ABOUT THE COHOMOLOGY OF THE LIE SUPERA LGE OF VECTOR FIELDS ON $\mathbb{R}^{n|n}$

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Abstract. In this paper, we compute the first space of cohomology of $\text{Vect}(\mathbb{R}^{n|n})$, the Lie superalgebra of vector fields on the supermanifold $\mathbb{R}^{n|n}$ with coefficients in $\mathcal{F}(\mathbb{R}^{n|n})$, the space of smooth functions on $\mathbb{R}^{n|n}$. We give a super analog of the cohomologies of vector fields that were studied for instance by D.B. Fuchs [2]. This work allows us to classify the deformations of the action of $\text{Vect}(\mathbb{R}^{n|n})$ on $\mathcal{F}(\mathbb{R}^{n|n})$.

1. Introduction

Let $\text{Vect}(\mathbb{R}^{n|n})$ be the Lie superalgebra of vector fields on the supermanifold $\mathbb{R}^{n|n}$ and $\mathcal{F}(\mathbb{R}^{n|n})$ be the space of smooth functions on the manifold $\mathbb{R}^{n|n}$. As $\mathcal{F}(\mathbb{R}^{n|n})$ can be identified with the supercommutative superalgebra $\Omega(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathbb{R}^n)$ of differential forms on $\mathbb{R}^n$, then $\text{Vect}(\mathbb{R}^{n|n})$ is identified with the superalgebra of superderivations of $\Omega(\mathbb{R}^n)$. So, $\text{Vect}(\mathbb{R}^{n|n})$ is identified to a sum of two copies of the space of tensor valued differential forms on $\mathbb{R}^n$, $\Omega = \bigoplus_{k} \Omega^k(\mathbb{R}^n, T\mathbb{R}^n)$, one with the Frölicher-Nijenhuis bracket $[[\ ,\ ]]$, the other one with the Richardson Nijenhuis bracket $[\ ,\ ]^\wedge$. We shall set $\mathfrak{g} = (\Omega, [[\ ,\ ]])$, and $\mathcal{R} = (\Omega, [\ ,\ ]^\wedge)$. For this identification, as well as relationship between the two brackets, see the book by Michor, Kolar and Slovac [3]. Here we compute $H^1(\text{Vect}(\mathbb{R}^{n|n}), \mathcal{F}(\mathbb{R}^{n|n}))$.

1.1. Notations and definitions.

1.1.1. Identification of $\text{Vect}(\mathbb{R}^{n|n})$. We shall first precise the structure of $\text{Vect}(\mathbb{R}^{n|n})$. The space $\mathcal{F}(\mathbb{R}^{n|n})$ of smooth functions on $\mathbb{R}^{n|n}$ can be identified with the graded commutative algebra

$$\Omega(\mathbb{R}^n) = \bigoplus_{s=0}^{n} \Omega^s(\mathbb{R}^n)$$

of differential forms on $\mathbb{R}^n$. We denote by $\text{Der}_s(\Omega(\mathbb{R}^n))$ the space of all graded derivations of degree $s$, i.e all linear mappings

$$D : \Omega(\mathbb{R}^n) \longrightarrow \Omega(\mathbb{R}^n)$$

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with \( D(\Omega^i(\mathbb{R}^n)) \subset \Omega^{i+1}(\mathbb{R}^n) \) and
\[
D(\varphi \wedge \psi) = D\varphi \wedge \psi + (-1)^{kl} \varphi \wedge D(\psi)
\]
for \( \varphi \in \Omega^i(\mathbb{R}^n) \) and \( \psi \in \Omega^k(\mathbb{R}^n) \). The space
\[
Der(\Omega(\mathbb{R}^n)) = \bigoplus_s Der_s(\Omega(\mathbb{R}^n))
\]
is a graded Lie superalgebra with the graded commutator:
\[
[D_1, D_2] := D_1 \circ D_2 - (-1)^{s_1s_2} D_2 \circ D_1
\]
for \( D_i \in Der_s(\Omega(\mathbb{R}^n)) \), for \( i \in \{1, 2 \} \). Then the space
\[
Vect(\mathbb{R}^n|n) := Der(\Omega(\mathbb{R}^n)).
\]
We call \( \Omega(\mathbb{R}^n, T\mathbb{R}^n) = \bigoplus_{s=0}^n \Omega^s(\mathbb{R}^n, T\mathbb{R}^n) \) the space of all vector valued differential forms. We shall frequently use the identification between \( \Omega^*(\mathbb{R}^n, T\mathbb{R}^n) \) and the completed tensor product over functions \( \Omega^*(\mathbb{R}^n) \otimes T\mathbb{R}^n \). So, by a slight abuse notations, we shall identify \( \omega \otimes X \) where \( \omega \in \Omega^*(\mathbb{R}^n) \) and \( X \in T\mathbb{R}^n \), with the corresponding tensor valued differential form.

A derivation \( D \in Der_s(\Omega(\mathbb{R}^n)) \) is algebraic if its restriction to \( \Omega^0(\mathbb{R}^n) \) vanishes identically. Then \( D(f\omega) = fD(\omega) \) for \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \). So, from C. Roger ([6] p 68), \( D \) is given by a tensor field. So, \( D \) induces a derivation \( D_x \in Der_s(\Omega(\mathbb{R}^n)) \) for each \( x \in \mathbb{R}^n \). It is uniquely determined by its restriction to 1-forms:
\[
D_x|T^*_x\mathbb{R}^n : T^*_x\mathbb{R}^n \longrightarrow \wedge^{s+1}T^*_x\mathbb{R}^n
\]
which we may view as an element \( K_x \in \wedge^{k+1}T^*_x\mathbb{R}^n \otimes T^*_x\mathbb{R}^n \) depending smoothly on \( x \in \mathbb{R}^n \). We write \( D = i_K \), where
\[
K \in C^\infty(\wedge^{s+1}T^*_x\mathbb{R}^n \otimes T^*_x\mathbb{R}^n) =: \Omega^{s+1}(\mathbb{R}^n, T\mathbb{R}^n).
\]
Note the defining equation: \( i_K(w) = w \circ K \) for \( w \in \Omega^1(\mathbb{R}^n) \).

The exterior derivative \( d \) is an element of \( Der_1(\Omega(\mathbb{R}^n)) \). In view of the formula
\[
\mathcal{L}_X = [i_X, d] = i_X \circ d + d \circ i_X
\]
for vector fields \( X \in Vect(\mathbb{R}^n) \), we define for \( K \in \Omega^s(\mathbb{R}^n, T\mathbb{R}^n) \) the Lie derivation \( \mathcal{L}_K \in Der_s(\Omega(\mathbb{R}^n)) \) by
\[
\mathcal{L}_K := [i_K, d] = i_K \circ d + (-1)^s d \circ i_K,
\]
then the mapping \( \mathcal{L} : \Omega(\mathbb{R}^n, T\mathbb{R}^n) \longrightarrow Der(\Omega(\mathbb{R}^n)) \) is injective, since \( \mathcal{L}_K f = i_k df = df \circ K \) for \( f \in C^\infty(\mathbb{R}^n, \mathbb{R}) \).

**Lemma 1.** [6] For any graded derivation \( D \in Der_k(\Omega(\mathbb{R}^n)) \), there exists an unique \( K \in \Omega^k(\mathbb{R}^n, T\mathbb{R}^n) \) and \( L \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n) \) such that \( D = \mathcal{L}_K + i_L \), where
\begin{enumerate}
  \item \( i_L(\omega \otimes X) := i_L(\omega) \otimes X \) and \( i_L(\omega) = \eta \wedge i_Y(\omega) \) for \( L = \eta \otimes Y \).
\end{enumerate}
The degree of \( D \) is denoted \( |D| \) and is equal to \( k \).
1.1.2. Richardson-Nijenhuis algebra. The injection
\[ i : \Omega^{+1}(\mathbb{R}^n, T\mathbb{R}^n) \longrightarrow \text{Der}^{+}(\Omega(\mathbb{R}^n)) ; \quad i([K, L]^\wedge) := [i_K, i_L] \]
is a graded Lie bracket on \( \Omega^{+1}(\mathbb{R}^n, T\mathbb{R}^n) \). So, we get a bracket on \( \Omega^{+1}(\mathbb{R}^n, T\mathbb{R}^n) \) which defines a graded Lie algebra structure with the grading as indicated. For \( K \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n) \) and \( L \in \Omega^{\ell+1}(\mathbb{R}^n, T\mathbb{R}^n) \) we have
\[ [K, L]^\wedge = i_K L - (-1)^{k\ell} i_L K. \]
The space \( \mathcal{R} = (\bigoplus \Omega^{+1}(\mathbb{R}^n, T\mathbb{R}^n), [\ , \ ]^\wedge) \) is called the Richardson-Nijenhuis algebra. It is a subalgebra of \( \text{Vect}(\mathbb{R}^{n}) \)

**Remark 2.** This Lie superalgebra is linked with \( \mathbb{R}^{0|n} \) the Lie superalgebra of vector fields on a purely odd space. More precisely, if one identifies as a space
\[ \Omega^{+1}(\mathbb{R}^n, T\mathbb{R}^n) = \text{Vect}(\mathbb{R}^{0|n}) \otimes C^\infty(\mathbb{R}^n) \]
with completed tensor product, then the Richardson-Nijenhuis bracket reads as follows: for \( K = a \otimes \xi \) and \( L = b \otimes \lambda \) with \( \xi, \lambda \) in \( \text{Vect}(\mathbb{R}^{0|n}) \) and \( a, b \) in \( C^\infty(\mathbb{R}^n) \), one has \([K, L]^\wedge = ab \otimes [\xi, \lambda]\), where \([\xi, \lambda]\) is the bracket of vector fields on the supermanifold \( \mathbb{R}^{0|n} \). So, it can be identified with the super Lie algebra of currents with value in \( \text{Vect}(\mathbb{R}^{0|n}) \).

1.1.3. Frölicher-Nijenhuis algebra. The bracket of \( \mathcal{L}_\theta \) and \( \mathcal{L}_\eta \) is still a derivation, which gives the Frölicher-Nijenhuis bracket by the following formula:
\[ \mathcal{L}_{[[\theta, \eta]]} = [\mathcal{L}_\theta, \mathcal{L}_\eta]. \]
For \( \theta = \alpha \otimes X \) and \( \eta = \beta \otimes Y \) with \( \alpha \in \Omega^k(\mathbb{R}^n) \), \( \beta \in \Omega^\ell(\mathbb{R}^n) \), \( X \) and \( Y \)
in \( \text{Vect}(\mathbb{R}^n) \) one has:
\[ [[\alpha \otimes X, \beta \otimes Y]] = \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge L_X \beta \otimes Y - \beta \wedge \mathcal{L} \alpha. \]
\[ + (-1)^k (d\alpha \wedge i_X \beta \otimes Y + i\mathcal{Y} \alpha \wedge d\beta \otimes X). \]
The space \( \mathfrak{F} = (\bigoplus \Omega^i(\mathbb{R}^n, T\mathbb{R}^n), [[ \ , \ ]] \) is called Frölicher-Nijenhuis algebra. It is a subalgebra of \( \text{Vect}(\mathbb{R}^{n|n}) \) (see [5]). The above formula has been obtained by Michor in [3].

We get, so, for \( \text{Vect}(\mathbb{R}^{n|n}) = \mathcal{R} \oplus \mathfrak{F} \) the following bracket:

**Lemma 3.** [6] For \( K_i \in \Omega^k(\mathbb{R}^n, T\mathbb{R}^n) \) and \( L_i \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n) \) where \( i \in \{1, 2\} \), we have:
\[ [\mathcal{L}_{K_1} + i_{L_1}, \mathcal{L}_{K_2} + i_{L_2}] = \mathcal{L}([[K_1, K_2]] + i_{L_1}(K_2) - (-1)^{k_1k_2} i_{L_2}(K_1)) \]
\[ + i([L_1, L_2]^\wedge + [[K_1, L_2]] - (-1)^{k_1k_2} [[K_2, L_1]]) \]

**Remark 4.** As a consequence of this lemma, for \( K \in \Omega^k(\mathbb{R}^n, T\mathbb{R}^n) \) and \( L \in \Omega^{k+1}(\mathbb{R}^n, T\mathbb{R}^n) \), one has
\[ [\mathcal{L}_K, i_L] = i([[K, L]]) - (-1)^{k_L} i(L) \]
and
\[ [i_L, \mathcal{L}_K] = \mathcal{L}(i_L K) - (-1)^k i([[L, K]]). \]
2. Mains results

With the notations of previous subsection, one has:

Proposition 5. If $n > 2$, the space of cohomology $H^1(\mathcal{F}, \Omega(\mathbb{R}^n))$ is one dimensional and is generated by the 1-cocycle given by:

$$c_1 : \mathcal{F} \rightarrow \Omega(\mathbb{R}^n)$$

$$\omega \otimes X \rightarrow d(i_X \omega)$$

Proposition 6. The space of cohomology $H^1(\mathcal{R}, \Omega(\mathbb{R}^n))$ is one dimensional and is generated by the 1-co cycle given by:

$$c_2 : \mathcal{R} \rightarrow \Omega(\mathbb{R}^n)$$

$$\omega \otimes X \rightarrow (-1)^{|\omega|-1} i_X \omega$$

where $|\omega|$ denotes the degree of $\omega$. □

This result can be deduced from C. Roger and P. Lecomte in [7]. Here we take an other proof and rectify their result.

The cohomology of $\text{Vect}(\mathbb{R}^n|\mathbb{R}^n) = \mathcal{F} + \mathcal{R}$ is given by:

Theorem 7. If $n > 2$, the space of cohomology $H^1(\text{Vect}(\mathbb{R}^n|\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n))$ is generated by the 1-cocycles

$$c : \text{Vect}(\mathbb{R}^n|\mathbb{R}^n) \rightarrow \mathcal{F}(\mathbb{R}^n|\mathbb{R}^n)$$

defined by

$$c(\mathcal{L}_K + i_L) = -c_1(K) + \partial \omega_1(K) + c_2(L)$$

where $\partial \omega_1(K) = \mathcal{L}_K(\omega_1)$, a coboundary on $\mathcal{F}$ with $\omega_1 \in \Omega^0(\mathbb{R}^n)$.

3. Proof of propositions 5 and 6 and theorem 7

Before proving the propositions and the theorem, we shall give some definitions and preliminary results.

3.1. Polynomial notation. (see [1] and [5])

Polynomial notation is very useful to handle computation with differential operators. It allows to apply polynomial computations for operators.

We suppose that $E \rightarrow M$ and $F \rightarrow M$ are vector bundles, with typical fibers $E_0$ and $F_0$, that $\Gamma(E)$ and $\Gamma(F)$ denote their spaces of smooth sections. Then fixing a local chart $(U, x_1, ..., x_n)$, we can identify $\Gamma(E)$ and $\Gamma(F)$ to $C^\infty(U, E_0)$ and $C^\infty(U, F_0)$ respectively. Then a differential operator of order $k$ can be written in following form:

$$f \mapsto \sum_{|\alpha| \leq k} A_\alpha(x) D^\alpha f(x)$$

where $D^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$ denotes partial derivatives with respect to $(x_1, ..., x_n)$, furthermore the mappings $A_\alpha$ is in $C^\infty(U, \mathcal{L}(E_0, F_0))$. Then
the symbolic polynomial associated to $A$ is defined by

$$P(\xi; X)(x) = \sum_{|\alpha| \leq k} A_{\alpha,x}(X)\xi^\alpha.$$  

For example if $X = \sum_i X_i \partial x_i \inVect(\mathbb{R}^n)$, where $\partial x_i = \frac{\partial}{\partial x_i}$, acting on a function $f \in C^\infty(\mathbb{R}^n)$ through the operator of Lie derivative:

$$f \mapsto L_X(f) = \sum_i X_i \frac{\partial f}{\partial x_i}$$

is represented by the polynomial function

$$\sum_i X_i\xi_i f = \langle X, \xi \rangle f.$$  

3.2. Preliminary results. Let

$$c_1 : \mathfrak{F} \longrightarrow \Omega(\mathbb{R}^n)$$

be a cochain, the condition of 1-cocycle applied to $c_1$ reads:

$$c_1([\alpha \otimes X, \beta \otimes Y]) - L_{(\alpha \otimes X)}c_1(\beta \otimes Y) = (-1)^{|\alpha||\beta|+1} L_{(\beta \otimes Y)}c_1(\alpha \otimes Y).$$

Remark that for every $\alpha \otimes X \in \mathfrak{F}$ and $\gamma \in \Omega^q(\mathbb{R}^n)$ one has

$$(2) \quad L_{(\alpha \otimes X)}(\gamma) = i_{(\alpha \otimes X)}d\gamma + (-1)^q d(i_{(\alpha \otimes X)}\gamma)$$

where $i_{(\alpha \otimes X)}\gamma = \alpha \wedge i_X\gamma$.

Lemma 8. If

$$c : \Omega(\mathbb{R}^n) \otimes Vect(\mathbb{R}^n) \longrightarrow \Omega(\mathbb{R}^n)$$

is a 1-cocycle, then $c$ is a differential operator.

Proof. This is a simple adaptation of the result of [8].

Lemma 9. Each cohomology class $[c]$ in $H^1(\mathfrak{F}, \Omega^k(\mathbb{R}^n))$ contains a 1-cocycle with constants coefficients.

Proof. We suppose that $c$ is a 1-cocycle, then its restriction to the Lie subalgebra $Vect(\mathbb{R}^n) \subset \mathfrak{F}$ of vector fields is also a 1-cocycle. The first cohomology space of the Lie algebra of vector fields is generated by ”div” and ”ddiv” so there exist $a, b \in \mathbb{R}$ and $\omega \in \Omega^k(\mathbb{R}^n)$ such that

$$c(X) = a \ \text{div}(X) + b \ \text{ddiv}(X) + \partial_X \omega \ \forall X \in Vect(\mathbb{R}^n).$$

Now, the 1-cocycle $c - \partial \omega$ vanishes on constant vector fields. Here we use the identification of the algebra of vector fields $Vect(\mathbb{R}^n)$ as a subalgebra of $\mathfrak{F}$ and the fact that the restriction of the action $L$ of $\mathfrak{F}$ on forms to Lie algebra $Vect(\mathbb{R}^n)$ coincides with the classical Lie derivative $L$. It follows from the relation of 1-cocycle:

$$(3) \quad L_X(c(K)) = c([X, K]) - L_K(c(X))$$

for $K \in \mathfrak{F}$, that $c$ commutes with the Lie derivative in the direction of constant vector fields:

$$(4) \quad L_X(c(K)) = c([X, K]).$$
A direct computation finishes the proof.

3.3. **Proof of proposition 5.** Since \(\mathfrak{g}\) is a graded Lie algebra and \(\Omega(\mathbb{R}^n)\) is a graded module by the degree of forms, the space of cohomology \(H^1(\mathfrak{g}, \Omega(\mathbb{R}^n))\) is graded, then we have

\[
H^1(\mathfrak{g}, \Omega(\mathbb{R}^n)) = \bigoplus_q H^1(\mathfrak{g}, \Omega(\mathbb{R}^n))_q
\]

where \(H^1(\mathfrak{g}, \Omega(\mathbb{R}^n))_q\) is the space of class of homogeneous cocycle \(c_{1,q}\) of degree \(q\) i.e transforms an argument of degree \(p\) on an argument of degree \(p+q\). The restriction of \(c_{1,q}\) to \(\Omega^p(\mathbb{R}^n) \otimes \text{Vect}(\mathbb{R}^n)\) is noted \(c_{1,p,q} : \Omega^p(\mathbb{R}^n) \otimes \text{Vect}(\mathbb{R}^n) \rightarrow \Omega^{p+q}(\mathbb{R}^n)\).

The condition of 1-cocycle applied to \(c_{1,p,q}\) can be written

\[
(5) \quad c_{1,p,q}(\lbrack [K_1, K_2]\rbrack) - L_{K_1}(c_{1,p,q}(K_2)) - (-1)^{|K_1||K_2|+1} L_{K_2}(c_{1,p,q}(K_1)) = 0
\]

for \(K_1\) and \(K_2\) in \(\mathfrak{g}\). Up to a coboundary, we may suppose that the restriction of \(c\) to \(\text{Vect}(\mathbb{R}^n) \cong \Omega^0(\mathbb{R}^n) \otimes \text{Vect}(\mathbb{R}^n) \subset \mathfrak{g}\) is a combination of "div" and "div". Hence, in equation (5), if we set \(K_1\) to be a linear vector field \(X\), since \(c_{1,p,q}(K_1)\) is constant, we directly obtain the relation

\[
c_{1,p,q}(L_X(K_2)) - L_X(c_{1,p,q}(K_2)) = 0,
\]

where \(L_X\) is the classical Lie derivative in the direction of \(X\). If \(\eta\) denote the derivative affecting \(K\), then one may write the symbolic form \(c_{1,p,q}(\eta, K)\) associated to \(c_{1,p,q}\) (see [5]). For \(X_1, ..., X_{n+q}\) in \(T\mathbb{R}^n\), the polynomial \(c_{1,p,q}(\eta, K)(X_1, ..., X_{n+q})\) is invariant with respect to the action of the algebra \(\text{gl}(n, \mathbb{R})\). The classical result of Weyl (see [1]) states that such invariant polynomials are generated by contractions. Hence one gets that the degree in \(\eta\) (say \(r\)) is equal to \(q+1\). Now, the polynomial \(c_{1,p,q}(\eta, K)(X_1, ..., X_{n+q})\) must be symmetric in \(\eta\) and antisymmetric in \(X_1, ..., X_{n+q}\), so, \(r \in \{0, 1, 2\}\). Hence, as a result of the invariance property, we obtain (where \(a_p, b_p, c_p\) and \(e_p\) are reals numbers):

\[
c_{1,p-1}(\eta, K) = a_p \tau(\eta, K) \text{ where } K = \alpha \otimes X \text{ and } \tau(\eta, K) = i_X \alpha; \\
c_{1,p,0}(\eta, K) = b_p \tau_1(\eta, K) + c_p \tau_2(\eta, K) \text{ where } \tau_1(\eta, K) = \eta \wedge \tau(\eta, K) \text{ and } \tau_2(\eta, K) = \langle K, \eta \rangle; \\
c_{1,p,1}(\eta, K) = d_p \tau_3(\eta, K) \text{ where } \tau_3(\eta, K) = e_p \eta \wedge < K, \eta >.
\]

Thus, we compute the coefficients \(a_p, b_p, c_p\) and \(e_p\) in accordance with the degree \(q\).

- **Case** \(q = -1\)
  In this case, the condition for \(c_{1,p,-1}\) to be a 1-cocycle forces \(a_p\) to be zero for all \(p\), if \(n > 1\).
  - **Case** \(q = 0\)
Take $c_{1,p,0}(\eta, K) = b_p \tau_1(\eta, K) + c_p \tau_2(\eta, K)$ and plug it in equation (5) one can show that $b_p$ is equal to $b_{p'}$ for all $p$ and $p'$ and $c_p$ must be zero for all $p$, if $n > 2$.

• Case $q = 1$

In this case we have

$$\delta \tau_3(\eta, K)(\alpha \wedge 1, X) = e_p \div X \wedge d\alpha$$

where $1 = \sum_{i=1}^n dx^i \otimes \partial x_i$. A straightforward computation shows that if $n > 2$, $e_p$ must be zero for all $p$.

3.4. Proof of proposition 6. Consider the mapping

$$c_2: \mathcal{R} \rightarrow \Omega(\mathbb{R}^n)$$

$$\omega \otimes X \rightarrow (-1)^{|\omega|-1} i_X \omega$$

where $|\omega|$ denotes the degree of $\omega$.

We shall prove that $c_2$ is a 1-cocycle. Then for $L_1 = \alpha \otimes X$, where $|\alpha| = |L_1| = l_1 + 1$ and $|i_{L_1}| = l_1$, and for $L_2 = \beta \otimes Y$ where $|\beta| = |L_2| = l_2 + 1$ and $|i_{L_2}| = l_2$, one has:

$$c_2([\alpha \otimes X, \beta \otimes Y]) - i_{i_{L_1}} c_2(\beta \otimes Y) - (-1)^{l_1 l_2} i_{i_{L_2}} c_2(\alpha \otimes X)$$

$$= c_2(\alpha \wedge i_X \beta \otimes Y + (-1)^{l_1 l_2+1} \beta \wedge i_Y \alpha \otimes X) - \alpha \wedge i_X ((-1)^{l_2} i_Y \beta)$$

$$- (-1)^{l_1 l_2+1} \beta \wedge i_Y ((-1)^{l_2} i_X \alpha)$$

$$= (-1)^{l_1 l_2} i_Y (\alpha \wedge i_X \beta) + (-1)^{l_1 l_2+1+1} i_X (\beta \wedge i_Y \alpha) - (-1)^{l_2} \alpha \wedge i_X i_Y \beta$$

$$- (-1)^{l_1 l_2+1} \beta \wedge i_Y i_X \alpha$$

$$= (-1)^{l_1 l_2} i_Y \alpha \wedge i_X \beta + (-1)^{l_1 l_2+1} \alpha \wedge i_Y i_X \beta - (-1)^{l_1 l_2+1+1} i_X \beta \wedge i_Y \alpha$$

$$- (-1)^{l_1 l_2+1} \beta \wedge i_X i_Y \alpha - (-1)^{l_2} \alpha \wedge i_X i_Y \beta + (-1)^{l_1 l_2+1} \beta \wedge i_Y i_X \alpha$$

$$= 0.$$

To prove that this 1-cocycle is unique we use the same method as in the Proposition 5.

3.5. Proof of theorem 7. Let

$$c: Vect(\mathbb{R}^{n|n}) \rightarrow \mathcal{F}(\mathbb{R}^{n|n})$$

be a 1-cocycle. The restriction of $c$ to the subalgebra $\mathfrak{g}$ (respectively $\mathcal{R}$) is a 1-cocycle over $\mathfrak{g}$ (respectively $\mathcal{R}$). According to propositions 5 and 6 the 1-cocycle $c$ reads

$$c(\mathcal{L}_K + i_L) = c(\mathcal{L}_K) + c(i_L)$$

and

$$\left\{ \begin{array}{l}
    c(\mathcal{L}_K) = a c_1(\mathcal{L}_K) + \partial \omega_1(K), \\
    c(i_L) = b c_2(i_L) + \bar{\partial} \omega_2(L),
\end{array} \right. \quad (6) \quad (7),$$

where $a, b$ are real constants and $\partial \omega_1$ and $\bar{\partial} \omega_2$ are coboundaries of $\mathfrak{g}$ and $\mathcal{R}$ respectively given by $\partial \omega_1(K) = L_K(\omega_1)$ and $\bar{\partial} \omega_2(L) = i_L(\omega_2)$. Besides, since the superalgebras $\mathfrak{g}$ and $\mathcal{R}$ are graded and $\mathcal{F}(\mathbb{R}^{n|n}) \cong \Omega(\mathbb{R}^n)$ is a graded module, too, the terms in the right hand of the
equation (6) (respectively (7)) must have the same degrees. Then we must have in equation (6):
\[ |c_1(L_K)| = |\partial \omega_1(K)|, \]
but \[ |c_1(L_K)| = |c_1(\alpha \otimes X)| = |\alpha| \] where \( K = \alpha \otimes X \in \mathfrak{g} \) and \( |\partial \omega_1(K)| = |L_K(\omega_1)| = |\alpha| + |\omega_1| \) (see equation (2)) then \( |\omega_1| = 0 \) besides \( \omega_1 \in \Omega^0(\mathbb{R}^n) \), moreover we must have in equation (7):
\[ |c_2(i_L)| = |\overline{\partial} \omega_2(L)| \]
but \[ |c_2(i_L)| = |c_2(\beta \otimes Y)| = |i_Y(\beta)| = |\beta| - 1 \] where \( L = \beta \otimes Y \in \mathcal{R} \), then \[ |\overline{\partial} \omega_2(L)| = |i_{\beta \otimes Y}(\omega_2)| = |\beta| + |\omega_2| - 1 \) (see equation (1)), one deduces that \( \omega_2 \in \Omega^0(\mathbb{R}^n) \). Since, \( i_L(\omega_2) = i_{\beta \otimes Y}(\omega_2) = \beta \wedge i_Y(\omega_2) = 0 \), one has \( \overline{\partial} \omega_2(L) = 0 \).

Now, the condition of 1-cocycle applied to \( c \) reads:
\[ \text{bc}_2([[K, L]], [i_L]) - (-1)^{kl} a c_1(i_L(K)) - b L_K(c_2(i_L)) + (-1)^{lk} a i_L(c_1(L_K)) = 0, \]
We use (2), we obtain that the right hand of equation (6) vanish and it becomes
\[ \text{bc}_2([[K, L]], [i_L]) - (-1)^{kl} a c_1(i_L(K)) - b L_K(c_2(i_L)) \]
\[ + (-1)^{lk} a i_L(c_1(L_K)) = 0. \]
Now, if we substitute the expressions of the 1-cocycles \( c_1 \) and \( c_2 \) in equation (7), we show that we must have \( a + b = 0 \). The result follows immediately.

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References
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