Bulletin de la Société Royale des Sciences de Liège, Vol. 84, 2015, p. 93 - 96

Conceiving a conjecture, and other stories about Eugène Charles Catalan

Preda MIHĂILESCU Preda[1]

Mathematisches Institut der Universität Göttingen, Göttingen, Germany

1. The person

Eugène Charles Catalan would have been 200 years old these days. Had he lived to reach this age, had he been of some immortal stock like a *Highlander mathematician*, what would he think of our ages? Would the passionate revolutionary believe that some of the *acquis révolutionnaires* were well preserved? Would he think that some of the goals for which he has spent much of his youth have been successfully achieved? And what would his opinion be, about the progress that mathematics has achieved meanwhile?

This and similar questions came to life in my mind, from reading the wonderful biography of Catalan Eugène Catalan. *Géomètre sans patrie Républicain sans République* written by François Jongmans. Oh, do not fear, I will not even try to respond to these questions. Instead, I will spend some time trying to explain what they may mean to us, travelling time in the inverse direction, so to speak.

Catalan was born on Belgian territory, at Bruges, in Napoleonian France. The family moved to Lille and then in the early twenties, to Paris, when he still was a child. He thus grew up in postrevolutionary Paris. His youth was marked by a strong interest for mathematics, and a heart for social emancipation. Paris was full of brotherships and societies, and, for sure, young Eugène was also a member of one or two. These were societies of school teenagers, such as the "Société des amis du peuple", who dedicated their time offering free meditations to younger colleagues from poor social conditions, whose families could not afford private lessons. He was not more than 15 of age when he started activating in the société. The political and the social, generous character were close to each other, and the republican convictions of young Catalan were outspoken. This was not a comfortable attitude in the times of reconstitution — although it appears almost romantic, from the perspective of the apparata of repression developed in the XXth century, the political police was well existing and active in Paris of the 1830'es. Unlike Galois however, Catalan never was enjailed. Unlike Galois also, Catalan was not son to a suburb mair, humiliated in a political-clerical conspiration, he was enfant naturel, son to a stepfather who was a not so rich jeweller. It was this father who, at that moment a member of the popular *gardes nationales*, prevented Eugène from joining the meeting for the funerals of général Lamarque, which were closely related and short after the ones of Évariste Galois. A fact less known to me and maybe to many

[1] Email: preda@uni-math.gwdg.de

mathematicians, Galois was indeed very popular in the republican millieu, and his funerals were the catalysator for the massive popular gathering mentioned; this has been the last important republican protest under Louis Philippe. About one hundred people were shot and the father of Catalan had been wise to prevent the son from attending.

Nevertheless, the republican sympathies on the one side, and the relatively humble social condition have not been a stimulus for Catalan's career. His mathematical qualities brought him with success through the prestigious École Polytechnique founded by Napoléon, and he obtained there a position of Lecteur. At a time when there were less than ten professors of mathematics at all, counting Cauchy and Liouville, Fourier and Carnot among them, this was not necessarily a bad situation — however, his promotion was clearly impedished by his republican views. In the revolution from 1848, he was the second after the poet Lamartine to join the commité provisoire. He left it weeks later, while some of the friends and companions he had there, developed later a political career. By that time Catalan had already had more remarkable results and even won some prizes of the prestigious Académie; interestingly, one was for being the first to develop correctly the apparatus and proofs for what we presently denote, in calculus, by Jacobian — in honour of Jacobi, who had considered the problem before Catalan, but reached the closed final form of the result, after him. Despite of being known as one of the important and promising mathematicians of his generation, it is only when he was offered a position in Liège, in his native Belgium, that Catalan could accomplish[2] his academic career and teach new generations in his spirit.

2. The mathematics If Catalan would be here to contemplate how times advanced, it is not only the brutal development of political police in the XXth century which would cause him pain. He might also be surprized, but not painfully, by the light that was shed on his mathematical legacy. His well deserved place among the major mathematicians of French expression in the XIXth is respected. However, his name is mostly recalled in connection to work on combinatorics and number theory: The Catalan numbers, the Catalan Conjecture, etc. While XIXth century's scientists were less specialized than today, it is undoubted that Catalan might have considered himself an analyst. A domain in which not only did he develop the formal procedure for multivariate integrals, and the related Jacobian matrix, but he dedicated much research to the investigation of elliptic functions, which was an important topic of his time.

Since I dedicated a short but important period of my research to the number theoretical conjecture that bears Catalan's name, it is of it that I will speak at the end of this short note. Let us take a phenomenological approach, and let our imagination work upon the spectacular incident of numbers:

$$3^2 - 2^3 = 9 - 8 = 1. \qquad (1)$$

[2] There remains one biographic question which is uncleared, to the best of my knowledge: what is the origin of the name? Does it suggest that some remote ancestor did come from far away *Cataluña* to Belgium? Is it even a frequent name there?

Numbers are considered by some people as abstractions, and talking about empirical and phenomenological approaches in the theory of numbers will strike those who think like that as surprizing. It is not, and many active number theorists consider their work and discipline as an empirical science of nature — of the nature of numbers. Looking back to our equality, we see a richness of intriguing symmetries. The investigative mind is assaulted by a series of questions which begin by *is it always so ...?* These are very important and useful questions, and children love to ask them — but then again, children are among the most intelligent adults, because they can avoid as often as possible to be so.

Let us see: how many "is it always so?" come to mind. The numbers three and two have double role: so, replacing them by indeterminate integers a, b, one may ask how often does one have $a^b - b^a = 1$? It is a relatively simple application of calculus, to see that $a = 3$, $b = 2$ is the only solution. Breaking the symmetry, one may keep either bases or exponents fixed, and comes to ask, what are *all the solutions* to thefollowing equations:

$$3^n - 2^m = 1 \tag{2}$$

$$y^2 = x^3 + 1. \tag{3}$$

The first equation (2) was solved by Isaac ben Gershon (also: Gershonides), a sephardic theologian and philosopher, who proved that (1) yields the only solution; the proof uses congruences and was translated by Paulo Ribenboim in modem mathematical notation, thus confirming its elegance and consistency, 650 years later. I encourage the reader to develop an own proof and try to find the one of ben Gershon, for comparison, in the Internet. The question itself was connected to Platonic harmony and is part of the book *De numeris harmonicis*, written by Gershon upon request of Philippe de Vitry, Bishop of Meaux.

The solutions to (3) are points on a quite special elliptic curve. Elliptic curves have been investigated in European mathematics after Fermat. The one under discussion was considered by Euler in the first half of the XVIII-th century, and he has shown that also in this case, (1) yields the only solution.

All these facts were known at the time of Catalan, and certainly to Catalan himself. When he came to ask, like an open minded child, what can one ask more about (1), he forgot the symmetrical role of two and three, and came to focus on the difference of one, between two non-trivial powers of integers. The *asymmetrized* question then lays at hand: how often are two consecutive integers both proper powers of other integers. As a formula, one asks how many solutions does the equation $x^u - y^v = 1$ have in integers? Catalan believed that also in this case, (1) yields the only solution. He asked the question first in a Parisian *Nouvelles Annales de Mathématiques*, in 1842; since the question was so appealing, he repeated it two years later in a much more prestigious journal, the famous *Journal de Crelle* from Berlin. The Catalan Conjecture was born. The story of the chain of intermediate steps towards its proof is ... an other story, and it will be told in part in the lecture.

Let us restrict here to mentioning that Catalan himself has left no contribution or attempts towards the solution of this number theoretical question, possibly also because his main interest was dedicated to analysis ... The elegance and simplicity of the formulation, led in time to the fact that Catalan's Conjecture attracted more and more research and interest, becoming in this respect some kind of a younger brother to the so celebrated *Fermat's Last Theorem*. Now both have led, in different measures, to development of mathematics, and are solved[3].

Thank you, Eugène Catalan!

[3] Let us mention, for those who are embarassed or otherwise uncomfortable with the finality of a proof, that many questions can be asked beyond Catalan's one: let me mention just precious few. If we replace *consecutive* by *at fixed*, *small distance*, we get a special case of a conjecture of Pillai: $x^u - y^v = k$ has finitely many solutions.

If we *dehomogenize*, we obtain a conjecture of Nagell and Ljunggren: $\frac{x^n - 1}{x - 1} = y^m$ has only two, known solutions. One can also consider the equation of Catalan in the rationals, or in a number field ... as some of the precious few generalizations to be mentioned here.