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The mathematical achievements of Eugène Catalan

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Résumé

Eugène Catalan a travaillé sur de nombreux sujets, qui relèvent maintenant de domaines aussi variés que la théorie des nombres, la combinatoire, la géométrie, l'analyse, le calcul intégral, l'algèbre, la mécanique, etc. Après une présentation globale, nous nous focaliserons sur certains de ses résultats pour comprendre comment Catalan intervenait dans la science de son temps et pour décrire les caractéristiques de sa pratique des mathématiques. Nous mettrons en particulier en lumière sa réactivité constante aux travaux des autres mathématiciens, qu'ils soient ou non célèbres, et son utilisation de l'analyse pour découvrir des ponts entre différents problèmes et transférer des idées d'un champ mathématique à un autre.

Mots-clefs : Catalan, histoire, théorie des nombres, séries, surfaces minimales

Abstract

Eugène Catalan worked on a variety of topics, now classified as falling under the headings of number theory, combinatorics, geometry, integral calculus, analysis, algebra, mechanics, etc. After a global survey, we shall focus on some of them, first of all to show how Catalan intervened in the mathematics of his own time, then to trace some characteristics of his mathematical practice. We shall display in particular his constant reactivity to the work of others, some famous, some not, and his use of analysis to locate bridges between problems and transfer ideas from one mathematical field to another.

Keywords: Catalan, history, number theory, series, minimal surfaces

MSC2010: 01A55, 11-03, 40-03, 52-03, 53-03

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Eugène Catalan published more than 380 mathematical articles on a variety of topics, from number theory to geometry to mechanics and probability. Many of these articles are very short, some contain simple conjectures, or "empirical theorems," as Catalan called them, and at least half of them were published in journals for teachers or amateurs, such as *Nouvelles Annales* or the Proceedings of the conferences of the "Association française pour l'avancement des sciences"; yet, these are neither the least creative nor the least celebrated. His earliest known paper was written at the beginning of the 1830s, before he was even a student at Polytechnique; his last one was published in 1894, the very year of this death. There exist a Catalan conjecture, Catalan polyhedra, Catalan surfaces, a Catalan number and (unrelated) Catalan numbers, etc. "À défaut d'un sonnet valant un long poème, Je produis, chaque jour, un nouveau théorème", he wrote, quite to the point, in 1885.[1] Such a variegated abundance defies synthesis; numerous results seem elementary but, when looked at closer range, reveal links, often implicit, with the work of contemporary mathematicians or a potential fruitfulness only revealed when revisited through later developments. The present article will present briefly a sampler of Catalan's work in order to display more concretely its variety and to show how Catalan intervened in the mathematics of his time, then it will try to trace some characteristics of his mathematical practice.

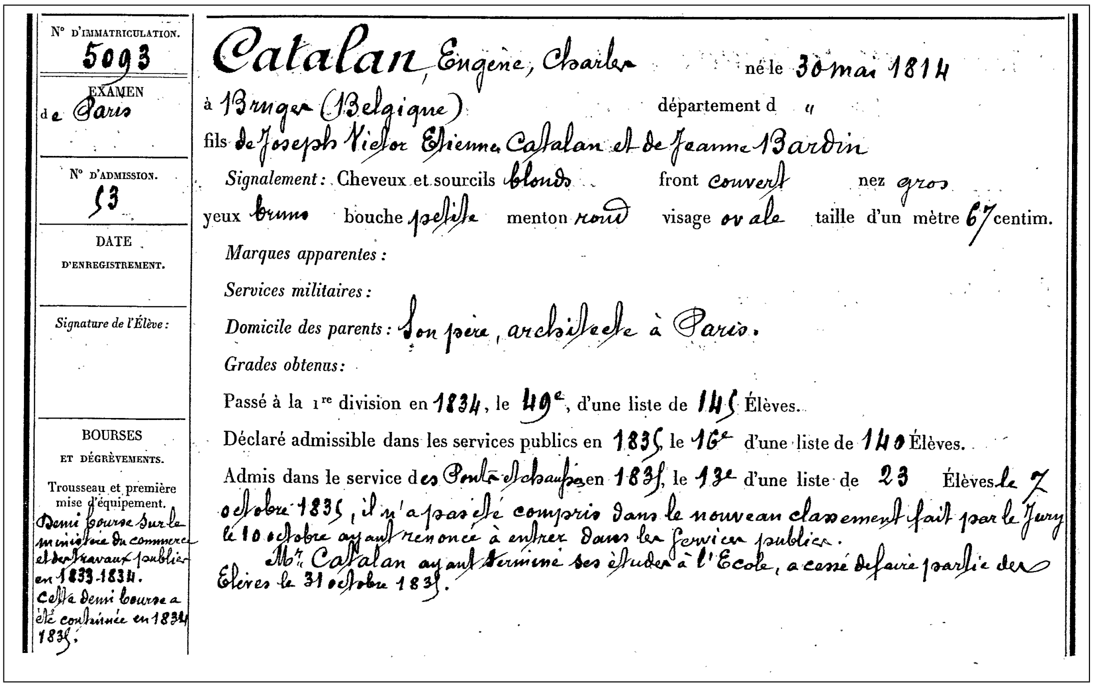
1 Catalan's life in a nutshell

Eugène Catalan was born an illegitimate child, Eugène Charles Bardin, in Bruges on May 30, 1814 and became Catalan only in 1821 after the marriage of his mother.[2] A year later, the family left for Lille, then for Paris; there, Catalan studied (and soon taught) perspective, construction, drawing and applied mathematics, while demonstrating an early interest in politics, his other lasting passion along with mathematics. In 1833, he passed the entrance examination at the École polytechnique; this introduced him to, among others, the professors Gabriel Lamé and Joseph Liouville, and despite some political vicissitudes, would have finally led him to a career of Ponts et Chaussées engineer, had he not decided to quit this path in favour of teaching positions, first in Châlons sur Marne, then in Paris.

The following decade saw a series of promising successes for Catalan: his marriage in 1836, his first mathematical papers, a job as *répétiteur* at Polytechnique in

[1] "For want of a sonnet worth a long poem, I produce each day a novel theorem." Quoted in [14, p. 110].

[2] This summary relies on the extensive biography by François Jongmans, [14] to which the reader is of course directed. Jongmans, in particular, depicts convincingly the modest situation of the family lying behind the somewhat misleading profession of the father which is given in most official documents as "architect". Complements to Catalan's biography can be found in [20], as well as in the very interesting communication by Jan Vandersmissen [19].



N° d'immatriculation. 5093

Examen de Paris

N° d'admission. 53

Date d'enregistrement.

Signature de l'Élève :

Bourses et dégrèvements.

Trousseau et première mise d'équipement.

Demi bourse sur le ministère du commerce et des travaux publics en 1833-1834. Cette demi bourse a été continuée en 1834 1835.

Catalan, Eugène, Charles né le 30 mai 1814

à Bruges (Belgique) département d "

fils de Joseph Victor Étienne Catalan et de Jeanne Bardin

Signalement : Cheveux et sourcils blonds front couvert nez gros

yeux bruns bouche petite menton rond visage ovale taille d'un mètre 67 centim.

Marques apparentes :

Services militaires :

Domicile des parents : son père, architecte à Paris.

Grades obtenus :

Passé à la 1re division en 1834, le 49e, d'une liste de 145 Élèves.

Déclaré admissible dans les services publics en 1835 le 16e d'une liste de 140 Élèves.

Admis dans le service des Ponts et chaussées en 1835, le 13e d'une liste de 23 Élèves. le 7 octobre 1835, il n'a pas été compris dans le nouveau classement fait par le Jury le 10 octobre ayant renoncé à entrer dans les services publics.

Mr Catalan ayant terminé ses études à l'École, a cessé de faire partie des Élèves le 31 octobre 1835.

Figure 1: Catalan's record at Polytechnique. Reproduced with the kind authorization of the Archives de l'École polytechnique (Palaiseau, France).

1838, a mathematical prize from the Brussels Academy in 1840, and, in the same year, his election to the Société philomatique, a melting pot for scientific stars of different generations and usually an antechamber to the French Academy of Sciences. One year later, Catalan defended a thesis in mathematical mechanics at the Faculty of Sciences in Paris, and in 1846 he obtained the first position in the difficult competitive examination for the *Agrégation de mathématiques*.

However, politics took its toll: Catalan was first pressured to quit Polytechnique, then after refusing to take the compulsory oath to Napoléon III in 1852, he was barred from all state positions. For more than a decade, he made a living by teaching in private institutions, until, in 1865, he was at last offered a professorship at the University of Liège. In 1874, he created with his friend Paul Mansion, professor at the University of Ghent an ephemeral mathematical journal, the *Nouvelle Correspondance mathématique*. Testimonies of Catalan's international recognition can be found in his election to numerous prestigious learned societies, the Société royale des sciences de Liège and the Académie royale de Belgique, the Academy of the Nuovi Lincei, the Imperial Academy of Saint-Petersburg, and others, a notable exception being the French Academy. Catalan became emeritus in 1884 and died ten years later, "a Republican, socialist and anti-catholic", says his will, [14, p. 113].

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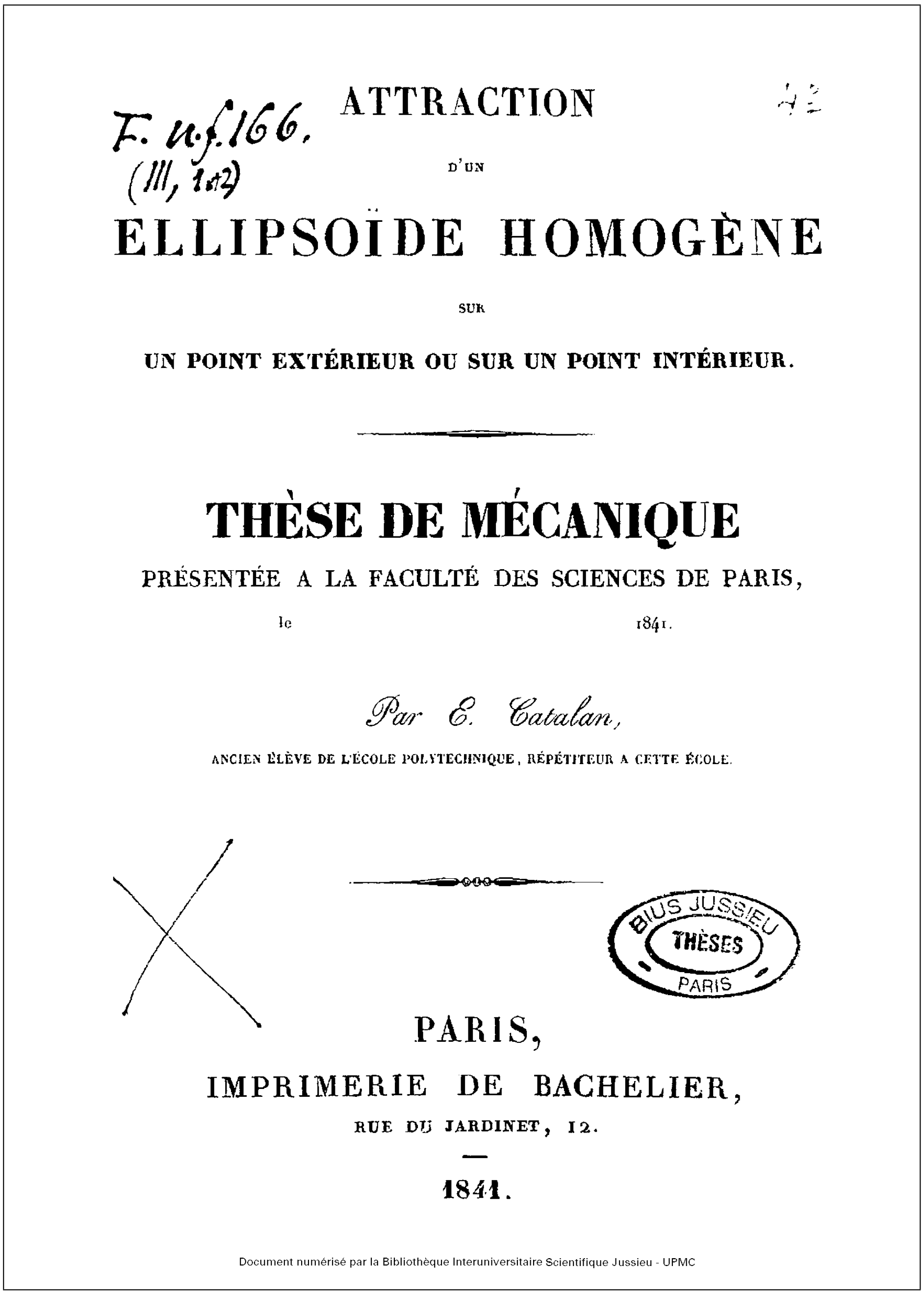


ATTRACTION

D'UN

ELLIPSOÏDE HOMOGÈNE

SUR

UN POINT EXTÉRIEUR OU SUR UN POINT INTÉRIEUR.

THÈSE DE MÉCANIQUE

PRÉSENTÉE A LA FACULTÉ DES SCIENCES DE PARIS,

le 1841.

Par E. Catalan,

ANCIEN ÉLÈVE DE L'ÉCOLE POLYTECHNIQUE, RÉPÉTITEUR A CETTE ÉCOLE.

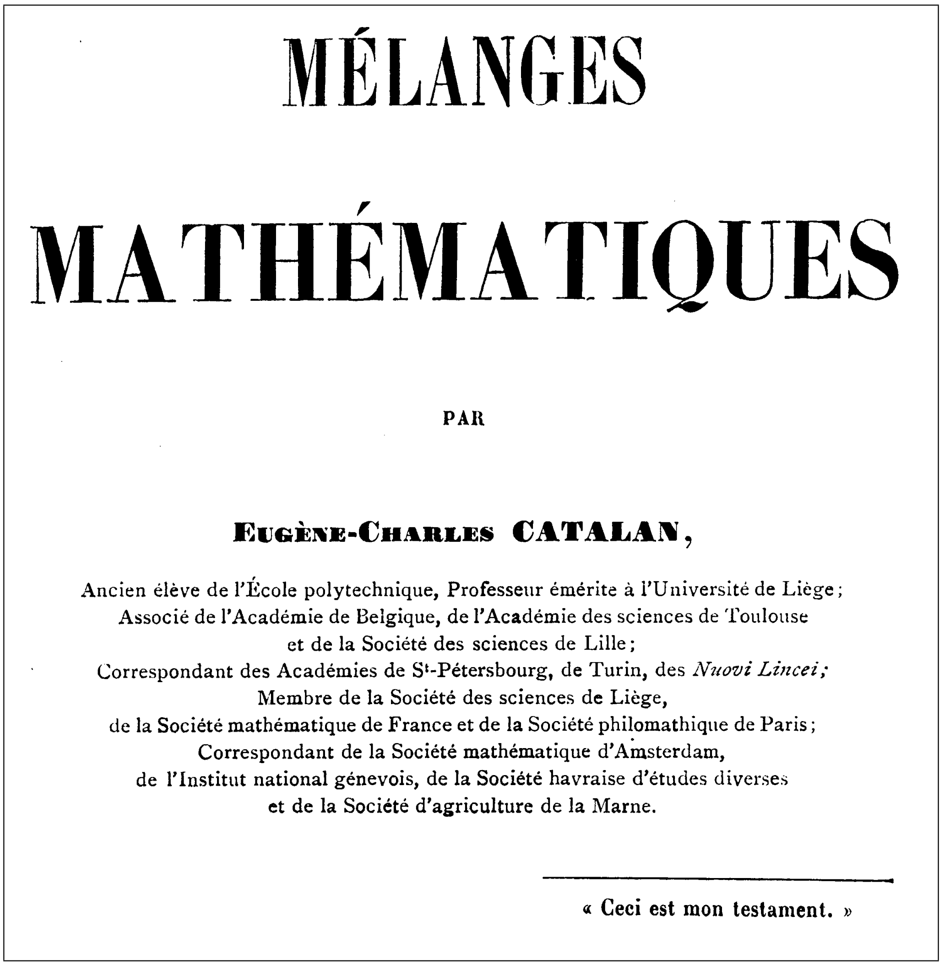
PARIS,

IMPRIMERIE DE BACHELIER,

RUE DU JARDINET, 12.

1841.

Figure 2: Title page of Catalan's thesis. Université Pierre et Marie Curie (UPMC), Bibliothèque universitaire Pierre et Marie Curie (BUPMC) (Paris, France).



MÉLANGES

MATHÉMATIQUES

PAR

EUGÈNE-CHARLES CATALAN,

Ancien élève de l'École polytechnique, Professeur émérite à l'Université de Liège;
Associé de l'Académie de Belgique, de l'Académie des sciences de Toulouse
et de la Société des sciences de Lille;
Correspondant des Académies de St-Pétersbourg, de Turin, des *Nuovi Lincei;*
Membre de la Société des sciences de Liège,
de la Société mathématique de France et de la Société philomathique de Paris;
Correspondant de la Société mathématique d'Amsterdam,
de l'Institut national génevois, de la Société havraise d'études diverses
et de la Société d'agriculture de la Marne.

« Ceci est mon testament. »

Figure 3: Catalan's three volumes of *Mélanges*: an epitome of his mathematical versatility.

2 Catalan's mathematics: a summary

As explained in the introduction, Catalan's mathematical production is profuse. The fifty main memoirs listed in his 1859 report[3] have blossomed, thirty years later (as seen at the end of the third volume of Catalan's *Mélanges* in 1888), into 383 articles and about 15 textbooks, to which should be added numerous (published) reports on the work of others, and of course, the three volumes of mathematical short miscellanea, the *Mélanges* themselves. Half of the articles were published in what Eduardo Ortiz has called "intermediary journals," that is, journals intended

[3]This handwritten report was found by Norbert Verdier in Catalan's file in the French National Archives, LH 448/55. A page is reproduced in [20].

for mathematics students and teachers, which flourished in the second half of the nineteenth century, like the *Nouvelles Annales de mathématiques* and Catalan's own *Nouvelle Correspondance mathématique*; but Catalan also wrote papers in the most prestigious research journals of his time: *Journal de l'École polytechnique*, *Journal de mathématiques pures et appliquées*, *Comptes rendus de l'Académie des sciences*, *Journal für die reine und angewandte Mathematik*, *Mémoires et Bulletins de l'Académie de Belgique*, *Annali di matematica pura ed applicata*, etc. The *Jahrbuch über die Fortschritte der Mathematik*, the ancestor of MathSciNet created in 1868, reviewed, from this date on, 224 mathematical research papers by Catalan; they were classified under such headings as number theory, series, special functions, integrals, elementary geometry, analytic geometry, theory of surfaces, orthogonal polynomials, algebraic curves, probability theory, differential equations, mechanics, astronomy and even history.

Such an abundance and variety appear also at more local levels. In number theory, for instance, Catalan worked on Diophantine equations, on prime numbers, on questions linked to the divisibility of integers or their decompositions as sums of squares or other exact powers [1].

Catalan's probably most famous statement belongs to this domain: it says that 8 and 9 are the only consecutive powers greater than 1. First published in 1842 as a simple query among others in the first issue of the *Nouvelles Annales*, then reproduced two years later in the international and celebrated *Journal für die reine und angewandte Mathematik*, the statement has a long history of partial proofs, false proofs, numerical computations, changes of frame, generalizations, until its (first) complete proof by Preda Mihăilescu in 2002 (published, as is only proper, in the *Journal für die reine und angewandte Mathematik* two years later...).

Although its status changed during the nineteenth century—first through the reception of Gauss's highly theoretical *Disquisitiones arithmeticae*, then through new connections with higher analysis and structural algebra—number theory still had during the last decades of the nineteenth century a following among mathematicians keen on such elementary, sometimes fruitful, statements;[4] Catalan's arithmetical work figures prominently in their research, as witnessed for instance by Édouard Lucas's *Récréations mathématiques* and *Théorie des nombres*. He himself was keen on what he called "empirical theorems". In the *Bulletin de la Société mathématique de France* of 1888, he discussed several of them: one, reproduced from the *Nouvelle Correspondance mathématique* states that the quantity $6n^2 + 6n - 3$, for all integer n, is the sum of three squares of integers; another considers, for each integer n, the sequence of integers, n_1, n_2, $\ldots$, such that n_1 is the sum of proper divisors of n, n_2 is the sum of proper divisors of $n - 1$, etc, and states that this sequence has a limit, which is either 1 or a perfect number (a number which is equal to the sum of

[4]On this group, see [12, 8, 9].

48. ***Théorème.*** **Deux nombres entiers consécutifs, autres que 8 et 9, ne peuvent être des puissances exactes. (Catalan.)**

49. Combien l'équation transcendante $2(1-\cos x)=x\sin x$, admet-elle de racines réelles positives?

50. Une corde étant inscrite dans une parabole, le produit des distances des extrémités de cette corde, à un diamètre quelconque est égal à la partie de ce diamètre interceptée entre la courbe et la corde, multipliée par le paramètre de l'axe principal.

Figure 4: Catalan's conjecture in the first issue of *Nouvelles Annales*, in an environment of elementary problems for students.

its proper divisors, like 6=1+2+3). This last conjecture is almost obviously false: when m and n are two amicable numbers (numbers such that each is the sum of the proper divisors of the other, like 220 and 284), the sequence n_i takes successively the two values m and n. Cycles of more than two terms also exist. A refined form was thus proposed by Leonard Dickson in 1913, stating that the sequence (n_k) is at least bounded: this, now called the Catalan-Dickson conjecture, is still open as of 2014…

However, Catalan's number-theoretical involvement was not limited to elementary techniques and problems. In his "Recherches sur quelques produits indéfinis," presented in 1871 to the Royal Academy of Belgium, Catalan proves formulas relative to the periods of elliptic functions, in the spirit of Jacobi. For instance, if one denotes by K and K' the complete elliptic integrals associated respectively to the modulus k and its complementary k' (for k a real number, $0 < k < 1$ and $k^2 + k'^2 = 1$), that is

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \quad K' = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}}$$

Catalan shows, [5, pp. 99–101], that, for $q = e^{-\pi\frac{K'}{K}}$,

$$\frac{q^8}{1-q^{16}} + 2^3\frac{q^{16}}{1-q^{32}} + 3^3\frac{q^{24}}{1-q^{48}} + 4^3\frac{q^{32}}{1-q^{64}} + \cdots = (q+q^9+q^{25}+\cdots)^8,$$

as well as

$$\left[\frac{q}{1-q^2}+3\frac{q^3}{1-q^6}+5\frac{q^5}{1-q^{10}}+7\frac{q^7}{1-q^{14}}+\cdots\right]^2$$
$$=\frac{q^2}{1-q^4}+2^3\frac{q^4}{1-q^8}+3^3\frac{q^6}{1-q^{12}}+\cdots.$$

He deduces from such relations a number of theorems on the decomposition of integers into sums of squares, here, for instance, that any multiple of 8, say $8n$, is the sum of 8 odd squares, and that the number of its decompositions, $8n = i_1^2 + i_2^2 + \cdots + i_8^2$, is equal to the sum of the cubes of the divisors d of n such that $n = di$, with i odd. Let us take for example $8n = 48 = 8 \cdot 6$. There are two factorizations, $n = 6$ as di, with odd i, either $6 = 6 \cdot 1$ or $6 = 2 \cdot 3$., Thus 48 has $6^3 + 2^3 = 224$ decompositions as a sum of eight odd squares; these decompositions are not necessarily all distinct, here they are provided by the permutations in the decompositions $48 = 25+9+9+1+1+1+1+1 = 9+9+9+9+9+1+1+1$.

Catalan's dexterity with formulas, in particular those related to series, was sometimes described as matching that of Euler himself. Catalan gave for instance an integral form for the Euler constant,

$$\gamma =: \lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n) = 1-\int_0^1 \frac{dx}{1+x}(x^2+x^4+x^8+\cdots),$$

and studied extensively what is now called the Catalan constant,

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots,$$

providing several identities useful for its effective computation. He also contributed to the theory of orthogonal polynomials or spherical functions, among others Legendre polynomials. For $n \geq 1$ an integer, these polynomials X_n of degree n are defined by:

$$X_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

They satisfy the differential equation

$$(1-x^2)f''(x) - 2xf'(x) + n(n+1)f(x) = 0,$$

and constitute a family of orthogonal polynomials, that is

$$\int_{-1}^1 X_n(x)X_m(x)dx = 0$$

iff $n \neq m$.

Here, again, Catalan's analytic research joins up with that of many of his contemporaries (sometimes up to the point of sparking priority issues): from his former student Charles Hermite to Pafnuti Tchebychev, Carl Neumann and Eduard Heine. Paul Mansion, who reviewed for the *Jahrbuch* Catalan's second memoir on the polynomials X_n, published in 1882, explained in a characteristic way (JFM 14.0429.02): "[This work] contains more than 200 formulas. A presentation of its content is thus impossible. Especially worth mentioning [...] :

$$\int_{-1}^{+1} \frac{\log(1+x)dx}{(1-2zx+z^2)^{\frac{3}{2}}} = \frac{2}{1-z^2}[\log 2 - (1-z)\log(1+z)]$$

$$\frac{1}{\sqrt{1-x^2}} = \Sigma X_n x^n,$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-z^2\sin^2\theta}} = \frac{1}{2}\sum_0^\infty z^{2n} \int_{-1}^{+1} \frac{X_{2n}dx}{\sqrt{1-x^2}}$$

$$\arcsin x = \frac{\pi}{2}\sum_0^\infty \left[\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\right]^2 (X_{2n+1} - X_{2n-1}).$$

According to a communication from E. Heine to the reviewer, several of these results are not only new, but also important for the theory of Legendre's polynomials [and related functions]."

Besides such explicit relations with a theoretical scope, on orthogonal polynomials, or on, for instance, Bernoulli numbers, Catalan also addressed issues of actual computations of definite integrals, series, etc. His work is thus very representative of mid-nineteenth century priorities: on one hand, explicit, concrete formulas intended to ease the application of analysis to natural sciences, in particular mechanics, and, on the other, a renewed attention to the foundation of the domain around issues of convergence for instance, mainly expressed in the framework of textbooks, such as his *Théorie élémentaire des séries* (1860), but also in short notes in journals for teachers or students.[5]

Catalan for instance, published and publicized convergence techniques due to an engineer, Leclert, who had studied Catalan's treatise on series. Leclert's idea was the following: if u_n is a convergent series, with a sum s, and α_n a sequence of real numbers such that

(1) $\alpha_n u_n$ converges to 0 and
(2) $a_n = \alpha_n - \alpha_{n+1}\frac{u_n}{u_{n+1}}$ converges to $A \neq 0$,

[5]A typical example is his letter to the editor of the *Nouvelles Annales* in which he explained why trigonometric series like $\cos a + \cos 2a + \cdots + \cos nx + \cdots$ do not converge, [4]. This concern for rigour, of which we approve nowadays, has of course its limits, well displayed in Catalan's life-long irony against the "partisans of divergent series," of which we do not.

then the sum s' of the "derived" series $u'_n = (1 - \frac{a_n}{A})u_n$ satisfies $s' = s - \frac{\alpha_1}{A}u_1$ and the sum s'' of the series $u''_n = (\frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n})\alpha_{n+1}u_{n+1}$ satisfies $s'' = s - \frac{\alpha u_1}{a_1}$.

Catalan remarks that the u'_n or the u''_n can be chosen "much smaller" than the u_n, thus accelerating the convergence of the series and the computation of the sum s, which is easily deduced from that of s' or s''. He then applies the procedure to a variety of examples. The Leibniz series for the computation of π,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

converges notoriously slowly. Catalan applies Leclert's idea to $u_n = \dfrac{(-1)^{n-1}}{2n-1}$ and first $\alpha_n = 1$, which thus provides $a_n = \frac{4n}{2n+1}$, $A = 2$, and $u'_n = \dfrac{(-1)^{n-1}}{4n^2-1}$.

He then iterates the procedure. Finally

$$\frac{\pi}{4} = \frac{1}{2} + \sum_1^\infty \frac{(-1)^{n-1}}{4n^2-1} = \cdots = \frac{4}{3} + 24\sum_1^\infty \frac{(-1)^{n-1}}{(4n^2-1)^2(4n^2-9)}$$

this last expression giving in only 4 computations $\pi = 3.141510\ldots$.

Another example is that of Catalan's constant itself:

$$G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

where thus $u_n = \dfrac{(-1)^{n-1}}{(2n-1)^2}$.

Catalan chooses $\alpha_n = \frac{2n-1}{2n-3}$, thus $a_n = 2\frac{(2n-1)^2}{(2n+1)(2n-3)}$, $A = 2$, etc., and finds, after iteration:

$$\begin{aligned} G &= \frac{5}{6} + 4\sum_1^\infty \frac{(-1)^{n-1}}{(2n-1)(2n+1)^2(2n+3)} = \cdots \\ &= \frac{2909}{3150} - 768\sum_1^\infty \frac{(-1)^{n-1}}{(2n-1)(2n+1)^2(2n+3)^2(2n+5)^2(2n+7)}. \end{aligned}$$

The last expression gives $G = 0.9156955941\ldots$ with only 16 terms (instead of some 50000 with the original expression).

Analysis was a leading domain of the French-speaking mathematical world [10], and was then perceived as a key to many topics, including geometrical problems; such applications would soon flourish in the form of differential geometry. Catalan made for instance an important contribution to the theory of minimal surfaces. Originally, in the eighteenth century, minimal surfaces had been introduced in the search for surfaces $z = f(x, y)$ with the smallest area which have a given curve as a boundary. A necessary condition is that

$$\frac{\partial}{\partial x}\left(\frac{z_x}{\sqrt{1 + z_x^2 + z_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{z_y}{\sqrt{1 + z_x^2 + z_y^2}}\right) = 0$$

or

$$(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy} = 0,$$

while, by Catalan's times, minimal surfaces were simply surfaces such that at each point "the two principal curvatures are equal and of opposite sign," that is, such that the mean curvature is zero.

Quite early in his career, in 1842–43, Catalan showed that, besides the plane, the only minimal surfaces which are ruled (i.e., generated by a straight line) are helicoidal conoids; for a suitable choice of coordinates, their parametrization is $x = u \cos v, y = u \sin v, z = hv$ (with positive constant h).

$$(26) \quad x - az = \frac{C}{m^2\beta^3}\left[g(\alpha^2+\beta^2)+h\alpha-(g\alpha+h)a\right] \text{arc tang} \frac{a-\alpha}{\beta} + Da + E,$$

$$(27) \quad y - gx - hz = -\frac{C}{m\beta} \text{arc tang} \frac{a-\alpha}{\beta} + F.$$

Figure 5: Catalan's equations for minimal ruled surfaces, *Journal de mathématiques pures et appliquées* 7, 1842, 211.

He obtained examples of algebraic minimal surfaces and those surfaces, now called Catalan minimal surfaces, which contain cycloids as geodesics. He also provided equations of a family of minimal surfaces deforming the catenoid (the only surface of revolution, besides the plane, to be minimal and the first minimal surface to be identified as such) into the helicoid.[6]

However, in a now familiar pattern, Catalan's geometrical achievements were not limited to differential geometry. In the 1860s, Catalan turned to the classification and construction of semi-regular polyhedra. These are of two types:[7] those whose angles are equal and whose faces are regular polygons, such that faces with the

[6]Numerous graphic representations of these surfaces can now be found on the Internet; for instance, `http://commons.wikimedia.org/wiki/File:Helicoid.svg` for the helicoid by Krishnavedala, or the file on "Catalan's Minimal Surfaces" by Anders Sandberg on `http://commons.wikimedia.org/`. For a featured animation of an isometric deformation between catenoids and helicoids, by Wickerprints, see Wikimedia Commons, `http://commons.wikimedia.org/wiki/File:Helicatenoid.gif`.

[7]The definition of semi-regular polyhedra varies according to authors. We reproduce Catalan's here.

same number of sides are equal; or those whose faces are equal and whose angles are regular, and even equal of they belong to the same number of faces. For instance, the truncated icosahedron, which is the point of departure for a standard soccer ball, has 60 vertices and 32 faces which are either pentagons or hexagons. As is the case for regular polyhedra, the basis of the study of the semi-regular polyhedra is the Euler relation, $V - F + E = 2$, where V, F, and E denote respectively the number of vertices, faces and edges of the polyhedron. Catalan discovered the Catalan polyhedra, which are the duals of the Archimedean polyhedra and explicitly determined all their metric relations.

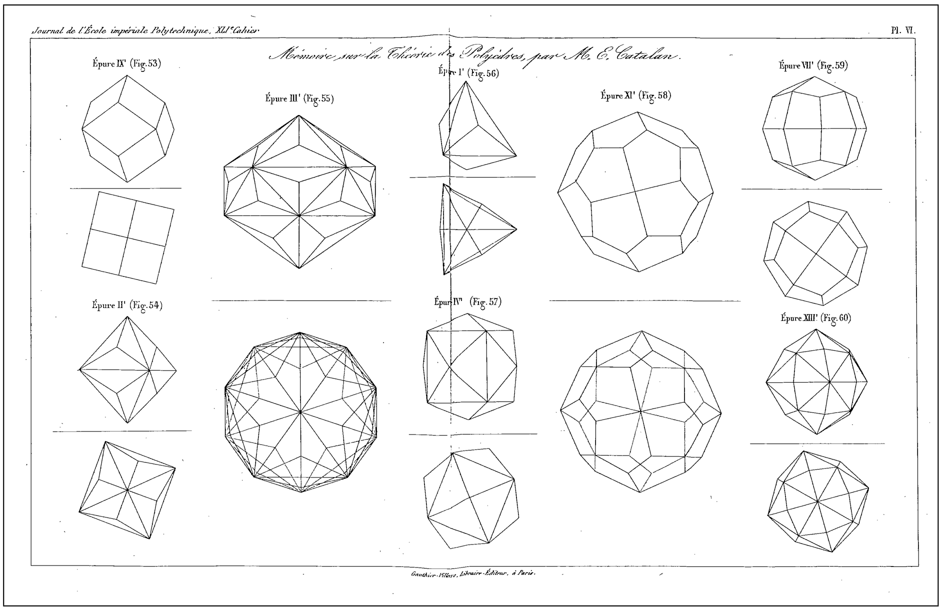


Figure 6: A plate from Catalan's study of semi-regular polyhedra, *Journal de l'École polytechnique* 24, 1865. Figure 55 displays a triakis icosahedron (in Catalan's terminology, hexecontaedra with triangular faces), with 60 faces and 32 vertices, 20 angles belonging to 3 faces and 12 belonging to 10 faces.

Many other results would have to be mentioned to do justice to Catalan's achievements, from his mathematical criticism of the legal distribution of heritage with respect to illegitimate children—a topic which resonated both with his own childhood and his political engagement—to his anticipation of Markov processes, when he discussed the probability expectation of a value as independent of any knowledge of past events. But underneath this profuse variety, verging on dispersion, constant and regular characteristics of Catalan's mathematical practice can be detected.

3 Catalan's mathematical practice

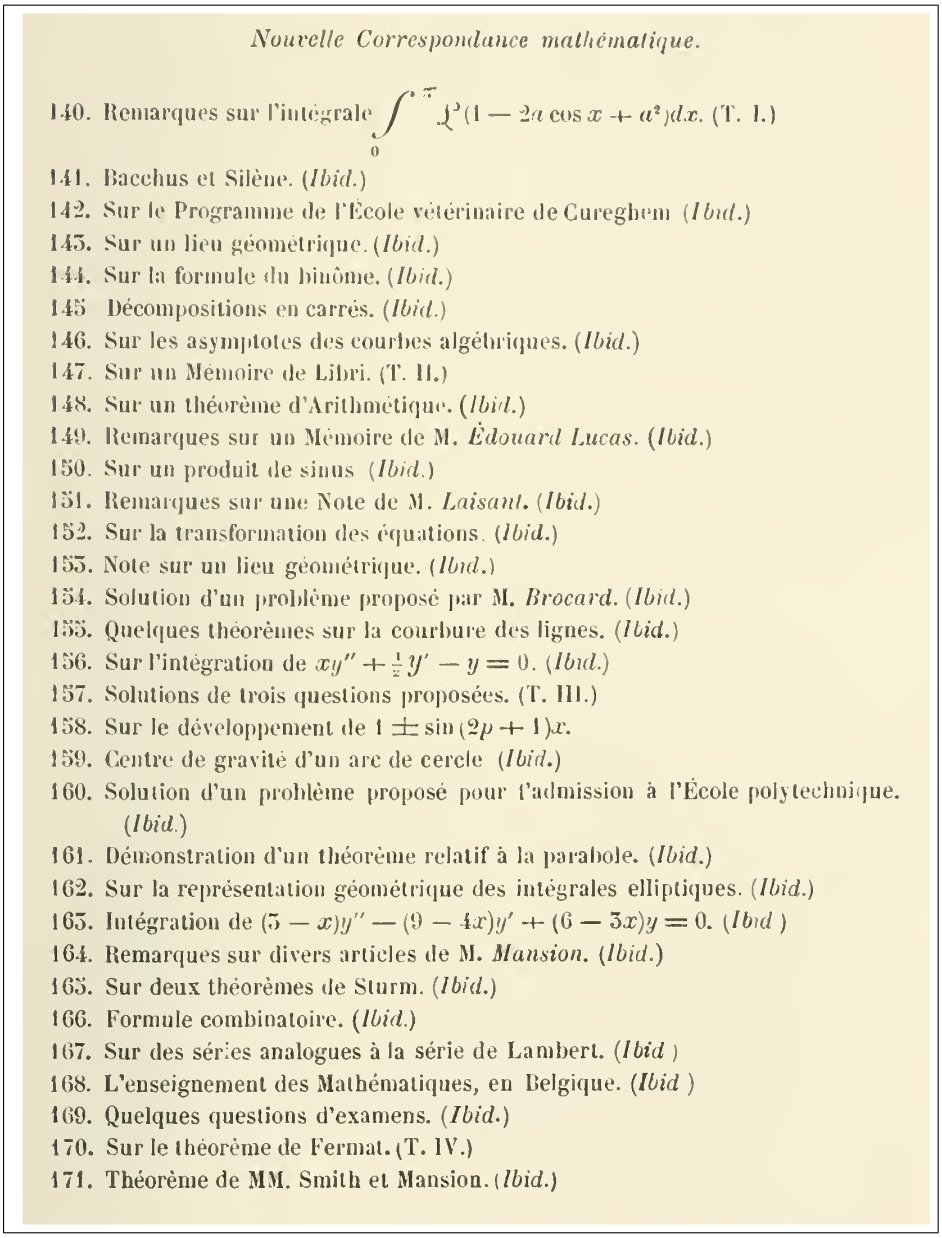
3.1 Reactivity and interference

Catalan's versatility is backed by one remarkable feature: his reactivity to external influences. His profile is not that of the lonely scientist who prefers to focus on his own project and material. Catalan interacted constantly with others, in a variety of ways. First of all, he responded to institutional stimuli: one of his earliest important memoirs, on the transformation of variables in multiple integrals, [2], answered a request by the Brussels Academy of Sciences for a memoir on "algebraic analysis, the topic of which is left to the choice of the competitors," and won the Golden Medal of the Academy in 1840. His work on polyhedra, discussed above, attempted to solve the problem proposed by the French Academy of Sciences for its 1861 Grand Prix de mathématiques: "To perfect in some important point the geometrical theory of polyhedra."[8] Less prestigious, but numerous, Catalan's feedback to school programmes and examinations also bears testimony to his involvement with outside challenges: to illustrate this, one can mention his new method—which may be "of interest to pupils, because of its simplicity,"—published in the *Nouvelles Annales* for computing the sum of the p-powers of the n first natural numbers (vol. 15, p. 231), or, in the third volume of the *Nouvelle Correspondance mathématique*, his derivation of the equation of a locus of points constructed from normals to a hyperbola and required from the 1876 candidates for the Polytechnique (vol. 3, pp. 27-29), as well as, in another register, his ironical comment on the mathematical entrance examinations for the Berlin military school: "Roots of the equation $(x-1)(x-2)(x-3)=0$. Time: 1/2 hour. But what could the candidates possibly be doing during this half hour?"

However, the main source of Catalan's inspiration is the work of other mathematicians, close to him or far away, famous or unknown, right or wrong. A simple look at the titles of his articles in the *Nouvelle Correspondance mathématique* offers abundant evidence, see document 3.1: "Remarks on a memoir by M. Edouard Lucas," "Remarks on a note by M. Laisant," "Remarks on various articles by M. Mansion," testify how Catalan's mathematics is entwined with that of others. Besides the engineer Leclert, already mentioned, one can also refer to both Liouville's and Heinrich Scherk's results on minimal surfaces, to the erroneous paper by a student at Polytechnique and the future Ponts-et-Chaussées engineer, Jules Bénard, which launched Catalan on probability questions[9], or Charles Hermite's properties of binomial coefficients, of which Catalan offered a new proof in the wake of his involvement with Bernoulli numbers and the Staudt theorem.

[8] The announcement of this prize can be read in the *Comptes rendus* of the Academy for the year 1858, vol. 46, p. 301. On Catalan's rather unfair failure to obtain the prize, see [14].

[9] On Bénard's paper, see [7, p. 348] and [15].

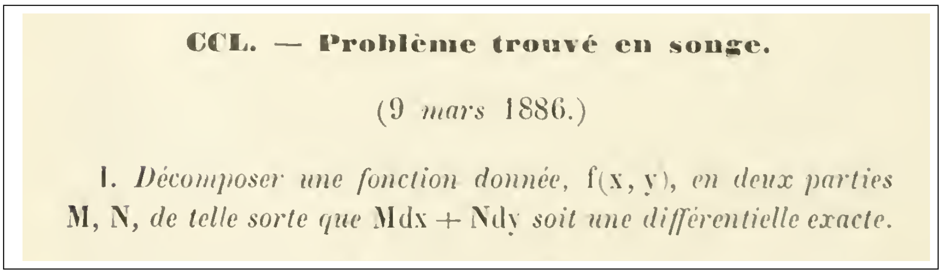


Nouvelle Correspondance mathématique.

140. Remarques sur l'intégrale $\int_0^{\pi} \mathcal{L}(1 - 2a\cos x + a^2)dx$. (T. I.)
141. Bacchus et Silène. (*Ibid.*)
142. Sur le Programme de l'École vétérinaire de Cureghem (*Ibid.*)
143. Sur un lieu géométrique. (*Ibid.*)
144. Sur la formule du binôme. (*Ibid.*)
145 Décompositions en carrés. (*Ibid.*)
146. Sur les asymptotes des courbes algébriques. (*Ibid.*)
147. Sur un Mémoire de Libri. (T. II.)
148. Sur un théorème d'Arithmétique. (*Ibid.*)
149. Remarques sur un Mémoire de M. *Édouard Lucas*. (*Ibid.*)
150. Sur un produit de sinus (*Ibid.*)
151. Remarques sur une Note de M. *Laisant*. (*Ibid.*)
152. Sur la transformation des équations. (*Ibid.*)
153. Note sur un lieu géométrique. (*Ibid.*)
154. Solution d'un problème proposé par M. *Brocard*. (*Ibid.*)
155. Quelques théorèmes sur la courbure des lignes. (*Ibid.*)
156. Sur l'intégration de $xy'' + \frac{1}{2}y' - y = 0$. (*Ibid.*)
157. Solutions de trois questions proposées. (T. III.)
158. Sur le développement de $1 \pm \sin(2p+1)x$.
159. Centre de gravité d'un arc de cercle (*Ibid.*)
160. Solution d'un problème proposé pour l'admission à l'École polytechnique. (*Ibid.*)
161. Démonstration d'un théorème relatif à la parabole. (*Ibid.*)
162. Sur la représentation géométrique des intégrales elliptiques. (*Ibid.*)
163. Intégration de $(3 - x)y'' - (9 - 4x)y' + (6 - 3x)y = 0$. (*Ibid.*)
164. Remarques sur divers articles de M. *Mansion*. (*Ibid.*)
165. Sur deux théorèmes de Sturm. (*Ibid.*)
166. Formule combinatoire. (*Ibid.*)
167. Sur des séries analogues à la série de Lambert. (*Ibid.*)
168. L'enseignement des Mathématiques, en Belgique. (*Ibid.*)
169. Quelques questions d'examens. (*Ibid.*)
170. Sur le théorème de Fermat. (T. IV.)
171. Théorème de MM. Smith et Mansion. (*Ibid.*)

Figure 7: Extract of a list of Catalan's articles: here a third of the titles directly refer to the work of another mathematician, dead (Fermat, Lambert, Libri, Sturm) or alive (Brocard, Laisant, Lucas, Mansion, Smith).

Such interactions were supported by an intense correspondence; according to the habits of the time, several of them appear as articles in mathematical journals.[10] Mathematical exchanges paralleled institutional ones: Baldassare Boncompagni and Victor Bouniakowski became corresponding members of the Liège Society of Sciences thanks to Catalan who, in turn, was presented by Boncompagni to the Academy of the Nuovi Lincei and by Bouniakowski to the Saint-Petersburg Academy. “The influence of the milieux cannot be and is not contested,” wrote Catalan in 1884 on the occasion of his retirement[6, p. iii]. He mentioned the Polytechnique and his teachers, but his letters and papers bear witness to a variety of milieux, matching the variety of his mathematical achievements; milieux between which mathematics did not circulate fluidly at the time: Association française pour l’avancement des sciences (AFAS) and other scientific societies, Polytechnique and engineering schools, Academies of sciences in various countries and universities, high-school institutions, all backed by a variety of types of publications, from research journals to communications at conferences to popularization papers or textbooks.[11] The way not only Catalan, but also his mathematics, circulated through these milieux, is thus both characteristic of him and quite remarkable.



CCL. — Problème trouvé en songe.

(9 *mars* 1886.)

I. *Décomposer une fonction donnée*, f(x, y), *en deux parties* M, N, *de telle sorte que* Mdx + Ndy *soit une différentielle exacte.*

Figure 8: An analytic problem found in a dream, in Catalan’s *Mélanges*.

3.2 Unity and transfers

Catalan numbers, as they are now called, offer a paradigmatic instance of such transfers. These numbers, $c_n = \frac{1}{n+1}\binom{2n}{n}$ originally appeared as the numbers of possible triangulations of a (plane, convex) polygon of $n + 2$ sides, a problem which Olry Terquem had brought to the attention of Liouville and his circle: a series of articles followed from 1838 on in the newly founded *Journal de mathématiques pures*

[10] A large part of Catalan’s correspondence is now on line: http://lib.ulg.ac.be/fr/content/eugene-catalan-1814-1894-mathematicien-l-ulg. I would like to thank Stéphanie Simon, at the Liège University Library, who kindly provided me in advance with several of the letters.

[11] On the partial separation of such institutions in France, see [10, 12, 11].

et appliquées, from Gabriel Lamé, Jacques Binet, Olinde Rodrigues and Catalan.[12] Catalan contributed in particular to extract the c_n from their original geometrical context: he showed that they are also the number of ways one can compute the product of n numbers without changing their order and he provided both new recurrent relations and analytic representations by means of Eulerian integrals. But these numbers would reappear much later at meetings of the AFAS; Catalan was then interested in another sequence, the coefficients P_n of the development of the complete elliptic integral K as a function of the complementary modulus k', such that $0 < k, k' < 1$, $k^2 + k'^2 = 1$:

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{\pi}{2} \sum_0^{\infty} P_s \frac{(1-k')^s}{16}$$

The sequence of integers P_n, which Catalan was able to link with another of his interests at the time, the Legendre polynomials mentioned earlier, via the relation

$$\frac{P_n}{8} = \frac{2}{\pi} \int_1^{\infty} \frac{X_n dx}{x^{n+1}\sqrt{x^2 - 1}}$$

satisfy a recurrence relation

$$n^2 P_n - 8(3n^2 - 3n + 1)P_{n-1} + 128(n-1)^2 P_{n-2} = 0.$$

Catalan showed that the P_n can be expressed as combinations of the squares of the c_n and thus provided new relations involving the $c - N$. But he also mobilized a member of the AFAS, the admiral and part-time mathematician Ernest de Jonquières, to compute the first terms of the sequence: $P_1 = 8$, $P_2 = 80$, $\ldots$, $P_{12} = 2^{16} \cdot 229011025, \ldots$ and to carry on with their study. He also launched further work on the c_n in connection with orthogonal polynomials by Ferdinand Caspary, a German mathematics teacher in professional schools who was also a correspondent of Hermite.

What warrants such bridges between mathematical milieux is here, as elsewhere, the unity created inside mathematics: Legendre polynomials are used by Catalan in number-theoretical and combinatorial questions, and reciprocally. Such transfers occur in many places in Catalan's work, between number theory and probability, between geometry and differential equations, between combinatorics and the theory of determinants. The study of toroids provided him with infinitely many integral solutions to the equation $x^3 + y^3 + z^3 = u^2$. In his work on polyhedra, he associated

[12] These articles are described in [16], as well as the political, institutional and mathematical links between their authors. Catalan's contribution and some developments are explained in [14, pp. 197]. More details on the history of these numbers, and on the attribution of Catalan's name to them, can be found in [18].

the pieces of a board game to the elements of polyhedra, thus transforming the classification of polyhedra into Diophantine analysis and vice versa.

We are still habituated to think that from the end of the nineteenth century on, unity in mathematics was moved forward by the adoption of a structural and axiomatic perspective.[13] Catalan's case testifies to alternative possibilities, in the "safe haven of actual mathematics," in the words of Leopold Kronecker. For instance, in the course of his work on minimal surfaces, Catalan found the equation of a geodesic on the surface $z(x,y)$:

$$(3)\ \frac{d^2y}{dx^2}(1+p^2+q^2)+(r+2u\frac{dy}{dx}+t\frac{dy^2}{dx^2})(q-p\frac{dy}{dx})=0,$$

where $dz = pdx + qdy$, $dp = rdx + udy$, $dq = udx + tdy$. He then characteristically comments: "according to M. Delaunay, the equation of the curve of given length bounding a maximal area, is

$$(4)\ \frac{d^2y}{dx^2}(1+p^2+q^2)+(r+2u\frac{dy}{dx}+t\frac{dy^2}{dx^2})(q-p\frac{dy}{dx})=\frac{1}{m}(\frac{ds}{dx})^3\sqrt{1+p^2+q^2}.$$

The left hand sides of equations (3) and (4) are identical: this circumstance indicates a necessary link between two problems which appeared to be different."

Ainsi, l'équation de la ligne minimum peut être mise sous la forme

$$(3)\quad \frac{d^2y}{dx^2}(1+p^2+q^2)+\left(r+2u\frac{dy}{dx}+t\frac{dy^2}{dx^2}\right)\left(q-p\frac{dy}{dx}\right)=0.$$

D'après M. Delaunay, l'équation de la courbe de longueur donnée, qui renferme une aire maximum, est

$$(4)\quad \left\{\begin{aligned} &\frac{d^2y}{dx^2}(1+p^2+q^2)+\left(r+2u\frac{dy}{dx}+t\frac{dy^2}{dx^2}\right)\left(q-p\frac{dy}{dx}\right)\\ &\qquad=\frac{1}{m}\left(\frac{ds}{dx}\right)^3\sqrt{1+p^2+q^2}.\end{aligned}\right.$$

Figure 9: Comparing two equations, [3, p. 152].

Catalan's dexterity in formulas and concrete analytical computations thus allows him to derive new results and build new bridges, some directly arising from identification between equations.

[13]That such a partial view does not allow a realistic understanding of the development of mathematics is discussed for number theory in [13].

Already in 1841, discovering that the same cubic equation,

$$t^3 - (a^2 + b^2 + c^2)t - 2abc = 0,$$

intervenes in two geometrical problems about quite distinct situations, Catalan commented: “This relation between two problems which are apparently so different, seems quite curious.” Here, as in more substantial work, are to be seen the permanent characteristics of Catalan’s mathematical contributions across domains and themes: a constant reactivity and energy; a pleasure in locating bridges between problems; all fed by a vivid curiosity for the surprises of mathematics.

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