}^{\ell})=\sum_{i=0}^{\ell} c^{\ell}_{i,\ell-i}D^i(a)\overline{D}^{\ell-i}+\theta\sum_{i=1}^{\ell+1} d^{\ell}_{i,\ell-i+1}D^i(a)\overline{D}^{\ell-i+1},$$

and by a change of notation, (5.1) is thus proved. □

Remark 5.2. *It is easy to prove that the operator T given in* (5.1) *is even. So, the* $\mathfrak{osp}(1|2)$*-invariance of T means that*

$$[\mathfrak{L}^{\lambda,\mu}_X, T]=\mathfrak{L}^{\lambda,\mu}_X\circ T-T\circ\mathfrak{L}^{\lambda,\mu}_X=0,\ \forall X\in\{X_\theta, X_{x\theta}, X_{x^2}\}. \tag{5.2}$$

Let us now impose the invariance condition with respect to the vector field X_θ

Theorem 5.3. *The algebra of* Aff*-supersymmetries is given by the set of linear operators $T:\mathfrak{D}^k_{\lambda,\mu}\longrightarrow\mathfrak{D}^k_{\lambda,\mu}$ such that*

$$T\left(a\overline{D}^{\ell}\right)=\sum_{i=0}^{\ell} T^{\ell}_i D^i(a)\overline{D}^{\ell-i}, \forall\ell\in\{0,1,\cdots,2k\}, \tag{5.3}$$

where T^{ℓ}_i are arbitrary constants.

Proof. If $\ell=2p$ is even, a priori, $T\left(a\overline{D}^{2p}\right)$ has the general form

$$T\left(a\overline{D}^{2p}\right)=\sum_{i=0}^{2p} c^{2p}_i D^i(a)\overline{D}^{2p-i}+\theta\sum_{i=1}^{2p+1} d^{2p}_i D^i(a)\overline{D}^{2p+1-i}.$$

By a direct computation we have $\mathfrak{L}^{\lambda,\mu}_{X_\theta}(a\overline{D}^{2p})=\left(\theta a'+\frac{1}{2}\overline{D}(a)\right)\overline{D}^{2p}$,

$$\begin{aligned}
&T\circ\mathfrak{L}^{\lambda,\mu}_{X_\theta}\left(a\overline{D}^{2p}\right)=\sum_{i=0}^{p} c^{2p}_{2i}\left(\theta D^{2i+2}(a)+\frac{1}{2}\overline{D}\circ D^{2i}(a)\right)\overline{D}^{2p-2i}\\
&+\sum_{i=0}^{p-1} c^{2p}_{2i+1}\left(D^{2i+2}(a)-\theta D\circ D^{2i+2}(a)+\frac{1}{2}D\circ\overline{D}(D^{2i}(a))\right)\overline{D}^{2p-2i-1}\\
&+\frac{1}{2}\theta\sum_{i=1}^{p} d^{2p}_{2i}\overline{D}\circ D^{2i}(a)\overline{D}^{2p-2i+1}+\theta\sum_{i=0}^{p} d^{2p}_{2i+1}\left(D^{2i+2}(a)+\frac{1}{2}D\circ\overline{D}(D^{2i}(a))\right)\overline{D}^{2p-2i}
\end{aligned}$$

and

$$\begin{aligned}
&\mathfrak{L}^{\lambda,\mu}_{X_\theta}\circ T\left(a\overline{D}^{2p}\right)=\sum_{i=0}^{p} c^{2p}_{2i}\left(\theta D^{2i+2}(a)+\frac{1}{2}\overline{D}\circ D^{2i}(a)\right)\overline{D}^{2p-2i}\\
&+\sum_{i=0}^{p-1} c^{2p}_{2i+1}\left(\theta D^{2i+3}(a)++\frac{1}{2}\overline{D}\circ D^{2i+1}(a)\right)\overline{D}^{2p-2i-1}+\sum_{i=1}^{p} d^{2p}_{2i}\left(\frac{1}{2}D^{2i}(a)-\frac{1}{2}\theta\overline{D}\circ D^{2i}(a)\right)\overline{D}^{2p-2i+1}\\
&+\sum_{i=0}^{p} d^{2p}_{2i+1}\left(\frac{1}{2}D^{2i+1}(a)-\frac{1}{2}\theta\overline{D}\circ D^{2i+1}(a)\right)\overline{D}^{2p-2i}.
\end{aligned}$$

By taking into consideration the above remark we have

$$[T, \mathfrak{L}^{\lambda,\mu}_{X_\theta}](a\overline{D}^{2p}) = \frac{1}{2}\sum_{i=1}^{p}\left(2\theta D^{2i}\circ \overline{D}(a) - D^{2i}(a)\right) d^{2p}_{2i}\overline{D}^{2p-2i+1}$$
$$+\frac{1}{2}\sum_{i=0}^{p}\left(\theta D^{2i+2}(a) - D^{2i+1}(a)\right) d^{2p}_{2i+1}\overline{D}^{2p-2i}.$$

So, the condition $[T, \mathfrak{L}^{\lambda,\mu}_{X_\theta}] = 0$ yields $d^{2p}_i = 0$ for $1 \le i \le 2p+1$. We get the same result when ℓ is odd.

□

6 The main theorem

Finally we come to the main result of the paper, which is the characterization of the algebra of orthosymplectic supersymmetries of modules of differential operators on the supercircle $S^{1|1}$. Since $[X_{x\theta}, X_{x\theta}] = 2X_{x^2}$, an operator T commutes with the action of the vector field X_{x^2} as soon as he commutes with the action of $X_{x\theta}$. Then, the last step is to impose the invariance condition with respect to the vector field $X_{x\theta}$. The upshot is the following:

Theorem 6.1. *The algebra $\mathfrak{I}^k_{\lambda,\mu}$ of orthosymplectic supersymmetries is given by the set of linear operators $T : \mathfrak{D}^k_{\lambda,\mu} \longrightarrow \mathfrak{D}^k_{\lambda,\mu}$ such that*

$$T\left(a\overline{D}^{\ell}\right) = \sum_{i=0}^{\ell} T^{\ell}_i D^i(a)\overline{D}^{\ell-i}, \forall \ell \in \{0,1,\cdots,2k\},$$

where T^ℓ_i are arbitrary constants satisfying the following equations:
forall ℓ in $\{1,2,\cdots,2k\}$,

$$\left\{\begin{array}{l} 0 = \left((1-(-1)^\ell)\lambda + \left[\frac{\ell}{2}\right]\right) T^{\ell-1}_{2i} - \left((1-(-1)^\ell)\lambda + \left[\frac{\ell}{2}\right] - i\right) T^{\ell}_{2i} + (2\mu - 2\lambda - \ell + i)c^{\ell}_{2i+1}, \\ 0 = \left((1-(-1)^\ell)\lambda + \left[\frac{\ell}{2}\right]\right) T^{\ell-1}_{2i+1} + \left((1+(-1)^\ell)\lambda + \left[\frac{\ell-1}{2}\right] - i\right) T^{\ell}_{2i+1} - (i+1)T^{\ell}_{2i+2}. \end{array}\right. \tag{6.1}$$

where $0 \le i \le \left[\frac{\ell-1}{2}\right]$ for the first equation and $0 \le i \le \left[\frac{\ell}{2}\right] - 1$ for the second one.

Proof. For $\ell = 2p$, $\ell \in \{1,2,\cdots,2k\}$, by a direct computation we get

$$\mathfrak{L}^{\lambda,\mu}_{X_{x\theta}}\left(a\overline{D}^{2p}\right) = \left(x\theta a' + \frac{1}{2}x\overline{D}(a) + (\mu-\lambda-p)\theta a\right)\overline{D}^{2p} + \frac{1}{2}(-1)^{|a|}pa\overline{D}^{2p-1},$$

$$T\circ\mathfrak{L}^{\lambda,\mu}_{X_{x\theta}}\left((a_0+\theta a_1)\overline{D}^{2p}\right) = \sum_{i=0}^{p} T^{2p}_{2i}\left(\frac{1}{2}x\theta a_0^{(i+1)} + (\mu-\lambda-p+\frac{1}{2}i)\theta a_0^{(i)} + \frac{1}{2}xa_1^{(i)} + \frac{1}{2}\theta a_1^{(i-1)}\right)\overline{D}^{2p-2i}$$
$$+\sum_{i=0}^{p-1} T^{2p}_{2i+1}\left(\frac{1}{2}xa_0^{(i+1)} + (\mu-\lambda-p+\frac{1}{2}i)a_0^{(i)} + \frac{1}{2}\theta(xa_1^{(i+1)} + a_1^{(i)}) + \frac{1}{2}ia_1^{(i)}\right)\overline{D}^{2p-2i-1}$$
$$+\sum_{i=0}^{p-1}\frac{p}{2}T^{2p-1}_{2i}(a_0^{(i)} + \theta a_1^{(i)})\overline{D}^{2p-2i-1} + \sum_{i=0}^{p-1}\frac{p}{2}T^{2p-1}_{2i+1}(a_1^{(i)} + \theta a_0^{(i+1)})\overline{D}^{2p-2i-2}$$

and

$$\mathfrak{L}^{\lambda,\mu}_{X_{x\theta}} \circ T\left(a\overline{D}^{2p}\right) = \sum_{i=0}^{p} T^{2p}_{2i}\left(x\theta D^{2i+2}(a) + \frac{1}{2}x\overline{D}\circ D^{2i}(a) + (\mu-\lambda-p+i)\theta D^{2i}(a)\right)\overline{D}^{2p-2i} +$$
$$\frac{1}{2}\sum_{i=0}^{p}(-1)^{|a|}(p-i)T^{2p}_{2i}D^{2i}(a)\overline{D}^{2p-2i-1}+$$
$$+\sum_{i=0}^{p-1}T^{2p}_{2i+1}\left(x\theta D^{2i+3}(a) + \frac{1}{2}x\overline{D}\circ D^{2i+1}(a) + (\mu-\lambda-p+i+\frac{1}{2})\theta D^{2i+1}(a)\right)\overline{D}^{2p-2i-1}+$$
$$\sum_{i=0}^{p-1}(-1)^{|a|}T^{2p}_{2i+1}D^{2i+1}(a)(\lambda+\frac{k-i-1}{2})\overline{D}^{2p-2i-2}.$$

Thus

$$[T,\mathfrak{L}^{\lambda,\mu}_{X_{x\theta}}]\left((a_0+\theta a_1)\overline{D}^{2p}\right) = \mathfrak{L}^{\lambda,\mu}_{X_{x\theta}}\circ T - T\circ\mathfrak{L}^{\lambda,\mu}_{X_{x\theta}}$$
$$= \frac{1}{2}\sum_{i=0}^{p-1}\left(pT^{2p-1}_{2i} - (p-i)T^{2p}_{2i} + (2\mu-2\lambda-2p+i)T^{2p+1}_{2i}\right)(a^{(i)}_0 - \theta a^{(i)}_1)\overline{D}^{2p-2i-1}$$
$$+\frac{1}{2}\sum_{i=0}^{p-1}\left(pT^{2p-1}_{2i+1} + (2\lambda+p-i-1)T^{2p}_{2i+1})(-1)^{|a|}(a^{(i)}_1 + \theta a^{(i+1)}_0)\right.$$
$$\left.+(i+1)T^{2p}_{2i+2}(a^{(i)}_1 - \theta a^{(i+1)}_0)\right)\overline{D}^{2p-2i-2}.$$

Similar expressions are obtained when ℓ is odd. Equations (6.1) are then readily available. □

7 The dimension of the superalgebra $\mathfrak{I}^k_{\lambda,\mu}$

Consider the graded $\mathcal{K}(1)$-module $\mathrm{gr}D_{\lambda,\mu}$, called the *space of symbols of differential operators* and denoted by $\mathcal{S}_{\mu-\lambda}$, associated with the filtration

$$\mathfrak{D}^0_{\lambda,\mu}\subset\mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu}\subset\mathfrak{D}^1_{\lambda,\mu}\subset\mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu}\subset\cdots\subset\mathfrak{D}^{k-\frac{1}{2}}_{\lambda,\mu}\subset\mathfrak{D}^k_{\lambda,\mu}\subset\cdots.$$

As a $\mathcal{K}(1)$-module, $\mathcal{S}_{\mu-\lambda}$ is a direct sum of density modules (see [6]):

$$\mathcal{S}_{\mu-\lambda} = \bigoplus_{i=0}^{\infty}\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}.$$

Note that this module depends only on the shift, $\delta=\mu-\lambda$, of the weights and not on μ and λ independently. The space of symbols of degree k is isomorphic to $\mathfrak{F}_{\mu-\lambda-k}$.
In [7], the authors look for $\mathfrak{osp}(1|2)$-isomorphisms

$$\sigma:\mathfrak{D}_{\lambda,\mu}\to\mathcal{S}_{\mu-\lambda},\qquad Q:\mathcal{S}_{\mu-\lambda}\to\mathfrak{D}_{\lambda,\mu},$$

where $Q=\sigma^{-1}$. These isomorphisms are called the $\mathfrak{osp}(1|2)$-equivariant *symbol map* and *quantization map*, respectively. They prove the existence and uniqueness (up to normalization) of the $\mathfrak{osp}(1|2)$-equivariant symbol map σ when the weights δ is not resonant, i.e. when $\delta\notin$

$\frac{1}{2}\mathbb{N}\backslash\{0\}$. They also prove that, for the resonant values of δ, there is no $\mathfrak{osp}(1|2)$-isomorphism between $\mathfrak{D}_{\lambda,\mu}$ and $\mathcal{S}_{\mu-\lambda}$, except for the special values:

$$\lambda = \frac{1-m}{4}, \qquad \mu = \frac{1+m}{4}, \tag{7.1}$$

where m is an odd natural number.
Obviously, when the $\mathfrak{osp}(1|2)$-modules $\mathfrak{D}_{\lambda,\mu}$ and $\bigoplus_{i=0}^{2k}\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}$ are isomorphic, a non trivial $\mathfrak{osp}(1|2)$-Intertwining operator of the module $\mathfrak{D}^k_{\lambda,\mu}$ exists if and only if there exists an $\mathfrak{osp}(1|2)$-equivariant endomorphism of the space $\bigoplus_{i=0}^{2k}\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}$.

In this section, we shall determine by induction on $k \in \frac{1}{2}\mathbb{N}^*$ the dimension of the space $\mathfrak{I}^k_{\lambda,\mu}$ for all λ, μ given that

$$T = \sum_{\ell=0}^{2k}\sum_{i=0}^{\ell} T_i^\ell D^i(\ .\ )\overline{D}^{\ell-i} \in \mathfrak{I}^k_{\lambda,\mu} \Longleftrightarrow T_{|\mathfrak{D}^s_{\lambda,\mu}} = \sum_{\ell=0}^{2s}\sum_{i=0}^{\ell} T_i^\ell D^i(\ .\ )\overline{D}^{\ell-i} \in \mathfrak{I}^s_{\lambda,\mu}; \forall s \in \frac{1}{2}\mathbb{N}^*, s \leq k.$$

We shall get that only situations where the algebra $\mathfrak{I}^k_{\lambda,\mu}$ has a dimension greater than $2k+1$ are resonant situations. This result is expected since the only $\mathfrak{osp}(1|2)$-equivariant endomorphisms of the space $\bigoplus_{i=0}^{2k}\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}$ are maps whose restrictions to the space $\mathfrak{F}_{\mu-\lambda-\frac{i}{2}}$ are multiples of the identity map. Indeed, this fact is an easy consequence of the fact that the only non trivial $\mathfrak{osp}(1|2)$-equivariant maps between spaces of densities are given by the multiples of the maps $\overline{D}^k$ between $\mathfrak{F}_{\frac{1-k}{4}}$ and $\mathfrak{F}_{\frac{1+k}{4}}$, where k is an odd natural number (see [7]).

Theorem 7.1. *Let $k \in \mathbb{N}^*$, then:*

$$\dim\left(\mathfrak{I}^k_{\lambda,\mu}\right) = \begin{cases} \dim\left(\mathfrak{I}^{k-\frac{1}{2}}_{\lambda,\mu}\right) + 2 & \text{if } (\lambda,\mu) = (\frac{-k+j}{2}, \frac{k+j-s+1}{2}) \text{ where } 1 \leq j < s \leq k \\ \dim\left(\mathfrak{I}^{k-\frac{1}{2}}_{\lambda,\mu}\right) + 1 & \text{otherwise.}\end{cases}$$

Proof. We put $\ell = 2k$ in equations (6.1) and denote by $x_i = T^{2k}_{2i-2}$ $(i = 1, \ldots, k+1)$, $x_{k+i+1} = T^{2k}_{2i-1}$ $(i = 1, \ldots, k)$. If we consider T^{2k-1}_{2i-1} and T^{2k-1}_{2i-1} as parameters, we obtain a linear system in the x_j-unknowns where $j = 1, \ldots, 2k+1$. This system accurately determines the relationship between $\dim\left(\mathfrak{I}^k_{\lambda,\mu}\right)$ and $\dim\left(\mathfrak{I}^{k-\frac{1}{2}}_{\lambda,\mu}\right)$.

Let M be the matrix associated to this system, then $M \in \mathcal{M}_{2k,2k+1}(\mathbb{R})$. We can write the matrix M in the following form

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} A = (a_{i,j}), B = (b_{i,j}) \in \mathcal{M}_{k,k+1}(\mathbb{R}),\ C = (c_{i,j}), D = (d_{i,j}) \in \mathcal{M}_k(\mathbb{R})$$

where :

$$\begin{cases} a_{i,i+1} = i \\ a_{i,j} = 0 \text{ otherwise} \end{cases}, \quad \begin{cases} b_{i,i} = -(k-i+1) \\ b_{i,j} = 0 \text{ otherwise} \end{cases}, \quad \begin{cases} c_{i,i} = -(2\lambda+k-i) \\ c_{i,j} = 0 \text{ otherwise} \end{cases}$$

and

$$\begin{cases} d_{i,i} = 2\mu - 2\lambda - 2k + i - 1 \\ d_{i,j} = 0 \text{ otherwise.} \end{cases}$$

Let for a given matrix $T \in \mathcal{M}_{n,m}(\mathbb{R})$ denote by $\widetilde{T}_i$ (respectively $\widetilde{T}^j$, respectively $\widetilde{T}_i^j$) the matrix obtained by removing in the matrix T the i^{th} row (respectively the j^{th} column, respectively the i^{th} row and the j^{th} column). To achieve the proof, we should determine the rank of the matrix M, for this we distinguish the following cases (all determinants are given up to a sign):
case 1: $\lambda \neq \frac{-k+j}{2}$ for all $j = 1, \dots, k$. $\widetilde{M}^{k+1} \in \mathcal{M}_{2k}(\mathbb{R})$. A direct computation gives:

$$\det(\widetilde{M}^{k+1}) = k! \prod_{i=1}^{k} (2\lambda + k - i).$$

Thus, in this case, clearly we have $rk(M) = 2k$.
case 2: $(\lambda, \mu) = \left(\frac{-k+j}{2}, \frac{k+1}{2}\right)$, where $j \in \{1, \dots, k\}$, then

$$\det(\widetilde{M}^{k+j+1}) = \prod_{i=1}^{k}(k - i + 1) \prod_{i=1}^{j-1}(2\lambda + k - i) \prod_{i=j+1}^{k} (2\mu - 2\lambda - 2k + i - 1).$$

We also have $rk(M) = 2k$ in this case.
case 3: $(\lambda, \mu) = \left(\frac{-k+j}{2}, \frac{k+j-s+1}{2}\right)$ where $1 \leq s < j \leq k$, We have:

$$\det(\widetilde{T}^{s+1}) = \frac{(k!)^2}{(k-s)!(s+1)!} \prod_{t=1}^{s}(2\lambda + k - t) \prod_{t=s+1}^{k} (2\mu - 2\lambda - 2k + t - 1)$$

and still we have the same conclusion.
case 4: $\lambda = \frac{-k+j}{2}$, where $j \in \{1, \dots, k\}$ and $\mu \neq \frac{k+j-s+1}{2}$, for all $s \in \{1, \dots, k\}$. Thus

$$\det(\widetilde{M}^j) = k! \prod_{i=1}^{k}(2\mu - 2\lambda - 2k + i - 1)$$

and then we conclude as before that $rk(M) = 2k$.
case 5: $(\lambda, \mu) = (\frac{-k+j}{2}, \frac{k+j-s+1}{2})$ where $1 \leq j < s \leq k$. First we consider $L = \widetilde{M}^{k+1}$ which is in our case a noninvertible matrix, then we extract from L the matrix (of order $2k - 1$) $\widetilde{L}_j^{k+j+1}$. The determinant of this matrix is:

$$\det(\widetilde{L}_j^{k+j+1}) = (k-1)! \prod_{\substack{i=1 \\ i \neq j}}^{k} (2\lambda + k - i),$$

we deduce immediately that $rk(M) \geq 2k - 1$. In order to obtain the equality, we denote by $\Gamma_i, i = 1, \dots, 2k+1$ the columns of the matrix M and we easily check the following relationships

$$\begin{cases} \displaystyle\sum_{t=1}^{j} \alpha_t \Gamma_t + \frac{k-t+1}{2\mu - 2\lambda - 2k + t - 1} \Gamma_{k+t+1} = 0 \\ \displaystyle\sum_{t=s+1}^{k+1} \beta_t \Gamma_t + \frac{t-1}{2\lambda + k - t + 1} \Gamma_{k+t} = 0 \end{cases}$$

where

$$\begin{cases} \alpha_t = \prod_{i=1}^{t-1} \frac{(k-i+1)(2\lambda+k-i)}{2\mu-2\lambda-2k+i-1}; \; 2 \leq t \leq j \\ \alpha_1 = 1 \end{cases}$$

and

$$\begin{cases} \beta_t = \prod_{i=s+1}^{t-1} \frac{(k-i+1)(2\lambda+k-i)}{2\mu-2\lambda-2k+i-1}; \; s+2 \leq t \leq k+1 \\ \beta_{s+1} = 1. \end{cases}$$

Theorem 7.1 is thus proved. □

Now, by a similar reasoning, we prove the following theorem:

Theorem 7.2. *Let $k \in \mathbb{N}^*$, then*

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = \begin{cases} \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 2 & \text{if} \quad (\lambda,\mu) = (\frac{-k+j-1}{2}, \frac{k+j-s+1}{2}) \quad \text{where} \quad 1 \leq j \leq s \leq k+1 \\ \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 1 & \text{otherwise.} \end{cases}$$

The following corollary gives us the dimension of the space $\mathfrak{I}_{\lambda,\mu}^{n}$ for $n \in \frac{1}{2}\mathbb{N}^*$.

Corollary 7.3. *Let $k \in \mathbb{N}^*$. Then*

i) $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \begin{cases} 2k+2 \text{ if } (\lambda,\mu) = (0,\frac{1}{2}) \\ 2k+2s-2j+1 \text{ if } (\lambda,\mu) = (\frac{-k+j}{2}, \frac{k+j-s+1}{2}), 1 \leq j < s \leq k \\ 2k+1 \text{ otherwise.} \end{cases}$

ii) $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = \begin{cases} 2k+3 \text{ if } (\lambda,\mu) = (0,\frac{1}{2}) \\ 2k+3 \text{ if } (\lambda,\mu) = (\frac{-k}{2}, \frac{k-s+2}{2}); \; 1 \leq s \leq k+1 \\ 2k+2s-2j+3 \text{ if } (\lambda,\mu) = (\frac{-k+j-1}{2}, \frac{k+j-s+1}{2}); \; 1 < j \leq s \leq k+1 \\ 2k+2 \text{ otherwise.} \end{cases}$

Proof. i) According to Theorem 7.2 we start by the case where

$$(\lambda,\mu) = \tfrac{1}{2}\left(-k+j, k+j-s+1\right), 1 \leq j < s \leq k.$$

We have:

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-\frac{1}{2}}\right) + 2 = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) + 2.$$

By writing

$$(\lambda,\mu) = \tfrac{1}{2}\left(-(k-1)+j_1-1, (k-1)+j_1-s_1+1\right)$$

where $j_1 = j \in \{1,\ldots,k-1\}$ and $s_1 = s-1 \in \{1,\ldots,k-1\}$ such that $j_1 \leq s_1$, by Theorem 7.2 we get

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-1}\right) + 2.$$

Thus

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-1}\right) + 4.$$

Now, if $s > j-1$, we write (λ, μ) as

$$(\lambda, \mu) = \frac{1}{2}\left(-(k-1)+(j-1), (k-1)+(j-1)-(s-2)+1\right),$$

and as before one has

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k-1}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-2}\right) + 4$$

and by iteration

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-(s-j)}\right) + 4(s-j).$$

In this step,

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k-(s-j)}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-(s-j)-\frac{1}{2}}\right) + 1 = \dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) + 2(k-s+j-1),$$

consequently we have

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) + 4(s-j) + 2(k-s+j-1) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) + 2(k+s-j-1).$$

Suppose now that $\lambda \neq \frac{-k+j}{2}$ for all $j \in \{1, \ldots, k\}$ then

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-\frac{1}{2}}\right) + 1 = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) + 1.$$

Clearly, we have $\lambda \neq \frac{-(k-1)+j-1}{2}$ for all $j \in \{1, \cdots, k-1\}$, then

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)}\right) + 1$$

and

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)}\right) + 2 = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-2)}\right) + 4 = \cdots = \dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) + 2(k-1).$$

For $\lambda = \frac{-k+j}{2}, j \in \{1, \ldots, k\}$ and $\mu \neq \frac{k+j-s+1}{2}, \forall s > j$, we have

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) + 1.$$

Moreover, $\lambda = \frac{-(k-1)+j-1}{2}$ and $\mu \neq \frac{(k-1)+j-t+1}{2}$ for all $t \in \{1, \ldots, k-1\}$ with $t \geq j$, so

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{(k-1)+\frac{1}{2}}\right) + 1 = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k-1}\right) + 2 = \cdots = \dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) + 2(k-1).$$

Finally, from the fact that $\dim\left(\mathfrak{I}_{\lambda,\mu}^{1}\right) = \begin{cases} 4 \text{ if } (\lambda,\mu) = (0, \frac{1}{2}) \\ 3 \text{ otherwise} \end{cases}$, we can easily deduce $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right)$.

ii) We have

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = \begin{cases} \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 2 \text{ if } (\lambda,\mu) = (\frac{-k+j-1}{2}, \frac{k+j-s+1}{2}),\ 1 \leq j \leq s \leq k+1 \\ \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 1 \text{ otherwise.} \end{cases}$$

Thus:

- If $(\lambda, \mu) = (\frac{-k+j-1}{2}, \frac{k+j-s+1}{2}) = (\frac{-k+(j-1)}{2}, \frac{k+(j-1)-(s-1)+1}{2})$, using Corollary 7.3 i), we get

$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = 2k+1$ when $j=1$ or $j=s$, that means $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = 2k+3$ in these cases and $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) = 2k+2(s-1)-2(j-1)+1$ if $1<j<s\leq k+1$ which leads to the fact that $\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = 2k+2s-2j+3$ in this case.

• If $(\lambda,\mu) = (0,\frac{1}{2})$ then

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 1 = (2k+2)+1 = 2k+3.$$

• Otherwise

$$\dim\left(\mathfrak{I}_{\lambda,\mu}^{k+\frac{1}{2}}\right) = \dim\left(\mathfrak{I}_{\lambda,\mu}^{k}\right) + 1 = (2k+1)+1 = 2k+2.$$

This achieves the proof of Corollary 7.3. □

References

[1] Agrebaoui B and Ben Fraj N, *On the cohomology of the Lie superalgebra of contact vector fields on* $S^{1|1}$, Bull. Soc. Roy. Sci. Liège **72**, 6, (2004), 365–375.

[2] Basdouri I and Ben Ammar M, *Cohomology of* $\mathfrak{osp}(1|2)$ *acting on linear differential operators on the supercircle* $S^{1|1}$. Lett. Math. Phys., 81, (2007), 239-251

[3] N. Ben Fraj and S. Omri, Deforming of the Lie superalgebra of contact vector fields on $S^{1|1}$ inside the Lie superalgebra of superpseudodifferential operators on $S^{1|1}$, J. Nonlinear. Math. Phys. 13(1) (2006), 19–33.

[4] F. Berezin, Introduction to superanalysis, Math. Phys. Appl. Math. 9 (1987). Reidel Dordrecht. operators and irreducible representations of Lie superalgebra of vector fields, Selecta. Math. Sov. 1(2) (1981), 143–160.

[5] D. B. Fuchs, cohomology of infinite-dimensional Lie algebra, Consultants Bureau, New York, 1987.

[6] H. Gargoubi, P. Monthonet and V. Ovsienko, Symetries of modules of differential operators, J. Nonlinear. Math. Phys. 12 (2005), 348–380.

[7] H. Gargoubi, N. Mallouli and V. ovsienko, Differential operators on supercircle: Conformally equivariant quantization and symbol calculus, Lett. Math. Phys. 79 (2007), 51–65.

[8] I. Shchepochkina, How to realize Lie algebras by vector fields, Theor. Math. Phys. 147(3) (2006), 821–838. math.RT/0509472.