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MUST THE VISIBILITY FUNCTION OF AN OPEN SET BE CONTINUOUS?

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Abstract. The visibility function of a set S was defined in 1972 by G. Beer. This function evaluates the Lebesgue outer measure of the star of each point of S. The continuity of this function for compact sets was studied by its creator. Recently, the present authors settled the problem of continuity in the boundary of a compact set S. The question of the continuity in the case where S is an open bounded set - proposed by G. Beer - is still unsettled. We show here that the continuity of the visibility function does not depend on the boundedness of the open set S.

Keywords: Visibility Function, Visibility in open sets.

1. Statement of the Problem.

Unless otherwise stated, all the points and sets considered here are included in E^d, real d-dimensional Euclidean space. The interior, closure, boundary, complement, affine hull and convex hull of a set S are denoted by *int* S, *cl* S, ∂S, S^C, *lin* S and *conv* S, respectively. The open segment joining x and y is denoted $(x\ y)$, and the substitution of one or both parentheses by square brackets indicates the adjunction of the corresponding endpoints. $R(x \to y)$ denotes the ray issuing from x and going through y. $B(x, \varepsilon)$ and $U(x, \varepsilon)$ are, respectively, the closed and the open balls centered at x and having radius ε. If p is a point and M is a set such that $p \notin M$, the *star hull* of M over p is the set $J(p, M) = \bigcup_{q \in M} [p\ q]$. More generally, $J(A, B) = \bigcup_{a \in A,\ b \in B} [a\ b]$.

We say that x *sees* y *via* S if $[x\ y] \subset S$. The *star* of x in S is the set *st* (x, S) of all the points of S that see x via S. The *visibility function* of a compact set $S \subset E^d$ is the function $\upsilon_S : S \to R^+$ defined by $\upsilon_S(x) = \mu_d\,(st\ (x, S))$ where 'μ_d' indicates the Lebesgue d-dimensional outer measure. In a measurable set, the star of a point may not be measurable, whence using outer measure becomes necessary. As we mentioned in the abstract, this function was defined and studied by G. Beer in several papers ([1], [2] and [3]). In [3], theorem 4, Beer characterized the global continuity of υ_S in terms of the parallel bodies of S.

One of the authors of the present article (see [4]) characterized the continuity points of υ_S in the boundary of a planar Jordan domain S. The general question of the continuity of this function in the boundary of a compact subset of E^d was settled by the present authors in two recent papers ([5] and [6]). The remaining question about the continuity of υ_S is the case where S is an open subset of E^d. Beer suggested (see [3]) that in this case the general continuity depends on the boundedness of S. We are able to answer this suggestion in the negative sense. The present note is devoted to the construction of two examples of open subsets of the Euclidean plane E^2, one unbounded and the second one bounded, having discontinuous visibility functions.

2. The Unbounded Case.

Clearly, this is the easy case. The reader is invited to formulate his own counterexample. Meanwhile, we present our example, intended as a sort of introduction to the bounded case.

Example 1. An open unbounded subset S_u of E^2 that has discontinuous visibility function.

Let Q be an open rectangle in the plane E^2 and let M be the line that bissects both vertical sides of Q. Denote x and z two different points of $Q \cap M$, and without loss of generality assume that the distance between these points is 1. Denote $x_1 = z$, x_2 the midpoint between x_1 and x, and inductively x_{i+1} the midpoint between x_i and x. Hence, the whole sequence $\{x_i\}$ is included in $Q \cap M$ and $\lim_{i\to\infty} x_i = x$. Let $\{y_i\}$ be the vertical projection of the sequence $\{x_i\}$ over the upper rim B of Q, and let $\varepsilon > 0$ be a fixed small number (positive but much smaller than 1). With these elements we can construct the *stripes* that will produce the discontinuity of the visibility function at x. Let B_1 be a small segment included in B and centered at y_1 , whose length will be fixed below. Over this base we build a long and thin vertical rectangle called T_1, that is the first *stripe*. This stripe is totally visible from x_1 but x can peep only a small triangular region of it, the first *nail* N_1. The area of N_1 depends on the width of T_1. Then we can fix this width in such a way that

$$\mu_2 (N_1) \leq \frac{\varepsilon}{2}$$

Furthermore, it is possible to fix the height of T_1 in such a way that

$$\mu_2 (T_1) = 1 \ .$$

In the same way, for each positive integer k let us define B_k as a small subinterval of B centered at y_k, and denote T_k a thin vertical stripe based on B_k, totally visible from x_k but such that x can see only a small triangular region N_k . It is possible to adjust the width and height in such a way that

$$\mu_2\left(N_k\right) \leq \frac{\varepsilon}{2^k} \qquad ; \qquad \mu_2\left(T_k\right) = 1$$

Then, we define

$$S_u = Q \cup \left(\bigcup_{k=1}^{\infty} T_k \right)$$

If we delete the outer rim of S_u (both in the base and in the strips) we obtain an open, unbounded and connected set. Besides, we have

$$\nu_S\left(x\right) = \mu_2\left(st\left(x, S_u\right)\right) \leq \mu_2\left(Q\right) + \varepsilon$$

while

$$\forall k \in \mathbb{Z}, \quad \nu_S\left(x_k\right) = \mu_2\left(st\left(x_k, S_u\right)\right) \geq \mu_2\left(Q\right) + 1$$

Hence, the visibility function of S_u is discontinuous at x.

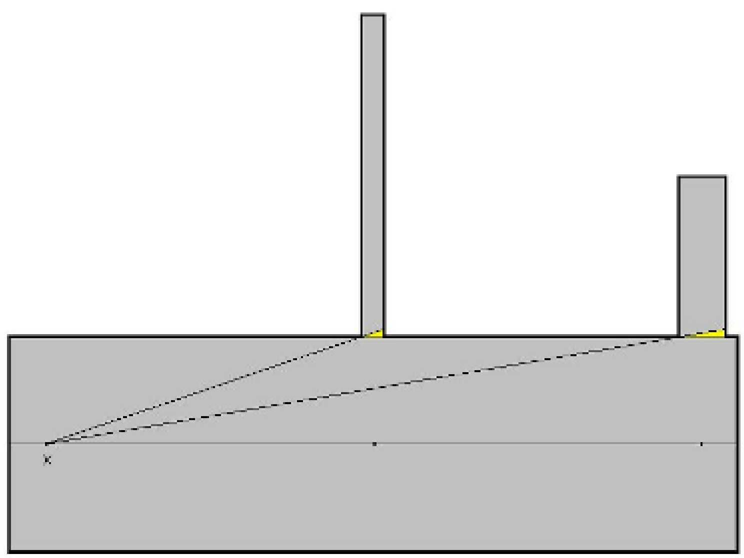

FIGURE 1. First two steps of the construction of the unbounded example: the first two vertical strips and their nails.

In the previous figure we can see the first two steps of construction of the set S_u with the basic rectangle Q, the first two strips T_1 and T_2 and the corresponding nails N_1 and N_2 painted in a different color.

3. The Bounded Case.

We intend to construct a planar open and bounded set whose visibility function has (at least) one point of discontinuity. The construction is similar to that in the unbounded case, but instead of the long and thin stripes we will use certain fan-like sets that we will define later. Since this construction is rather involved, we divide it into three steps.

3.1. **Open Coverings of a Pathological Cantor Dust.** In a previous paper [5] we had introduced the term *pathological Cantor dust* to mean a bounded subset of the line $\mathbb{R}$ that has positive Lebesgue measure and empty interior.The construction of such a set mimics the construction of the Cantor Ternary Set. In the present case we intend to obtain a set K such that $K \subset [0\ 1]$ and $\mu_1(K) = \frac{1}{2}$. Let us define $K_0 = [0\ 1]$ and obtain K in countably many steps as follows. K_1 is obtained from K_0 by deletion of a centered open interval of length $\frac{1}{4}$. Hence K_1 is the union of 2 closed intervals. Now, if we delete from K_1 two open intervals of length $\frac{1}{4^2}$ each, centered in the connected components of K_1, we obtain K_2 formed by the union of 4 closed intervals. And the construction goes on inductively. Finally, we define

$$K = \bigcap_{n=1}^{\infty} K_n \ .$$

K is a disconnected set and has empty interior. Moreover, the amount of length cropped off from K_0 is

$$\sum_{n=1}^{\infty} \frac{2^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \sum_{n=2}^{\infty} \frac{1}{2^n} = \frac{1}{2}$$

Hence, $\mu_1(K) = \frac{1}{2}$. Let us define the *gauge* of K_n as the length of any of the 2^n closed intervals that composed it. It is clear that

$$\lim_{n\to\infty} gauge\,(K_n) = 0$$

Whence, given $\delta > 0$ there exists a first index n such that $gauge\,(K_n) \leq \delta$. As we just have seen, K_n is the union of finitely many *closed* intervals. Let G_n be the union of the same intervals but stripped from its endpoints, i.e. open intervals. Hence, G_n is an open set and $\mu_1(G_n) = \mu_1(K_n)$. Furthermore, G_n *almost covers* K , i. e. it covers K with the exception of finitely many points. We describe G_n calling it an *open almost-covering of* K having gauge less than δ. If we consider K included in a line L and located on the Euclidean plane E^2, we can pick a point $p \notin L$. The set

$$F_n(p) = int\ J(p, G_n)$$

is the *open fan* generated by p and G_n . It is useful that we take a closer look at these open fans.

In Figure 2 we see a picture of $F_2(p)$. More generally, $F_n(p)$ has 2^n blades and $2^n - 1$ voids. Each of these elements, either a blade or a void, is a triangle having height equal to the distance from p to L, and one of the intervals of the construction of K as the base (either an included interval or a discarded one). Whence the individual and total area of the blades are completely under control, if we know this distance and the gauge of K_n. We also remark that these triangles (the

FIGURE 2. An open Fan generated by a G-set and a point.

blades) are completely stripped of its boundaries, with the exception of the apex p.

3.2. **Building the Open Planar Set.** The first steps of our open bounded set S coincide with those of the unbounded case. The rectangle Q, the line M, the points x, $x_1, x_2, \cdots$ are exactly as in Section 2. Let L be a line parallel to M, located below Q and such that its distance to Q equals the height of Q. Let us consider on the line L and located "in front" of Q, a pathological Cantor dust K constructed as we saw in the previous paragraph. A further restriction on the relative position of K and Q is that any ray issuing from the points x, $x_1, x_2, \cdots$ and going through any point of K must leave Q through its bottom (and not through a vertical side of Q). As we mention at the beginning of Section 3, we intend to add to Q open fans instead of the stripes that we had used in Section 2. Denote $\alpha = \mu_2(Q)$ and $\beta = dist\,(L, Q) = height(Q)$, by construction. Let ε be a *small* positive number (small compared with α and β), say $\varepsilon = \frac{1}{10^3}\inf\{\alpha;\beta\}$, for example. Now we consider the point x_1 and pick a certain positive number δ_1 whose snallness will be determined (by a finite number of geometrical constraints) in the next paragraph. As we have remarked above, there exists a positive integer $n(\delta_1)$ such that $K_{n(\delta_1)}$ has gauge not greater than δ_1. Let $G_{n(\delta_1)}$ be the open almost-covering of K generated by $K_{n(\delta_1)}$ and denote

$$F\,(1) = J\left(x_1, G_{n(\delta_1)}\right) \qquad ; \qquad S_1 = Q \cup F\,(1)$$

In the same way, consider x_2 and pick $\delta_2 > 0$ small enough, and construct

$$F\,(2) = J\left(x_2, G_{n(\delta_2)}\right) \qquad ; \qquad S_2 = Q \cup F\,(2)$$

The definition of $F(n)$ for each positive integer n follows inductively. Finally we denote

$$S = \bigcup_{n=1}^{\infty} S_n$$

Remark 2. (1) The set S is clearly bounded, since it is included in the set *conv* $(Q \cup K)$. Furthermore, it is open, since it is a union of open sets.
(2) By construction, it holds

$$\forall n \quad st(x_n, S) \supset S_n$$

(3) It is easy to verify that

$$\nu_S(x_n) = \mu_2(st(x_n, S)) \geq \alpha + \tfrac{1}{3}\beta$$

In the next paragraph we intend to prove that the measure of $st(x, S)$ is a number much smaller than that mentioned in the third item of the previous remark.

3.3. **Peeping.** The word *"peeping"* describes the visibility of a point of S outside its explicit visual range. In the case of each point x_i of the sequence, the explicit visual range is $S_i = Q \cup F_i$, whereas the limit point x has explicit visual range Q. The explicit visual range of a point is the subset of S that was intended to be visible from this point. But the points involved usually can see a little more (recall the nails in the unbounded case). We try to minimize this *"little more"* in the case of x to show that the evaluation of $\nu_S(x)$ is clearly less than the value of this function on the points of the sequence $\{x_i\}$. Our basic question is:

"*What is the measure of the set* $st(x, S) \sim Q$?"

There are two different ways that x can peep outside Q, by means of *nails* or by *bridges*:

Nails.: This a phenomenon totally analogous to that mentioned in the construction of the unbounded example. Each blade of each fan that emerges from the basic rectangle Q has a small triangle - its *nail* - that is visible from x. The measure of all the nails (finitely many) of the fan F_i can be controlled by the gauge $g_i = gauge(G_i)$ where G_i is the open almost-covering that generates F_i . Exactly in the same way as we proceed in the unbounded case, we can fix g_i small enough such that the total measure of all the nails of F_i be less that $\dfrac{\varepsilon}{2^i}$. Hence, the total amount of peeping by nails would be less than ε .

Bridges.: The basic difference between the construction of this example and the previous one is that the stripes of the unbounded case were parallel (and pairwise disjoint), whereas the blades of different fans are not. This generates the possibility of a different (and somewhat more involved) way of peeping. A

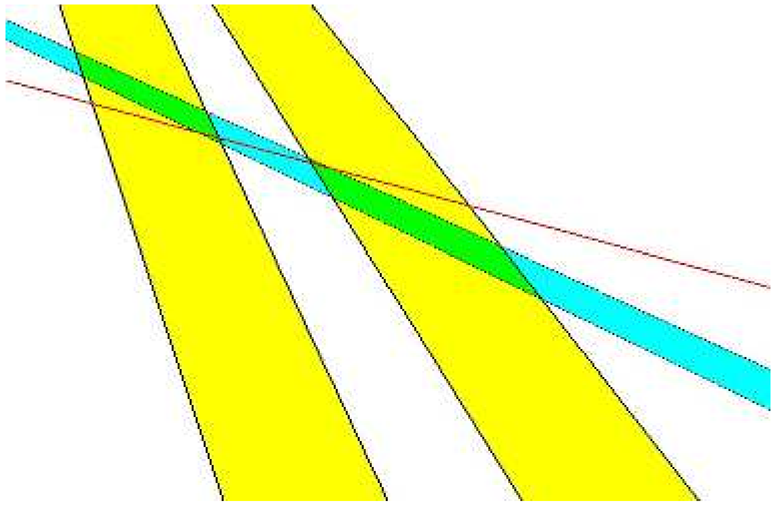

FIGURE 3. A ray issued from point x uses a bridge to pass from one blade to another inside S.

bridge between two adjacent blades B^i_m and B^i_{m+1} of the fan F_i is a portion of another blade, belonging to a different fan F_j with $i < j$, that crosses the gap between those two blades.

In Figure 3 we see these elements and a third type of object, a ray issued from x that uses the bridge to cross from the blade B^i_{m+1} to the blade B^i_m without falling outside S. This is, precisely, the type of peeping that we try to avoid. We remark that the three types of elements considered (the blades of fan F_i, the transversal blade from the fan F_j and the ray issued by x) have different inclinations, since they come from three points (x_i, x_j and x) with distinct locations on a horizontal line. If the bridge would have been a single segment, then no ray from x would have crossed the void between B^i_{m+1} and B^i_m without falling out of S. The same happens if the bridge produced by the transversal blade is narrow enough in comparison with the gap that it crosses. But the width of the gap is known, since it depends on the gauge of F_i. For the same reason, the narrowness of the bridge depends on the gauge of F_j. Hence, if we take the gauge g_j small enough with respect to g_i, no ray coming from x can cross a bridge produced by a blade of F_j between two adjacent blades of F_i. Observe that in the construction of the fan F_j there are finitely many of these constraints, those produced by the crossing of its blades with the blades issued by the previous points x_i with $i < j$. Hence, it is possible to fix the gauge of each fan in such a way that the phenomenon of peeping by *bridge-crossing* does not appear.

Thus, we have proved that

$$\nu_S(x) \leq \alpha + \varepsilon$$

whereas (see part 3 of Remark 2) we know that

$$\forall i \qquad \nu_S(x_i) \geq \alpha + \frac{1}{3}\beta$$

Hence, ν_S **is discontinuous at the point** x.

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