BARRELLEDNESS AND DUAL STRONG SEQUENCES IN LOCALLY CONVEX SPACES

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Abstract

Given a dual pair \((E, F)\), we consider a series of bidual enlargements of \(F\), define new concepts of dual strong sequence and dual strong union and analyze their connections to barrelledness.

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1 Introduction: nonbarrelled locally convex spaces

Given a locally convex space \(E\), we introduce the notions of a dual strong sequence and dual strong union. The actualization of these notions is motivated by exploring sets with "nice" useful properties such as compactness or equivalence of different topologies. ([8], [10], [26], [27]). Among with importance and value in mathematics, sets with "nice" properties are essential in applicative areas, such as economics, engineering, physics, where global infinite-dimensional locally convex spaces are widely used. Examples of "nice" properties employed in applications can be found in [1] or [20]. In the forthcoming article we show that dual strong sequences are a natural resource for some of the "nice" properties. In the present article we investigate the relationship between the dual strong sequences and barrelledness conditions.

A dual strong sequence is a universal structure in the sense that it appears in any locally convex nonbarrelled space. The collection of nonbarrelled spaces is considerable. Apparently the most important are the spaces of continuous functions \(C(T)\), widely used in different applications. It is known that the space \(C(T)\) of real continuous functions on a completely regular Hausdorff space \(T\) is barrelled for the topology of pointwise convergence if and only if each bounded subset of \(T\) is finite, ([6]), which implies that for a barrelled space \(C(T)\) the topologies of uniform convergence on finite, compact and bounded sets of \(T\) coincide. Thus if \(T\) admits an infinite bounding (or compact) subbasis then \(C(T)\) is nonbarrelled. The generalizations of Buck's strict topologies on \(C(T)\) explored by Russ are neither barrelled nor even \(\sigma\)-barrelled, however they enjoy the Banach-Mackey property, ([27], Proposition 2.1. \(\sigma\)-barrelled = \(\sigma\)-quasi-barrelled). Emphasizing their importance, we notice that each of the strict topologies of [27] is

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localizable on any absorbing disk (i.e., it is the finest locally convex topology equivalent to the given one, see also [20]). We also point out that in the spaces of continuous vector valued functions there exist strict topologies with important properties of angular and metrizability of compact sets, [8, Corollary 1.11]. Regarding the spaces of vector valued functions, we mention the result of [16] that the space \( c_0(X) \) with the sup norm is barrelled if and only if \( X \) is barrelled. Notice that \( c_0(I) \) can be regarded as the generalization of the well-known space \( c_0(N) = c_0(X) \) of convergent to zero sequences in \( X \) (see [17], 1.1, p. 463). As to general locally convex spaces, every separable Banach space admits a finest nonbarrelled Mackey topology of sup type, [12, Example 3.5]. Every Banach space admits a strictly increasing sequence of nonbarrelled norms, [21, Proposition 6.4)). The projective tensor product \( X \otimes \mathcal{Y} \) is nonbarrelled, as well as \( E \otimes \mathcal{Y} \) for a metrizable non-normable space \( E \), [9]. According to Luxemburg, the \( \ell_1 \)-direct sum \( (\ell_1, X) \) of normed spaces \( X \) is barrelled if and only if each \( X \) is barrelled, [23, 4.9.17]). Since any necessary and sufficient requirement for being barrelled provided as well a condition for being nonbarrelled, we suggest [20] as a valuable resource for further nonbarrelled examples.

Working in the general frame of nonbarrelled locally convex Hausdorff spaces, we develop a broad generic approach, presenting, a fairly complete account of concise results and leaving the study of tangible spaces to further investigations.

2 Preliminary notations and definitions

We follow the definitions and notations of [17] or [22]. Let \( E \) be an infinite-dimensional vector space over the field \( \mathbb{K} \) of the real or complex numbers, \( E^* \) the algebraic dual of \( E \) and \( F \) a \( (E^*, E) \)-dense subspace of \( E^* \). Given a dual pair \((E, F)\), we denote by \( \mu(E, F), \varphi(E, F), \sigma(E, F) \) the strong, Mackey and weak topologies on \( E \), respectively. We denote by \( \sigma(E, F) \) the topology on \( E \) of uniform convergence on all strongly bounded sets of \( F \). All topologies on \( E \) will be considered locally convex and Hausdorff.

A disk \( D \) is an absolutely convex set. A singleton \( \{a \in E | a = 1 \} \) for some nonzero \( x \in E \) is a disk as well as \( E \) itself. The set \( \{0 \} \) is a trivial disk. All coming next disks will be nontrivial. A disk is absorbing (barrelled), if it absorbs any element (bounded set) of \( E \). A barrel is a closed absorbing disk. For a disk \( D \) we denote by \( E_D \) the linear hull of \( A \), equipped with its gauge \( g_A \). The disk \( D \) is barrelled, resp. Banach, if \( E_\mathcal{D} \) is barrelled, resp. Banach. It is finite-dimensional, resp. infinite-dimensional, if \( E_\mathcal{D} \) is finite-dimensional, resp. infinite-dimensional.

A family \( \mathcal{A} \) of disks in \( E \) is a set of disks covering \( E \) such that for any \( A, B \in \mathcal{A} \) and \( a, \beta < \varepsilon \) there is \( C \in \mathcal{A} \) satisfying: \( aA + \beta B \subseteq C \). We consider the following families:

1. \( \mathcal{B} \) – the family of all \( \sigma(E, F) \)-closed bounded disks of \( E \).
2. \( \mathcal{B}_E \) – the family of all \( \sigma(E, F) \)-closed bounded disks of \( E \).
3. \( \mathcal{NAB} \) – the family of all \( \sigma(E, F) \)-closed bounded ballerellded disks of \( E \).
4. \( \mathcal{B} \) – the family of all \( \sigma(E, F) \)-closed bounded Banach disks of \( E \).
The name of types 3, 4, 5, 6 is a semantic abbreviation of the property the family carries, for example, $NBAR = NormedBAR(). Notice that a family of type $(n+1)$ is embedded into the family of type $(n)$ for $n = 1, 2, 3, 4, 5$.

There are special notions for a locally convex dual pair $(E, F)$ with some of the families equal. We say that $E$ is Banach-Mackey if $B = B'$ in $E$. Since barrels and closed bounded disks of a dual pair $(E, F)$ are connected by polarity (the polar of a barrel is a closed bounded disk and vice versa) we observe that $E$ is Banach-Mackey if and only if $F$ is Banach-Mackey. We say that $E$ is locally barrelled, resp. locally complete (or $\mathcal{H}$-complete, see [24]) if $B = NBAR$, resp. $B = BAN$ in $E$. We say that $E$ is locally quasi-barrelled, resp. locally quasi-complete, if $B' = NBAR$, resp. $B' = BAN$, in $E$. The space $E$ is dual locally (quasi) barrelled, resp. dual locally (quasi) complete, resp. (quasi) barrelled for $\mu(E, F)$ if $B = NBAR$ ($B' = NBAR$), resp. $B = BAN$ ($B' = BAN$), resp. $B = WC$ ($B' = WC$) in $E$. Dual locally (quasi) complete and (quasi) barrelled spaces are standing new; to each other, which is not a coincidence, because we know that these properties are affiliated, ([24], [28]).

Later we prove (Proposition 5.2) that for a dual strong union the case of $B = NBAR$ in $E$ and the barrelledness of $E$ are connected as well. If $B = FIN$ in $E$, then $F = E$ and $\sigma(E, F) = \beta(E, F)$ is the finest locally convex topology on $E$. If $B \neq FIN$ in $E$, then $F$ admits an enlargement in $E$ and $E$ admits a finer locally convex topology. Different enlargements of $F$ (including countable enlargements) and their connection to the existence of infinite-dimensional bounded sets in $E$ were studied in ([12], [29], [30]). We also proved in ([32]) that any locally convex space of dimension at most $c$ with an infinite-dimensional bounded Banach disk admits a hyperplane satisfying $B \neq BAN = FIN$. Although the $NBAR$ family is not muddled, any dense countable-codimensional sub-space in a Banach space is an example of a space with a nonempty $NBAR$ family such that $B = NBAR = BAN$. Since a locally complete metrizable space is complete (see [30], p. 7), it follows that a dense countable codimensional subspace of a Frechet space provides another example with $B = NBAR \neq BAN$. Selecting a locally complete space with an infinite dimensional bounded set of dimension at most $c$ and applying the method of [32] we obtain a hyperplane satisfying $FIN = BAN \neq NBAR = B$. We also mention the result of Vakhintovich on an increasing sequence of subspaces in a specific Banach space with at least one (therefore all but a finite number) of the subspaces dense and barrelled, ([36], Theorem 1, p. 46). The result of [36] is related to the measure theory thus incorporating the spaces with $B = NBAR$ into the eminent coalition of applicative spaces.

3 The concepts of a dual strong sequence and dual strong union

Let $(E, F)$ be a dual pair. Starting with $F = G_1$ and $\beta_1 = \beta(E, G_1)$ and keeping on with $G_2 = (E, \beta_1)^\prime$ and $\beta_2 = \beta(E, G_2)$ we perceive a structure described in the following definition.
Definition 3.1. Let \((E, F)\) be a dual pair. We say that \((G_n)_{n \in \mathbb{N}}\) is the dual strong sequence of \((E, F)\) if \(G_1 = F, \beta_n = \beta(E, G_n), G_{n+1} = (E, \beta_n)\), for every positive integer \(n \in \mathbb{N}\). We say that \(G = \cup \{G_n : n \in \mathbb{N}\}\) is the dual strong union of \((E, F)\).

The space \(E\) and its dual strong union \(G\) constitute a dual pair \((E, G)\), enabling it to generate a new dual strong sequence, which leads us to the next definition.

Definition 3.2. Let \((E, F)\) be a dual pair. Let \(N_k = \{0\} \cup \mathbb{N}\). Consider the following structure on \(E^*\) for \(k \in N_k, n \in \mathbb{N}\):

1. \(F = G_{k1}\),
2. \(\beta_{kn} = \beta(E, G_{kn})\),
3. \(G_{k(n+1)}\) is the dual of \((E, \beta_{kn})\),
4. \(G_{k(n+1)} = \cup \{G_{kn} : n \in \mathbb{N}\}\).

We say that \((G_{kn})_{n \in \mathbb{N}}\) is the \(k\)-th dual strong sequence and \(G_{n\cdot k}\) the \(n\)-th dual strong union of \((E, F)\). We say that \((G_{kn})_{n \in \mathbb{N}}\) resp. \((G_{kn})_{n \in \mathbb{N}}\) is the initial dual strong sequence, resp. union, of \((E, F)\) and denote \((G_{kn})_{n \in \mathbb{N}} = (G_{kn})_{n \in \mathbb{N}}\).

Remark 3.1. In the context of Definition 3.2, the following holds for \(k \in N_k, n \in \mathbb{N}\):

(a) the set of subspaces \((G_{kn})\) of \(E^*\) is (well) ordered by inclusion:

\[G_{kn+1} \subseteq G_{kn} \text{ if and only if } (\langle k < m \rangle \vee (\langle k = m \rangle \wedge (n \leq p)))\],

(b) \(\beta_{kn} \leq \beta_{kn+1}\) for any \(k, m, n, p, \) satisfying \(\{\langle k < m \rangle \vee (\langle k = m \rangle \wedge (n \leq p))\} \).

(c) the \(n\)-th dual strong sequence of \((E, F)\) is the initial dual strong sequence of \((E, G_{kn+1})\) for any \(n \in \mathbb{N}\).

Definition 3.3. The ordered set \((G_{kn})\) of Definition 3.2 is named the generalised dual strong sequence of \((E, F)\). The set \(\cup \{G_{kn} : k \in N_k, n \in \mathbb{N}\} = \cup \{G_{kn} : k \in N_k\}\) is the generalised dual strong union of \((E, F)\).

Remark 3.2. Let \((G_{kn})\) be the generalised dual strong sequence of \((E, F)\).

(a) \((G_{kn})\) is the bidual of \((G_{kn}, \beta(E, G_{kn}))\), for any \(k \in N_k, n \in \mathbb{N}\).

(b) For any fixed pair \((k, n)\), \(k \in N_k, n \in \mathbb{N}\), the tail \(\{G_{kn} : m \in \mathbb{N}, p \in \mathbb{N}\}\), such that \((\langle k < m \rangle \vee (\langle k = m \rangle \wedge (n \leq p)))\) is the generalised dual strong sequence of \((E, G_{kn})\).

Definition 3.4. A subspace \(L\) is boundedly completed (quasi-distinctively) in \((E, F)\), if any element (bounded set) of \((E, F)\) is contained in the closure of a bounded subset of \((L, F)\). The space \((E, F)\) is boundedly completed (quasi-distinctively), if it is boundedly completed (quasi-distinctively) in the completion of \((E, F)\).
Boundedly completed and quasi-distinguished spaces were defined and investigated in [12], [13], [33]. Certainly a quasi-distinguished subspace of $E$ is also boundedly completed in $E$. There exists an example of a metrizable, hence boundedly completed, but not quasi-distinguished locally convex space; [2]. Any subspace of a locally convex Fréchet-Urysohn space (i.e., a space in which a subset is closed if and only if it is sequentially closed) is boundedly completed. A discussion and examples of nonmetrizable Fréchet-Urysohn spaces can be found in [7].

**Remark 3.3.** Given the $k$-th dual strong sequence of $(E, F), G_n$, is boundedly completed in $(G_k, \sigma(G_k, E))$, $k \in \mathbb{N}, n \in \mathbb{N}$.

The next remark states that the barreledness of $E$ is the “bottom line” for the dual strong sequence of $(E, F)$. Recall that a locally convex space $E$ is distinguished, if $\beta(E, F) = \beta(E, E')$, or equivalently, if $(E, \beta(E, E'))$ is barreled, [18].

**Remark 3.4.** Using the terms of Definition 3.2, the following statements are equivalent:

(i) there exist $k \in \mathbb{N}, n \in \mathbb{N}$ such that $G_n$ is quasi-distinguished in $(G_k, \sigma(G_k, E));$

(ii) there exist $k \in \mathbb{N}, n \in \mathbb{N}$ such that $(G_n, \beta(G_n, E))$ is distinguished;

(iii) there exist $k \in \mathbb{N}, n \in \mathbb{N}$ such that $G_n = G_m$ for positive integers $m, h, p, r$ satisfying $m, h > k$ and $p, r > n$;

(iv) there exist $k \in \mathbb{N}, n \in \mathbb{N}$ such that $\beta_{mn} = \beta_{mn}$ for positive integers $m, h, p, r$ satisfying $m, h > k$ and $p, r > n$;

(v) there exist $k \in \mathbb{N}, n \in \mathbb{N}$ such that $(E, \beta_{mn})$ is barrelled.

We conclude this section with the following “perpetuum mobile” observation.

**Remark 3.5.** The formal process of generating dual strong sequences and dual strong unions can be initialized and continued by substituting $F = \cup \{G_n : k \in \mathbb{N}, n \in \mathbb{N}\}$ and using the steps of Definition 3.2.

### 4 Bounded sets of the dual strong sequence and dual strong union

For different families $A, B$ of bounded disks in a locally convex space $E$ we define the usual (partial) order relation: $A \leq B$ if and only if for any $A \in A$ there exists $B \in B$ such that $A \subseteq B$. If $A \leq B$ and $B \leq A$ we write $A \approx B$.

For $E_1 \subseteq E_2 \subseteq E_3$, a $\sigma(E, F_1)$-closed bounded disk $B$ of $E$ is $\sigma(E, F_2)$-closed but not necessarily bounded. In a similar fashion, a $\sigma(E, F_2)$-closed bounded disk $B$ is bounded but not necessarily closed in $(E, \sigma(E, F_3))$. By using $A \subseteq B$ ($A \approx B$) instead of $A \leq B$ ($A \approx B$) we indicate that there is no closure related confusion and exactly the same closed bounded disks participate in both $A$ and $B$. 

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The next proposition establishes the connections between the families of type 1, 2, 5, 6 for the generalized dual strong sequence of \((E, F)\). The families of type 3, 4 will be treated in the forthcoming article. For a generalized dual strong sequence \((G_{kn})_k \ni \in \mathbb{N}_0, k \in \mathbb{N}_0, n \in \mathbb{N}_0\), we denote by \(A_{kn}\) a family of bounded disks in \((E, \sigma(E, G_{kn}))\).

**Proposition 4.1.** Let \((G_{kn})_k \ni \in \mathbb{N}_0, k \in \mathbb{N}_0, n \in \mathbb{N}_0\) be the generalized dual strong sequence of \((E, F)\). Then for each \(k \in \mathbb{N}_0\) the following properties hold:

1. \(B_{kn} \supseteq B_{kn+1}\);
2. \(B_{kn} \supseteq B_{kn+1}\);
3. \(W_{kn} \supseteq W_{kn+1}\);
4. \(FIN_{kn} = FIN_{kn+1}\);

Proof. (a) Obviously \(B_{kn} \supseteq B_{kn+1}\). If \(B \in B_{kn}\), then \(B\) is closed and bounded in \((E, B_{kn})\). The topology \(B_{kn}\) is compatible with the duality \((E, G_{kn+1})\), hence \(B \in B_{kn+1}\). On the other side, if \(B \in B_{kn+1}\), then \(B\) is bounded for \((E, B_{kn})\), which is the sole condition for \(B\) to be a closed subset of \(E\) in the topology \(B_{kn}\). Therefore, the closure of \(B\) in \((E, \sigma(E, G_{kn}))\) belongs to \(B_{kn+1}\) and to \(B_{kn+1}\) as well, hence \(B_{kn+1} \subseteq B_{kn+1}\).

(b) If \(B \in B_{kn+1}\), then \(B^c \in G_{kn+1}\), and \(B^c \in G_{kn+1}\). Hence \(B^c \cap G_{kn}\) is a compact subset of \(G_{kn}\), which implies the compactness of \(B^c \cap G_{kn}\). Consequently, \(B\) is a bounded subset of \(G_{kn}\).

(c) A compact disk is compact in any compact topology, by the conclusion of the proof.

(d) is obvious.

Loosely speaking, when the chain \(G_{kn}\) increases in the sense of Remarks 3.1 (a) and 3.5, the bounded families \(B_{kn}\) of \((E, \sigma(E, G_{kn}))\) shrink, however restrained by the \(FIN\) family of \(E\). The members of \(WC_{kn}\) family of \(E\) change their status, becoming noncompact members of the \(H\) family of \(E\). On the other side the bounded families of \((G_{kn}, \sigma(G_{kn}, E))\) flourish, their expansion being restricted by Remark 3.4 and \(E\).

The following proposition describes the structure of bounded sets in the dual strong union.

**Proposition 4.2.** Let \((E, F)\) be a dual pair, \((G_{kn})_k \ni \in \mathbb{N}_0\) the initial dual strong sequence, and \(G\) the initial dual strong union. Any closed bounded disk in \((G, \sigma(G, E))\) is either compact or a countable union of an increasing sequence of compact disks in \((G, \sigma(G, E))\).

Proof. If a bounded closed disk \(A\) contained in some \(G_k\) then the closure of \(A\) in \((G_{kn+1}, \sigma(G_{kn+1}, E))\) is weakly compact in \(G_{kn+1}\) and in \(G\). Otherwise for a \((G, \sigma(G, E))\)-closed bounded disk \(A\) we denote \(A_k = A \cap G_k\). Then \(A = \bigcup A_k, k \in \mathbb{N}\). Since \(A\) is closed in \((G, \sigma(G, E))\), the disk \(A_k\) is closed in \((G_k, \sigma(G_k, E))\) for any \(k \in \mathbb{N}\). If \(A_k\) is the closure of \(A_k\) in \((G_k, \sigma(G_k, E))\), we have \(A_k \subseteq A \subseteq A_{kn+1}\) for each \(n \in \mathbb{N}\) and therefore \(A = \bigcup A_k\). Since \(A_k\) is \((G_k, \sigma(G_k, E))\)-bounded, \(A_k\) is \((G_k, \sigma(G_k, E))\)-compact, hence \(E^c, E\)-complete and therefore compact in \((G, \sigma(G, E))\). We proved that \(A\) is a countable union of an increasing sequence \(A_k\) of compact disks in \((G, \sigma(G, E))\).
Proposition 4.2 and Remark 3.1 (c) yield the following result.

**Proposition 4.3.** Let $G_{\alpha i}$ be the $n$-th dual strong union of a dual pair $(E, F)$. The following statements hold for each $n \in \mathbb{N}$.

(a) Any closed bounded disk in $(G_{\alpha i}, \sigma(G_{\alpha i}, E))$ is either compact or a countable union of an increasing sequence of compact disks.

(b) Any barrel in $(E, \mu(E, G_{\alpha i}))$ is either a $0$-neighborhood or a countable intersection of a decreasing sequence of $0$-neighborhoods.

**Proof.** (a) Follows from Proposition 4.2 and Remark 3.1 (c).

(b) If $A = \cup A_n : n \in \mathbb{N}$ is a closed bounded disk in $G_{\alpha i}$, then $A^* = \cap A_n : n \in \mathbb{N}$ is a barrel and $A_n^*$ is a $0$-neighborhood in $(E, \mu(E, G_{\alpha i}))$. □

### 5 Barreledness and the dual strong union

Suppose $\{U_n : n \in \mathbb{N}\}$ is a sequence of barrels in $E$ such that every $x \in E$ belongs to all but a finite number of $U_n$. Then $U = \cap U_n : n \in \mathbb{N}$ is a barrel in $E$. According to Mazur, $E$ is $C$-barreled if for any sequence $\{U_n : n \in \mathbb{N}\}$ of absolutely convex closed $0$-neighborhood such that for every $x \in E$ there exists $n_0 \in \mathbb{N}$ such that $x \in U_n$ for each $n \geq n_0$, the barrel $U = \cap U_n : n \in \mathbb{N}$ is a $0$-neighborhood, (23), Definition 8.2.6. According to Webb $(E, \tau)$ is $C_0$-barreled if any weakly convergent to zero sequence in $(E, \tau)$ is $\tau$-equi-continuous, (Webb used the term sequentially barreled, see [35]). According to Riesz, $(E, \tau)$ has property $(L)$ if for any absorbing disk $A$ in the finest locally convex topology that coincides on $A$ with $\tau$ Riesz mentioned that property $(L)$ "is a certain weakening of barreledness", (27), p. 180. Saxon and Sanchez Ruiz affiliated the dual local completeness with a weak barreledness condition by proving that a dual locally complete Mackey space is $C$-barreled, (285, Theorem 3.2). Qia consolidated the collection of barreled properties by observing the equivalence of $C$-barreledness, dual local completeness, property $(L)$ and $C_0$-barreledness, extending the equivalence results to the quasi-barreled case as well, (24, Theorems 4 and 5).

The next proposition gives the NBAR, NBN and WC families of the dual strong union. We need the following result of Valdivia : (35, Theorem 6). Let $\{U_n : n \in \mathbb{N}\}$ be an increasing sequence of closed disks in a barreled space $E$ such that $E = \cup U_n : n \in \mathbb{N}$. Then any bounded net is absorbed by some $U_{n_0}$. □

**Proposition 5.1.** Let $(E, F)$ be a dual pair and $G_{\alpha i}$ the $n$-th dual strong union for some $n \in \mathbb{N}$. If $B$ is a closed bounded disk in $(G_{\alpha i}, \sigma(G_{\alpha i}, E))$ such that $E_B$ is barreled then $B$ is weakly compact.

**Proof.** Let $B$ be a closed disk in $(G_{\alpha i}, \sigma(G_{\alpha i}, E))$. By Proposition 4.3, $B$ is either compact or $\cap (\cup A_n : n \in \mathbb{N})$, each $A_n$ being $\sigma(G_{\alpha i}, E)$-compact. If $B = \cup (\cup A_n : n \in \mathbb{N})$ then $E_B = \cup (\cup A_n : n \in \mathbb{N})$ is a normed barrelled space satisfying the conditions of (35, Theorem 6). Hence $B$ is absorbed by some $A_n$ therefore $B$ is $\sigma(G_{\alpha i}, E)$-compact. □
The proposition we just proved claims that $\mathbb{D} = \mathbb{B} = \mathbb{W}$ in any $n$-th dual strong union $(G_n, (\sigma(G_n), E))$, allowing to join these results from [22], p. 180) by suggesting the dual locally $(\forall x)$ barreledness is a certain weakening of $(\forall x)$ barreledness.

**Proposition 5.2.** Let $(E, F)$ be a dual pair and $G_n$ the $n$-th dual strong union for some $n \in \mathbb{N}$. The following statements are equivalent:

(i) $(E, \mu(E, G_n))$ is locally $\alpha$-barreled,

(ii) $(E, \mu(E, G_n))$ is locally completely (or dual $\theta$-complete, see [24]),

(iii) $(E, \mu(E, G_n))$ is $\alpha$-barreled,

(iv) $(E, \mu(E, G_n))$ has property $(L)$,

(v) $(E, \mu(E, G_n))$ is $C$-barreled,

(vi) $(E, \mu(E, G_n))$ is barreled.

Proof. (i) $\Rightarrow$ (vi) follows from Proposition 5.1. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) : see the discussion on page 1433 and Theorem 4 of [24]. (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i) : obvious. □

We say that $(A_n)_{n \in \mathbb{N}}$ is an absorbent sequence in a locally convex space $E$ if it is an increasing sequence of disks whose union $A = \bigcup \{A_n : n \in \mathbb{N}\}$ is absorbing. It is bounded-absorbing if any bounded subset of $E$ is absorbed by some $A_n$.

In [9, Theorem 1] De Wilde and Houot proved that given a dual pair $(E, F)$, any absorbent sequence of closed disks in $E$ is bounded-absorbing for $(E, \sigma(E, F))$, combining [9, Theorem 1] with a mild restriction of having the Banach-Mackey property in $E$ or equivalently in $F$, we conclude that in a Banach-Mackey space any absorbent sequence of closed disks is bounded-absorbing.

If $E$ admits an absorbent sequence $(A_n)_{n \in \mathbb{N}}$ of closed disks, then $E = \bigcup \{nA_n : n \in \mathbb{N}\}$ allowing the UBL related inquiry, [34], see also [23]. Indeed the result of De Wilde-Houot activated a vigorous research, inspiring new concepts, such as the property $(B)$ of Ross, (see the detailed conclusion), or the properties $(L)$ and $(Lb)$ (see [27], p. 180). Given a dual pair $(E, F)$, Theorem 1 of [9] amalgamates the following concepts: $E$ is Banach-Mackey $\Longleftrightarrow E$ has property $(B)$ $\Longleftrightarrow F$ is Banach-Mackey $\Longleftrightarrow F$ has property $(B)$. The paper [9] of De Wilde and Houot appeared almost simultaneously with the famous paper [35] of Valdivia, where similar results were established for barreled spaces. Our next proposition is based on the De Wilde-Houot result. In order to be self-contained we repeat the original proof of [9]. The notations of [9] follow the tradition of [17]. We change some of them in the spirit of [22] and [23].

**Theorem 5.1.** (De Wilde-Houot). Given a dual pair $(E, F)$, any absorbent sequence of closed subsets of $(E, \sigma(E, F))$ is bounded-absorbing in $(E, \sigma(E, F))$.

Proof. (see [9], p. 237). Let $(A_n : n \in \mathbb{N})$ be an absorbent sequence of closed sets in $(E, \sigma(E, F))$ and $B$ be a subset of $E$. Suppose $B \not\subseteq A_n$ for all $n \in \mathbb{N}$. Then there exist $a_n \in B$ and $f_n \in A_n'$ such that $|f_n(a_n)| > 1$. Since $(A_n : n \in \mathbb{N})$ is increasing and $\bigcup \{A_n : n \in \mathbb{N}\}$ is absorbing, $(f_n : n \in \mathbb{N})$ is $\sigma(E, E)$-bounded. If $B$ is strongly
bounded, then $B$ is a compact subset of $F$, hence $B'$ absorbs the set $\{x_n : n \in N\}$. But $\{x_n : n \in N\} \subseteq B'$ and $B'$ is unbounded on $\{x_n : n \in N\}$, hence the contradiction, therefore $B$ is absorbed by some $A_n$.

Proposition 5.3. Let $(G_n), k \in N, n \in N$ be the generalized dual strong sequence of $(E, F)$. Any absorbent sequence of closed disks in $(E, \sigma(E, G_n))$ for some $k \in N, n \in N$ is bounded-absorbed in $(E, \sigma(E, G_{n+1}))$ for any $m, n$, satisfying $G_{n+1} \subseteq G_m$ (i.e., for the tail of $G_{n+1}$).

Proof. Let $(A_n : n \in N)$ be an absorbent sequence of closed disks in $(E, \sigma(E, G_n))$ for some $k \in N, n \in N$. Then any $B \in B_n$, is absorbed by some $A_n$. Hence by Proposition 4.1(a) any $B \in B_{n+1}$ is absorbed by some $A_n$. The conclusion follows by noticing that $B_{n+1} \subseteq B_n$ and the disks $(A_n : n \in N)$ are closed in $(E, \sigma(E, G_{n+1}))$, for any tail $G_{n+1}$ of $G_{n+1}$.

To stress the significance of the generalized dual strong sequence we restate Proposition 5.3 and obtain:

Proposition 5.4. Let $(G_n), k \in N, n \in N$ be the generalized dual strong sequence of $(E, F)$. Any absorbent sequence of closed disks in $(E, \sigma(E, F))$ is bounded-absorbed in $(E, \sigma(E, G_n)), n \geq 1$.

6 The scope of the associated barrelled topology

Given a locally convex nonbarrelled space $(E, \tau)$, we say that $\tau_{\mu}$ is the associated barrelled topology for $\mu$ if $\tau_{\mu}$ is the weakest barrelled topology finer than $\tau$. If $(E, \tau)$ is barrelled, then $\tau_{\mu} = \tau$. The next proposition extends Theorem 1 of [3].

Proposition 6.1. Let $(E, \mu(F), F)$ be a dual pair, $(G_n), n \in N$ the initial dual strong sequence, $G$ the initial dual strong union and $\tau_{\mu}$ the associated barrelled topology for $\mu(F, F)$. Then $\tau_{\mu}$ is the associated barrelled topology for $\mu$ and for $\beta(E, G), n \in N$.

Proof. If $(E, \mu(F, F))$ is barrelled, then $\tau_{\mu} = \mu(F, F)$ and $F = G_n$ for each $n \in N$, therefore $\tau_{\mu} = \mu(F, F) = \beta$. For a nonbarrelled $(E, \mu(F, F))$ we notice that $\tau_{\mu} \geq \beta = \beta(E, F)$. By Remark 3.5, $F = G_n$ is boundedly completed in $(E, \sigma(E, G_n))$, therefore we have $\beta_n \geq \beta(E, G_n)$ which implies $\tau_{\mu} \geq \beta(E, G_n)$. But then $\tau_{\mu} \geq \beta(E, G_n)$, hence $\tau_{\mu} \geq \beta(E, G_n) = \beta_n$. Since $G_n$ is boundedly completed in $(E, G_{n+1}, \sigma(E, G_n))$, we conclude by induction that $\tau_{\mu} \geq \beta_n$ for any $n \in N$.

Let $\tau_{\mu, n}$ be the associated barrelled topology for $\mu(E, F)$. Consider the duals $(E, \mu(E, F), \tau_{\mu}) = \mu(E, \tau_{\mu}) = \tau_{\mu}$. If some $k \in N$, $(E, \beta_n)$ is barrelled, Remark 3.4 provides $G_n = G = F$ for each $n \geq k$, therefore $\tau_{\mu} = \beta_k = \mu(E, G) = \tau_{\mu, n}$. If $(E, \beta_k)$ is nonbarrelled for each $n \in N$, then $G_n \subseteq F$ and therefore $F \subseteq G_n$. It follows, that $\tau_{\mu} \geq \mu(E, G)$, and therefore $\tau_{\mu} \geq \tau_{\mu, n}$. On the other hand, $F \subseteq G_n$, therefore $\tau_{\mu} \subseteq \tau_{\mu, n}$, and we conclude that $\tau_{\mu} = \tau_{\mu, n}$.

The next proposition illuminates the horizon of the associated barrelled topology.
Proposition 6.2. Let $(G_k), k \in N, n \in N,$ be the generalized dual strong sequence for a dual pair $(E, F)$. Let $\tau_{\omega}$ be the associated barrelled topology for $\mu(E, F)$. Then $\tau_{\omega}$ is the associated barrelled topology for any $\beta(E, G_{\omega}), k \in N, n \in N$.

Proof. By Proposition 6.1, $\tau_{\omega}$ is the associated barrelled topology for $\beta(E, G_{\omega}), n \in N.$ Suppose that for some $k \in N$ the topology $\tau_{\omega}$ is the associated barrelled topology for $\beta(E, G_{\omega})$ and $\beta(E, G_{\omega+1}), n \in N.$ Then keeping in mind Remark 3.1 (c) and using the arguments of Proposition 6.2, we conclude that $\tau_{\omega}$ is the associated barrelled topology for $\beta(E, G_{\omega+1})$ and $\beta(E, G_{\omega+2}), n \in N.$ □

Combining Remark 3.4 with Proposition 6.2 we obtain:

Proposition 6.3. In the setting of Proposition 6.2, the following is equivalent.

(i) $\tau_{\omega}$ is the associated barrelled topology for $\mu(E, F)$;

(ii) $\tau_{\omega}$ is the associated barrelled topology for any $\beta(E, G_{\omega}), k \in N, n \in N$;

(iii) $\tau_{\omega}$ is the associated barrelled topology for any $\beta(E, G_{\omega}), k \in N, n \in N$.

7 A remark on Perez Carreras - Bonet - Qiu completeness scheme

In [24], Qiu extends the completeness scheme of Perez Carreras and Bonet [23], describing the relationship between different notions of completeness, each connected to some kind of compactness and/or barrelledness. We suggest to amplify the scheme of Qiu with the notions of $B$-completeness, $B_{\omega}$-completeness and local barrelledness. A locally convex space $E$ is $B$-complete ($B_{\omega}$-complete) if every (dense) subspace $l \subset E'$ with a closed intersection $L \cap U'$ for any closed equicontinuous disk $U'$ of $E'$, is itself closed in $E'$ (coincides with $E'$). The $B$- and $B_{\omega}$-completeness are related to the minimal barrelled topologies via the Closed Graph - Open Mapping Theorem of Ptak. We also mention the correspondence between the barrelled topologies on $E$ and its closed subspaces, (see [11], [13], [15]). As to the local barrelledness notion, we believe that Proposition 5.1 and 5.2 of this article justify the inclusion.

Qiu [24] begins the scheme with completeness and concludes with local completeness. Seeing the revised scheme in the comprehensive context of Baire categorization, we suggest starting from $B$-completeness and closing with the local barrelledness.

$B$-complete $\Rightarrow B_{\omega}$-complete $\Rightarrow$ complete $\Rightarrow$ continuing as in [24] $\Leftrightarrow \omega \Rightarrow B$.

The newly introduced parts of the scheme are irreversible. Examples of complete but not $B_{\omega}$-complete spaces are trivial, along with examples of locally barrelled but not locally complete spaces. An example of a $B_{\omega}$-complete space which is not $B$-complete was obtained by Vellhovia in [37].
8 Questions

We conclude this article with some questions and suggestions for further research. It seems natural to investigate the conditions for terminating the generalised dual strong sequence in specific locally convex spaces, thus attaining the associated barrelled topology. It also seems promising to combine the fine-grained research done by Rosenthal and its implications for differentiability and boundedness in tangibly locally convex spaces with the general results of this article (see [3, 4, 10, 16] for insights and references).

Given a dual pair (E, F), it seems natural to suggest that differentiability of a strongly bounded function on F by the dual enclosure of F, for example. Banach-Mackey or separability, it looks interesting because the bounded sets can be drastically transformed by a binary (see [14] on an illustration). In view of Proposition 4.3 another reasonable question arises: does a closed bounded disk in the dual strong union possess a quasi-weak drop property? We refer to [19] and [26] for the necessary setting, definitions and references.

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References


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