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SPECTRAL MEASURES IN CLASSES OF FRÉCHET SPACES

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Abstract

A detailed investigation is made of the canonical atomic spectral measure defined in such Fréchet spaces as the Köthe echelon sequence spaces and the sequence spaces ℓ^{p+}, as well as the (non-atomic) "natural" spectral measures in such Fréchet spaces of measurable functions as the space of locally p-th power integrable functions on $\mathbb{R}$ and L_{p-} on $[0,1]$. Of particular interest are questions concerned with the range of the spectral measure, whether or not it has finite variation (for certain operator topologies), the Radon-Nikodým property of the underlying spaces involved and, most importantly, does the spectral measure admit *unbounded* integrable functions?

1 Introduction and preliminaries

The theory of Boolean algebras of projections/spectral measures in Banach spaces was initiated by W. Bade, N. Dunford and others, [12], and is by now well understood. In contrast, there is a distinct lack of concrete, non-trivial examples in the non-normable setting, even within the class of *Fréchet* (locally convex) spaces. An attempt to rectify this (to some extent) can be found in [5]. The aim of this paper is two-fold. Firstly, we wish to summarize the main results of [5] and secondly, to expand on these results and elaborate further on some closely related topics. In order to do so, we begin with some general notation and definitions, so that the questions (some answers) and examples can be properly formulated.

Let X be a locally convex Hausdorff space (briefly, lcHs) and $L(X)$ denote the space of all continuous linear operators from X to itself. The space $L(X)$ is denoted by $L_s(X)$ (resp. $L_b(X)$) when it is equipped with the topology τ_s (resp. τ_b) of uniform

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convergence on all finite (resp. bounded) subsets of X. A function $m : \Sigma \to Y$ (with Y a lcHs) is called a *vector measure* if it is σ-additive; here Σ is a σ-algebra of subsets of some non-empty set Ω. If Y coincides with $L_s(X)$ or $L_b(X)$, for some lcHs X, then m is called an *operator-valued measure* (in X). An operator-valued measure $P : \Sigma \to L_s(X)$ is a *spectral measure* if it is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$ for all $E, F \in \Sigma$) and if $P(\Omega) = I$, the identity operator on X. This is an extension of the notion of the resolution of the identity of a normal operator in Hilbert space. An operator-valued measure $Q : \Sigma \to L_s(X)$ is called *boundedly σ-additive* (or τ_b-countably additive) if it is σ-additive as an $L_b(X)$-valued measure. For X a Banach space and $P : \Sigma \to L_s(X)$ a spectral measure, the bounded σ-additivity of P can only occur in trivial cases, due to the fact that $\|R\| \geq 1$ for every non-zero projection $R \in L(X)$. In the setting of non-normable X the situation can be quite different, [18], [23].

We will be exclusively interested in certain Fréchet spaces X. Since $L_s(X)$ is then quasicomplete, it follows from general vector measure theory that the range of any $L_s(X)$-valued operator measure is a relatively weakly compact subset of $L_s(X)$, [14, IV Theorem 6.1], that is, relatively compact for the weak operator topology. In particular, the barrelledness of X then ensures that the range is always an *equicontinuous* subset of $L(X)$. Since $L_b(X)$ is also quasicomplete, [15, §39 Theorem 6.5], the range of any $L_b(X)$-valued operator measure is relatively weakly compact in $L_b(X)$. For $L_s(X)$-valued spectral measures, their range is always an (equicontinuous) Bade σ-complete Boolean algebra of projections, [18, Lemma 3.1]. The range is a Bade complete Boolean algebra of projections if and only if it is a closed subset of $L_s(X)$, [18, Proposition 3.5]. For instance, if X is separable or there exists a cyclic vector, then this is always the case, [18, Proposition 3.9]. If the spectral measure happens to be purely atomic with countably many atoms, then its range is actually a compact subset of $L_s(X)$; apply Theorem 10 of [13] in the quasicomplete lcHs $E := L_s(X)$.

Let Y be a lcHs with topology determined by a family of continuous seminorms $\mathcal{N}$. Let $Y/q^{-1}(\{0\})$ be the quotient normed space determined by $q \in \mathcal{N}$ and Y_q denote its Banach space completion. The norm in Y_q is denoted by $\|\cdot\|_q$ and the canonical quotient map of Y onto $Y/q^{-1}(\{0\})$ is denoted by ρ_q. Of course, for elements $y \in Y \hookrightarrow Y_q$, we have $\|y\|_q = q(y)$. Given any Y-valued vector measure m defined on a measurable space (Ω, Σ), the continuity of ρ_q ensures that $m_q := \rho_q \circ m$ is a vector measure on Σ with values in $Y/q^{-1}(\{0\}) \hookrightarrow Y_q$, for each $q \in \mathcal{N}$. For the definition of the *variation measure* $|m_q| : \Sigma \to [0, \infty]$ of the Banach-space-valued measure m_q, see [10, pp.2-3]. The variation $|m_q|$ is called finite if $|m_q|(\Omega) < \infty$. We say that the vector measure m has *finite variation* if m_q has finite variation for every $q \in \mathcal{N}$.

The only lcH-spaces Y relevant to this paper will be X, $L_s(X)$ and $L_b(X)$, where X is a Fréchet space. In this case, if $\{q^{(n)}\}_{n=1}^{\infty}$ is any sequence of continuous seminorms determining the topology of X, then the topology τ_s of $L_s(X)$, respectively τ_b of $L_b(X)$, is determined by the family of seminorms

$$q_x^{(n)} : T \mapsto q^{(n)}(Tx), \qquad T \in L_s(X), \tag{1}$$

for all $x \in X$ and $n \in \mathbb{N}$, respectively,

$$q_B^{(n)} : T \mapsto \sup_{x \in B} q^{(n)}(Tx), \qquad T \in L_b(X), \tag{2}$$

for all bounded sets $B \subseteq X$ and $n \in \mathbb{N}$.

Let Q be any $L_s(X)$-valued spectral measure defined on some measurable space (Ω, Σ). A Σ-measurable function $f : \Omega \to \mathbb{C}$ is called *Q-integrable* if there exists an operator $\int_\Omega f dQ \in L(X)$, necessarily unique, such that f is integrable with respect to each complex measure $\langle Qx, x' \rangle : E \mapsto \langle Q(E)x, x' \rangle$, $E \in \Sigma$, for $x \in X$ and $x' \in X'$ (the continuous dual space of X), and

$$\left\langle \left(\int_\Omega f \, dQ \right) x, x' \right\rangle = \int_\Omega f \, d\langle Qx, x' \rangle, \qquad x \in X, \ x' \in X'.$$

The operator $\int_E f dQ := Q(E)(\int_\Omega f dQ)$ then satisfies, for each $E \in \Sigma$,

$$\left\langle \left(\int_E f \, dQ \right) x, x' \right\rangle = \int_E f \, d\langle Qx, x' \rangle, \qquad x \in X, \ x' \in X'.$$

The space of all Q-integrable functions is denoted by $\mathscr{L}(Q)$. An element $f \in \mathscr{L}(Q)$ is called *Q-null* if $\int_\Omega f dQ = 0$. Each continuous seminorm $q_x^{(n)}$ of the form (1) determines a seminorm $q_x^{(n)}(Q)$ in $\mathscr{L}(Q)$ via the formula

$$q_x^{(n)}(Q) : f \mapsto \sup_{E \in \Sigma} q^{(n)} \left(\left(\int_E f \, dQ \right) x \right), \qquad f \in \mathscr{L}(Q). \tag{3}$$

The family of seminorms $\{q_x^{(n)}(Q) : n \in \mathbb{N}, x \in X\}$ makes $\mathscr{L}(Q)$ into a lcs. The quotient space of $\mathscr{L}(Q)$, modulo the space of all Q-null functions, is a lcHs which is denoted by $\mathscr{L}^1(Q)$. A Σ-measurable function $f : \Omega \to \mathbb{C}$ is called *Q-essentially bounded* if

$$\|f\|_Q := \inf\{\sup\{|f(w)| : w \in E\} : E \in \Sigma, \ P(E) = I\} < \infty.$$

The Banach algebra of all (equivalence classes of) Q-essentially bounded functions is denoted by $\mathscr{L}^\infty(Q)$. Since $L_s(X)$ is quasicomplete, we have $\mathscr{L}^\infty(Q) \subseteq \mathscr{L}^1(Q)$, [14, p.26], with a continuous inclusion.

Some basic questions concerning a particular spectral measure $Q : \Sigma \to L_s(X)$, with X a particular Fréchet space, are the following ones.

Q1. What can be said about the range $Q(\Sigma) := \{Q(E) : E \in \Sigma\}$ as a subset of $L_s(X)$?

Q2. Does Q have finite variation in $L_s(X)$?

Q3. Is Q boundedly σ-additive and, in the case when it is, does Q have finite variation in $L_b(X)$?

Q4. In relation to Q2 and Q3 what is the connection with the Radon-Nikodým property of X, $L_s(X)$ and/or $L_b(X)$?

Q5. What locally convex space properties does $\mathscr{L}^1(Q)$ possess and is the containment $\mathscr{L}^\infty(Q) \subseteq \mathscr{L}^1(Q)$ strict or not (it is always an equality when X is a Banach space)?

For X a Banach space, the answers are essentially known and can be found in [12],[26], and the references therein. For X a non-normable Fréchet space, only partial answers are known in general and, as alluded to above, there is a distinct lack of good examples available. The aim here is to discuss the results and examples of [5] in relation to the above questions and to elaborate further. It is to be expected (and is indeed the case) that both the geometric and analytic properties of the Fréchet space X play a crucial role as do individual features of the particular spectral measure Q involved.

To proceed further we need to introduce various (particular) Fréchet spaces. Let Γ denote either $\mathbb{N}$ or $\mathbb{N} \times \mathbb{N}$ or any infinite subset of these. An increasing sequence $A = (a_n)_{n \in \mathbb{N}}$ of strictly positive functions $a_n : \Gamma \to (0, \infty)$ is called a *Köthe matrix* on Γ, where by increasing we mean $0 < a_n(i) \leq a_{n+1}(i)$ for all $i \in \Gamma$ and $n \in \mathbb{N}$. Of course, each $a_n \in \mathbb{C}^\Gamma$, for $n \in \mathbb{N}$. Elements $x \in \mathbb{C}^\Gamma$ are denoted by $x = (x_i)$. Interpreting elements of $\mathbb{C}^\Gamma$ as functions on Γ, it is clear what the notation such as $a_n x$, for $n \in \mathbb{N}$ and $x \in \mathbb{C}^\Gamma$, and a_m/a_n for $m, n \in \mathbb{N}$, means. To each $p \in [1, \infty)$ is associated the linear space

$$\lambda_p(A) := \{x \in \mathbb{C}^\Gamma : q_n^{(p)}(x) := \Big(\sum_{i \in \Gamma} a_n(i)|x_i|^p\Big)^{1/p} < \infty, \quad \text{for all } n \in \mathbb{N}\}.$$

We also require the linear space

$$\lambda_0(A) := \{x \in \mathbb{C}^\Gamma : a_n x \in c_0(\Gamma), \quad \text{for all } n \in \mathbb{N}\},$$

equipped with the seminorms $q_n^{(0)}(x) := \sup_{i \in \Gamma} a_n(i)|x_i|$, for each $n \in \mathbb{N}$. The spaces $\lambda_p(A)$ for $p \in \{0\} \cup [1, \infty)$, are called *Köthe echelon spaces* (of order p): they are all separable Fréchet spaces relative to the increasing sequence of seminorms $q_1^{(p)} \leq q_2^{(p)} \leq \ldots$. For the general theory of such spaces we refer to [1],[2],[16], for example. We recall a particular class of Köthe matrices A, the so called *Köthe-Grothendieck* (briefly, KG-) matrices. Here $\Gamma = \mathbb{N} \times \mathbb{N}$ and $a_n : \Gamma \to \mathbb{C}$, for $n \in \mathbb{N}$, must satisfy:

$$a_n(i,j) = 1, \text{ for all } j, n \in \mathbb{N} \text{ and } i > n. \tag{KG-1}$$

$$\sup_{j \in \mathbb{N}} a_n(n,j) = \infty, \text{ for all } n \in \mathbb{N}. \tag{KG-2}$$

$$a_p(i,j) = a_q(i,j), \text{ for all } i, j \in \mathbb{N} \text{ and all } p, q \geq i. \tag{KG-3}$$

The original KG-matrix corresponds to

$$a_n(i,j) = \begin{cases} j & \text{for } i \leq n \text{ and } j \in \mathbb{N} \\ 1 & \text{for } i > n \text{ and } j \in \mathbb{N}, \end{cases} \qquad n \in \mathbb{N}.$$

Another class of Fréchet spaces of relevance consists of the spaces $\ell^{p+} := \bigcap_{q>p} \ell^q$, for $p \in [1, \infty)$. Each one is a separable Fréchet space when equipped with the sequence of seminorms given by $q_{k,p}(x) := \big(\sum_{n=1}^\infty |x_n|^{\beta(k)}\big)^{1/\beta(k)}$, for $x \in \ell^{p+}$, where $\beta(k) := p + \frac{1}{k}$ for $k \in \mathbb{N}$. This class of spaces has been thoroughly investigated in [17].

All of the above sequence spaces are contained in the Fréchet space $\omega = \mathbb{C}^\mathbb{N}$, consisting of all $\mathbb{C}$-functions on $\mathbb{N}$ equipped with the product topology.

For each $p \in [1, \infty)$, let $L^p_{\text{loc}}(\mathbb{R})$ denote the space all all (equivalence classes of) Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $q_p^{(n)}(f) := \Big(\int_{-n}^n |f(t)|^p \, dt\Big)^{1/p} < \infty$ for every $n \in \mathbb{N}$. Each space $L^p_{\text{loc}}(\mathbb{R})$ is a separable Fréchet space when equipped with the sequence of seminorms $q_p^{(1)} \leq q_p^{(2)} \leq \ldots$.

Finally, we recall the separable Fréchet spaces $L_{p-} := \bigcap_{1 \le r < p} L^r([0,1])$, for $p \in (1, \infty)$, equipped with the seminorms $q_{p,m}(f) := \|f\|_{\beta(m)}$ $= \left(\int_0^1 |f(t)|^{\beta(m)}\, dt\right)^{1/\beta(m)}$ for every $f \in L_{p-}$ and any increasing sequence $1 \le \beta(m) \uparrow p$ as $m \to \infty$. We have, with continuous inclusions, that $L^p([0,1]) \hookrightarrow L_{p-} \hookrightarrow L^r([0,1])$ for every $1 \le r < p$. Each of the spaces L_{p-}, for $p \in (1, \infty)$, is reflexive und none of them is Montel. For further properties of this class of Fréchet spaces we refer to [6].

Some relevant spectral measures in the above Fréchet spaces are as follows. If λ denotes one of the sequence spaces $\lambda_p(A)$, for $p \in \{0\} \cup [1, \infty)$, or one of the sequence spaces ℓ^{p+}, for $p \in [1, \infty)$, then the set function given by

$$P(E) : x \mapsto x\chi_E, \qquad x \in \lambda, \tag{4}$$

for $E \in 2^{\mathbb{N}}$, defines a spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda)$; it is called the *canonical spectral measure* in λ, [19]. For each $p \in [1, \infty)$, the set function given by

$$\widehat{P}(E) : f \mapsto f\chi_E, \qquad f \in L^p_{\mathrm{loc}}(\mathbb{R}), \tag{5}$$

for $E \in \mathscr{B}(\mathbb{R})$ (the σ-algebra of Borel subsets of $\mathbb{R}$), defines a spectral measure $\widehat{P} : \mathscr{B}(\mathbb{R}) \to L_s\left(L^p_{\mathrm{loc}}(\mathbb{R})\right)$. Similarly, for each $p \in (1, \infty)$, the set function given by

$$\widetilde{P}(E) : f \mapsto f\chi_E, \qquad f \in L_{p-} \tag{6}$$

for $E \in \mathscr{B}$ (the σ-algebra of Borel subsets of $[0,1]$), defines a spectral measure $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$. The canonical spectral measures P given by (4) are all *purely atomic* with countably many atoms (indeed, $P(E) = \sum_{n \in E} P(\{n\})$ for $E \in 2^{\mathbb{N}}$), whereas the spectral measures $\widehat{P}$ and $\widetilde{P}$ (as given by (5) and (6)) have no atoms.

2 Q1: The range of spectral measures

For the spectral measures (4), (5) and (6), as defined in the appropriate Fréchet space, we conclude from the earlier (general) remarks of Section 1, together with the fact that all the Fréchet spaces involved are *separable*, that their ranges $P(2^{\mathbb{N}})$, $\widehat{P}(\mathscr{B}(\mathbb{R}))$ and $\widetilde{P}(B)$ are all *Bade complete* Boolean algebras of projections.

It was noted in [5, Proposition 2.10] that every spectral measure P (as given by (4)) has a *cyclic vector*. The same is true of the spectral measures $\widehat{P}$ (resp. $\widetilde{P}$), as given by (5) (resp. (6)); the constant function $\mathbf{1}$ on $\mathbb{R}$ (resp. on $[0,1]$) is a cyclic vector (as the $\mathscr{B}(\mathbb{R})$-simple (resp. $\mathscr{B}$-simple) functions are dense in each space $L^p_{\mathrm{loc}}(\mathbb{R})$ (resp. L_{p-})).

Since each P (as given by (4)) has countably many atoms, it follows from earlier remarks that $P(2^{\mathbb{N}})$ is a *relatively compact* (even compact) subset of $L_s(\lambda)$. This is *not* the case for $\widehat{P}$ and $\widetilde{P}$ as given by (5) and (6). For, if $\widehat{P}(\mathscr{B}(\mathbb{R}))$ was relatively compact in $L_s(L^p_{\mathrm{loc}}(\mathbb{R}))$, then continuity of the map $T \mapsto T\chi_{[0,1]}$ from $L_s(L^p_{\mathrm{loc}}(\mathbb{R}))$ into $L^p_{\mathrm{loc}}(\mathbb{R})$ would imply that $\{\widehat{P}(E)\chi_{[0,1]} = \chi_E : E \in \mathscr{B}\}$ is relatively compact in $L^p_{\mathrm{loc}}(\mathbb{R})$. But, the relative topology of $L^p_{\mathrm{loc}}(\mathbb{R})$ in $\widehat{P}([0,1])L^p_{\mathrm{loc}}(\mathbb{R}) \subseteq L^p([0,1])$ is precisely that of the Banach space $L^p([0,1])$. Since $L^p([0,1]) \hookrightarrow L^1([0,1])$, the set $\{\chi_E : E \in \mathscr{B}\}$ would be relatively compact in $L^1([0,1])$, which is not the case, [10, Example 2, p.61]. So,

$\widehat{P}(\mathscr{B}(\mathbb{R}))$ is not relatively compact in $L_s(L^p_{\text{loc}}(\mathbb{R}))$ for every $p \in [1,\infty)$. Similarly, if $\widetilde{P}(\mathscr{B})$ was relatively compact in $L_s(L_{p-})$, then continuity of the map $T \mapsto T\mathbf{1}$ from $L_s(L_{p-})$ into L_{p-} would imply that $\{\widetilde{P}(E)\mathbf{1} = \chi_E : E \in \mathscr{B}\}$ is relatively compact in L_{p-}. Since $L_{p-} \hookrightarrow L^1([0,1])$, it would again follow that $\{\chi_E : E \in \mathscr{B}\}$ is relatively compact in $L^1([0,1])$, which is not the case. So, $\widetilde{P}(\mathscr{B})$ is not relatively compact in $L_s(L_{p-})$, for every $p \in (1,\infty)$.

3 Q2: Finite τ_s-variation

We begin with the following result.

Proposition 1 (i) *The canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\ell^{p+})$, as given by (4), fails to have finite variation for every $p \in [1,\infty)$.*
(ii) *The spectral measure $\widehat{P} : \mathscr{B}(\mathbb{R}) \to L_s(L^p_{\text{loc}}(\mathbb{R}))$, as given by (5), fails to have finite variation for every $p \in (1,\infty)$ but, does have finite variation in $L_s(L^1_{\text{loc}}(\mathbb{R}))$.*
(iii) *The spectral measure $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$, as given by (6), fails to have finite variation for every $p \in (1,\infty)$.*

Proof. Part (i) is Proposition 4.7 of [5].

(ii) Let $p = 1$. Consider a typical seminorm (see (1)) determining the topology of $L_s(L^1_{\text{loc}}(\mathbb{R}))$, say $q(T) := \int_{-n}^{n} |(Tf)(t)|\, dt$, $T \in L(L^1_{\text{loc}}(\mathbb{R}))$ for some fixed (but arbitrary) $n \in \mathbb{N}$ and $f \in L^1_{\text{loc}}(\mathbb{R})$. Let $\{E(m)\}_{m=1}^{k}$ be any Borel partition of $\mathbb{R}$. Then

$$\sum_{m=1}^{k} q\left(\widehat{P}(E(m))\right) = \sum_{m=1}^{k} \int_{-n}^{n} |f(t)\chi_{E(m)}(t)|\, dt \le \int_{-n}^{n} |f(t)|\, dt < \infty$$

and so, $|\widehat{P}|_q(\mathbb{R}) < \infty$. Accordingly, $\widehat{P}$ has finite variation in $L_s(L^1_{\text{loc}}(\mathbb{R}))$.

For $p > 1$, let $n = 1$ and $f = \chi_{[0,1]}$. Then

$$q(T) := \left(\int_{-1}^{1} |(T\chi_{[0,1]})(t)|^p\, dt\right)^{1/p}, \qquad T \in L_s(L^p_{\text{loc}}(\mathbb{R})),$$

is a continuous seminorm of the form (1). For the partition $E(m) := [\frac{(m-1)}{k}, \frac{m}{k}]$, with $1 \le m \le k$, and $E(0) = \mathbb{R} \setminus [0,1]$, we have

$$\sum_{m=0}^{k} q\left(\widehat{P}(E(m))\right) \ge \sum_{m=1}^{k} \left(\int_{0}^{1} |\chi_{E(m)}(t)|^p\, dt\right)^{1/p} = \sum_{m=1}^{k} k^{-1/p} = k^{1/p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Since $k^{1/p'} \to \infty$ as $k \to \infty$, it follows that $|\widehat{P}|_q(\mathbb{R}) = \infty$ and hence, $\widehat{P}$ does not have finite variation in $L_s(L^p_{\text{loc}}(\mathbb{R}))$.

(iii) For any fixed (but arbitrary) $r \in (1,p)$ and with $f = \mathbf{1}$, we see from (1) that $q(T) := \left(\int_0^1 |(T\mathbf{1})(t)|^r\, dt\right)^{1/r}$, for $T \in L_s(L_{p-})$, is a continuous seminorm in $L_s(L_{p-})$. For the partition $\{E(m)\}_{m=1}^{k}$ of $[0,1]$ as given in the proof of part (ii) we have, by an analogous calculation as in (ii), that $\sum_{m=1}^{k} q(\widetilde{P}(E(m))) = k^{1/r'} \to \infty$ as $k \to \infty$. So, $|\widetilde{P}|_q([0,1]) = \infty$ and hence, $\widetilde{P}$ does not have finite variation in $L_s(L_{p-})$. □

The finite variation of the canonical spectral measure in Köthe echelon spaces can also be precisely described. For τ_s the complete answer is given by the following result, [5, Proposition 4.2].

Proposition 2 *Let A be a Köthe matrix.*

(i) *Let $p \in \{0\} \cup (1, \infty)$. Then the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_p(A))$, as given by (4), has finite variation if and only if $\lambda_p(A)$ is nuclear.*

(ii) *The canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_1(A))$ always has finite variation.*

Characterizations of when $\lambda_p(A)$ is nuclear are known, [16, Proposition 28.16]. As noted in Remark 4.3 of [5], the space $\lambda_1(A)$ is *not* always nuclear (eg. for any KG-matrix this is the case).

As indicated by Proposition 2, nuclearity plays a fundamental role for this class of examples. This will again be seen in relation to Q3. Accordingly, the following result is relevant; it was stated (without proof) in [5, Proposition 4.8]. For the sake of completeness we now include a proof.

Proposition 3 *For a Fréchet space X the following assertions are equivalent.*

(i) *X is nuclear.*

(ii) *$L_s(X)$ is nuclear.*

(iii) *$L_b(X)$ is nuclear.*

Proof. (i) ⇒ (iii). Since Fréchet spaces are dual nuclear (i.e. their strong dual is nuclear), [21, p.78], it follows from [21, Proposition 5.5.1] that $L_b(X)$ is nuclear.

(iii) ⇒ (i) and (ii) ⇒ (i) follow from the facts that X is a closed subspace of both $L_s(X)$ and of $L_b(X)$ and that a subspace of a nuclear space is again nuclear, [21, Proposition 5.1.1].

(i) ⇒ (ii). Fix $x_1, \ldots, x_t \in X$ and an absolutely convex neighbourhood U of 0 in X. Choose elements $u_1, \ldots, u_t \in X'$ such that $\langle x_j, u_i \rangle = \delta_{ij}$ for $1 \le i, j \le t$. Since X is nuclear, there is an absolutely convex neighbourhood V of 0 in X and a sequence $\{b_n\}_{n=1}^{\infty} \subseteq X'$ satisfying $\sum_{n=1}^{\infty} q_{V^\circ}(b_n) < \infty$ (where $V^\circ \subseteq X'$ is the *polar* of V and $q_{V^\circ}(x') := \sup_{x \in V} |\langle x, x' \rangle|$) such that

$$|\langle y, b \rangle| \le q_U(y) \le \sum_{n=1}^{\infty} |\langle y, b_n \rangle|, \qquad y \in X, \tag{7}$$

for all $b \in U^\circ$, [21, Proposition 4.1.4]. Here q_U denotes the *Minkowski functional* of U.

For $1 \le m \le t$ and $n \in \mathbb{N}$, define a linear functional $A_{m,n} : L_s(X) \to \mathbb{C}$ by $T \mapsto \langle T x_m, b_n \rangle$ for $T \in L_s(X)$. It is immediate from (1) that each $A_{m,n}$ is τ_s-continuous, that is, $A_{m,n} \in (L_s(X))'$. For the absolutely convex neighbourhood of 0 in $L_s(X)$ given by

$$W := \{S \in L(X) : S x_m \in V \text{ for all } 1 \le m \le t\}$$

we have, for each $1 \le m \le t$ and $n \in \mathbb{N}$, that

$$q_{W^\circ}(A_{m,n}) = \sup_{S \in W} |\langle S x_m, b_n \rangle| \le \sup_{y \in V} |\langle y, b_n \rangle| = q_{V^\circ}(b_n).$$

Accordingly,

$$\sum_{m=1}^{t}\sum_{n=1}^{\infty} q_{W^\circ}(A_{m,n}) \le \sum_{m=1}^{t}\sum_{n=1}^{\infty} q_{V^\circ}(b_n) < \infty.$$

Define

$$\widetilde{W} := \{T \in L(X) : Tx_m \in U \text{ for all } 1 \le m \le t\},$$

which is also an absolutely convex neighbourhood of 0 in $L_s(X)$, and let $q_{\widetilde{W}}$ be its Minkowski functional. It follows from (7) that

$$\begin{aligned} q_{\widetilde{W}}(T) &= \max_{1\le m\le t} \sup_{b\in U^\circ} |\langle Tx_m, b\rangle| \le \max_{1\le m\le t} \sum_{n=1}^{\infty} |\langle Tx_m, b_n\rangle| \\ &= \max_{1\le m\le t} \sum_{n=1}^{\infty} |\langle T, A_{m,n}\rangle| \le \sum_{m=1}^{t}\sum_{n=1}^{\infty} |\langle T, A_{m,n}\rangle| \end{aligned}$$

for every $T \in L(X)$. By [21, Proposition 4.1.4], applied in the lcHs $L_s(X)$, it follows that $L_s(X)$ is nuclear. □

4 Q3: Bounded σ-additivity and finite τ_b-variation

Concerning the bounded σ-additivity of any $L(X)$-valued operator measure we have the following general result. It is formulated in [5, Proposition 3.1(i)] for sequence spaces, but the same proof applies here.

Proposition 4 *Let X be a Fréchet Montel space. Then every $L_s(X)$-valued operator measure is boundedly σ-additive in $L_b(X)$.*

For the canonical spectral measure in Köthe echelon spaces the previous proposition is optimal, as seen by the following result; see [5, Corollary 3.2(ii)].

Proposition 5 *For some (all) $p \in \{0\}\cup[1,\infty)$ and any Köthe matrix A, the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\lambda_p(A))$ is boundedly σ-additive in $L_b(\lambda_p(A))$ if and only if $\lambda_p(A)$ is a Montel space (equivalently, $\lambda_1(A)$ is reflexive).*

Characterizations of when $\lambda_p(A)$ is Montel are known, [16, p.329 & p.334]. Moreover, for any KG-matrix A it follows that P *fails* to be boundedly σ-additive in $L_b(\lambda_p(A))$ for every $p \in \{0\} \cup [1,\infty)$; see Corollary 3.2(ii) of [5].

Proposition 6 (i) *For every $p \in [1,\infty)$, the canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\ell^{p+})$ fails to be boundedly σ-additive in $L_b(\ell^{p+})$.*

(ii) *For every $p \in [1,\infty)$, the spectral measure $\widehat{P} : \mathscr{B}(\mathbb{R}) \to L_s(L^p_{\mathrm{loc}}(\mathbb{R}))$, as given by (5), fails to be boundedly σ-additive in $L_b(L^p_{\mathrm{loc}}(\mathbb{R}))$.*

(iii) *For every $p \in (1,\infty)$, the spectral measure $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$, as given by (6), is boundedly σ-additive in $L_b(L_{p-})$.*

Proof. (i) is Corollary 3.2(i) of [5].

(ii) Let $B := \{f \in L^p_{\text{loc}}(\mathbb{R}) : f = 0 \text{ in } \mathbb{R} \setminus [-1,1] \text{ and } q_p^{(1)}(f) \leq 1\}$, where $q_p^{(1)}(f) := \left(\int_{-1}^{1} |f(t)|^p \, dt\right)^{1/p}$, for $f \in L^p_{\text{loc}}(\mathbb{R})$, is a continuous seminorm in $L^p_{\text{loc}}(\mathbb{R})$. Then B is a bounded subset of $L^p_{\text{loc}}(\mathbb{R})$ because $q_p^{(n)}$ coincides with $q_p^{(1)}$ on B, for every $n \in \mathbb{N}$. For every Borel subset $E \subseteq [-1,1]$ of positive Lebesgue measure, the continuous seminorm q on $L_b(L^p_{\text{loc}}(\mathbb{R}))$ – see (2) – given by $q(T) := \sup_{f \in B} q_p^{(1)}(Tf)$, for $T \in L(L^p_{\text{loc}}(\mathbb{R}))$, satisfies

$$q\left(\widehat{P}(E)\right) = \sup_{f \in B} \left(\int_{-1}^{1} |\chi_E(t) f(t)|^p \, dt\right)^{1/p} = 1.$$

Accordingly, $\widehat{P}$ cannot be boundedly σ-additive in $L_b(L^p_{\text{loc}}(\mathbb{R}))$.

(iii) A typical seminorm for τ_b (see (2)) is given by

$$q(T) := \sup_{f \in B} \|Tf\|_r, \qquad T \in L(L_{p-}),$$

where $B \subseteq L_{p-}$ is any bounded set, $r \in [1,p)$ is arbitrary and $\|\cdot\|_r$ denotes the usual norm in the Banach space $L^r([0,1])$. So, fix any such B and r. Let $t \in (r,p)$ be arbitrary and choose $s \geq 1$ according to $\frac{1}{r} = \frac{1}{t} + \frac{1}{s}$. Since B is bounded in $L^t([0,1])$, the generalized Hölder inequality (e.g. [11, p.527]) implies, for each $E \in \mathscr{B}$, that

$$q\left(\widetilde{P}(E)\right) = \sup_{f \in B} \|\chi_E f\|_r \leq \sup_{f \in B} \|f\|_t \lambda(E)^{1/s} = K_t \lambda(E)^{1/s}$$

for some constant $K_t > 0$ (with λ denoting Lebesgue measure in $[0,1]$). It follows that $q(\widetilde{P}(E(m))) \to 0$ as $m \to \infty$ whenever $\{E(m)\}_{m=1}^{\infty} \subseteq \mathscr{B}$ is decreasing to $\emptyset$. This is precisely the bounded σ-additivity of $\widetilde{P}$ in $L_b(L_{p-})$. □

As noted in Section 1, for $p \in (1,\infty)$ the Fréchet spaces L_{p-} are *not* Montel. So, part (iii) of Proposition 6 does not follow from Proposition 4.

Some of the spectral measures P and $\widetilde{P}$ (as given by (4) and (6)) are boundedly σ-additive. Accordingly, one can ask if these spectral measures have finite τ_b-variation. A general result in this direction is the following one, [5, Proposition 4.1].

Proposition 7 *Let X be a nuclear Fréchet space. Every $L_s(X)$-valued operator measure is necessarily boundedly σ-additive and has finite variation in both $L_s(X)$ and $L_b(X)$.*

Since a boundedly σ-additive $L_b(X)$-valued operator measure with finite τ_b-variation also has finite τ_s-variation in $L_s(X)$, we conclude from Proposition 1(iii) and Proposition 6(iii) that the spectral measures $\widetilde{P} : \mathscr{B} \to L_b(L_{p-})$, for $p \in (1,\infty)$, all *fail* to have finite τ_b-variation.

The question of finite τ_b-variation for the boundedly σ-additive canonical spectral measures $P : 2^{\mathbb{N}} \to L_b(\lambda_p(A))$, with $\lambda_p(A)$ Montel (see Proposition 5), is answered by the following result; see Corollary 4.4 and Proposition 4.5 of [5].

Proposition 8 *Let A be a Köthe matrix, $p \in \{0\} \cup [1,\infty)$ and $\lambda_p(A)$ be Montel. Then the boundedly σ-additive canonical spectral measure $P : 2^{\mathbb{N}} \to L_b(\lambda_p(A))$ has finite variation if and only if $\lambda_p(A)$ is nuclear.*

5 Q4: The Radon-Nikodým property

Many spectral measures $Q : \Sigma \to L_s(X)$ have finite τ_s-variation and are absolutely continuous with respect to some finite measure $\mu : \Sigma \to [0, 1]$. The question arises of whether or not there exists a μ-integrable function $G : \Omega \to L_s(X)$ such that $Q(E) = \int_E G\, d\mu$ for $E \in \Sigma$? Measures Q which have such a density G often have special features not exhibited by general spectral measures (e.g. relatively τ_s-compact range). In this section we investigate the *Radon-Nikodým property* (briefly, RNP) for X-valued vector measures and $L(X)$-valued operator measures, with special emphasis on our list of particular examples.

Let (Ω, Σ, μ) be a probability measure space and X be a Fréchet space. A function $G : \Omega \to X$ is called *strongly μ-measurable* if there exists a sequence $G_n : \Omega \to X$, for $n \in \mathbb{N}$, of Σ-simple functions such that $G(w) = \lim_{n\to\infty} G_n(w)$ in X, for μ-a.e. $w \in \Omega$. The obvious definition of "Bochner μ-integrable functions" is the one suggested by the Banach space case. Namely, a strongly μ-measurable function $G : \Omega \to X$ is called *Bochner μ-integrable* if $\int_\Omega (q_k \circ G)\, d\mu < \infty$, for each $k \in \mathbb{N}$, where $\{q_n\}_{n=1}^\infty$ are continuous seminorms determining the topology of X. This is equivalent to the definition in [27, p.282]. Namely, a function $G : \Omega \to X$ is Bochner μ-integrable if there exist Σ-simple functions $G_n : \Omega \to X$, for $n \in \mathbb{N}$, such that $G(w) = \lim_{n\to\infty} G_n(w)$, in X, for μ-a.e. $w \in \Omega$ and

$$\lim_{n\to\infty} \int_\Omega (q_k \circ (G - G_n))\, d\mu = 0, \qquad k \in \mathbb{N};$$

see [20, Lemma 2.5]. The "integral over E" can then be defined by

$$\int_E G\, d\mu := \lim_{n\to\infty} \int_E G_n\, d\mu, \qquad E \in \Sigma,$$

using the completeness of X and the obvious definition of $\int_E G_n\, d\mu$, for $n \in \mathbb{N}$. This is independent of the sequence $\{G_n\}_{n=1}^\infty$. The *indefinite integral* $G \cdot \mu : \Sigma \to X$ defined by $E \mapsto \int_E G\, d\mu$, for $E \in \Sigma$, is then a vector measure of finite variation, [20, Lemma 2.8].

A vector measure $m : \Sigma \to X$ is called *absolutely continuous* with respect to μ (denoted by $m \ll \mu$) if $m(E) = 0$ whenever $E \in \Sigma$ satisfies $\mu(E) = 0$. Given a vector measure $m : \Sigma \to X$ of finite variation, if there exists a Bochner μ-integrable function $G : \Omega \to X$ such that $m = G \cdot \mu$, then G is called the *Radon-Nikodým derivative* of m with respect to μ. The Fréchet space X has the RNP if for every (complete) probability measure space (Ω, Σ, μ) each vector measure $m : \Sigma \to X$ satisfying $m \ll \mu$ has an X-valued Radon-Nikodým derivative with respect to μ. It is known that every reflexive Fréchet space has the RNP, a result credited to D.R. Lewis; see [7, Corollary 3.1]. For further relevant references we refer to [3], [4], [7], [8], [28], which also apply to more general lcH-spaces X (not necessarily metrizable) with the various definitions given above adapted in a natural way (the number of seminorms needed may no longer be countable). An immediate application to our list of examples is the following fact.

Proposition 9 *Each of the separable, reflexive Fréchet spaces $\lambda_p(A)$, for any $1 < p < \infty$ and Köthe matrix A, has the* RNP. *The same is true of the separable, reflexive* *spaces ℓ^{p+}, for $1 \le p < \infty$, the separable, reflexive spaces $L^p_{\text{loc}}(\mathbb{R})$, for $1 < p < \infty$, and the separable, reflexive spaces L_{p-} for $1 < p < \infty$.*

Examples of non-reflexive Köthe echelon spaces with the RNP will be obtained by using the following three observations.

Proposition 10 *Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of Banach spaces with the* RNP. *Then the countable product $X = \prod_{k=1}^{\infty} X_k$ is a Fréchet space with the* RNP.

Proof. Let $\|\cdot\|_k$ denote the norm of X_k, for $k \in \mathbb{N}$, and $\mathbb{T}_k : X \to X_k$ be the canonical projection onto the k-th coordinate. Then

$$q_n(x) := \sum_{k=1}^{n} \|x_k\|_k, \qquad x = (x_1, x_2, \ldots) \in X,$$

for $n \in \mathbb{N}$, is a fundamental, increasing sequence of seminorms determining the topology of X.

Let $m : \Sigma \to X$ be any vector measure with finite variation on a complete probability measure space (Ω, Σ, μ) such that $m \ll \mu$. For each $k \in \mathbb{N}$, the vector measure $\mathbb{T}_k \circ m : \Sigma \to X_k$ is absolutely continuous with respect to μ. Accordingly, there exists a Radon-Nikodým derivative $G_k : \Omega \to X_k$ and a sequence of Σ-simple functions $G_k^n : \Omega \to X_k$, for $n \in \mathbb{N}$, satisfying

$$\lim_{n\to\infty} G_k^n(w) = G_k(w), \ w \notin A_k \text{ for some } \mu\text{-null set } A_k \in \Sigma, \qquad (1)_k$$

$$\int_{\Omega} \| G_k(\cdot)\|_k \, d\mu < \infty, \qquad \text{and} \qquad (2)_k$$

$$(\mathbb{T}_k \circ m)(E) = \int_E G_k \, d\mu, \qquad E \in \Sigma. \qquad (3)_k$$

Define $G : \Omega \to X$ by $G(w) := (G_1(w), G_2(w), \ldots)$, for $w \in \Omega$, and $G^{(n)} : \Omega \to X$ by $G^{(n)}(w) := (G_1^n(w), \ldots, G_n^n(w), 0, 0, \ldots)$, for $w \in \Omega$ and $n \in \mathbb{N}$. Then $\{G^{(n)}\}_{n=1}^{\infty}$ is a sequence of X-valued, Σ-simple functions with the property that $G^{(n)} \to G$ pointwise μ-a.e. on Ω (as $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = 0$). Accordingly, G is strongly μ-measurable. Moreover, $(2)_k$ yields

$$\int_{\Omega} (q_n \circ G) \, d\mu = \sum_{k=1}^{n} \int_{\Omega} \|G_k(\cdot)\|_k \, d\mu < \infty, \qquad n \in \mathbb{N},$$

that is, G is Bochner μ-integrable.

To see that $m = G \cdot \mu$ it suffices to show that equality holds in each coordinate. But, for $k \in \mathbb{N}$, we have by $(3)_k$ that $\mathbb{T}_k \circ m = G_k \cdot \mu$ and, by definition of G and $\mathbb{T}_k$, that $\mathbb{T}_k \left(\int_E G \, d\mu\right) = \int_E G_k \, d\mu$ for each $E \in \Sigma$. □

Remark. An obvious modification of the previous proof shows that the countable product of Fréchet spaces with the RNP also has the RNP.

The following result is recorded in [4, Proposition 3.4] but, with a rather unclear reference to a result in [28]. For the sake of completeness we include a proof.

Proposition 11 *Let Y be a separable Fréchet space with the* RNP. *If X is a closed subspace of Y, then also X has the* RNP.

Proof. Let $m : \Sigma \to X$ be any vector measure with finite variation on a complete probability measure space (Ω, Σ, μ) such that $m \ll \mu$. If $j : X \to Y$ denotes the canonical inclusion, then $j \circ m : \Sigma \to Y$ is a vector measure with $(j \circ m) \ll \mu$. By hypothesis there is a Bochner μ-integrable function $G : \Omega \to Y$ satisfying $(j \circ m)(E) = \int_E G \, d\mu$ for $E \in \Sigma$. Accordingly, $\int_E G \, d\mu = m(E) \in X$, for each $E \in \Sigma$, with $G : \Omega \to Y$ strongly μ-measurable. Since Y is a *Suslin space*, it follows that $G(w) \in X$ for μ-a.e. $w \in \Omega$, [28, VIII p.68]. Since X is also Suslin (being a separable Fréchet space), to check that $G : \Omega \to X$ is strongly μ-measurable it suffices to verify that $w \mapsto \langle G(w), x' \rangle$ is Σ-measurable on Ω, for each $x' \in X'$, [28, Theorem 1]. This follows easily from the strong μ-measurablility of G (in Y) and the Hahn-Banach theorem. Accordingly, G is an X-valued Radon-Nikodým derivative for m relative to μ. □

As a simple application of Proposition 11 we show that $L^1_{\text{loc}}(\mathbb{R})$ does *not* have the RNP. Indeed, let $Y = L^1_{\text{loc}}(\mathbb{R})$ and $X = \{f \in L^1_{\text{loc}}(\mathbb{R}) : f = 0 \text{ in } \mathbb{R} \setminus [-1, 1]\}$, in which case X is a closed subspace of Y. Since X is isomorphic to the Banach space $L^1([-1, 1])$ and this space fails to have the RNP, [10, p.219], Proposition 11 implies that Y also fails to have the RNP.

Proposition 12 *Let X be a Fréchet space which is the projective limit $X = \text{proj}_n X_n$ of separable Banach spaces $\{X_n\}_{n=1}^{\infty}$ with the* RNP. *Then also X has the* RNP.

Proof. X is isomorphic to a closed subspace of $\prod_{n=1}^{\infty} X_n$ and so the conclusion follows from Proposition 10 and Proposition 11. □

We can now decide about the Köthe echelon spaces to which Proposition 9 does not apply.

Proposition 13 *Let A be a Köthe matrix.*

(i) $\lambda_1(A)$ *always has the* RNP.

(ii) $\lambda_0(A)$ *has the* RNP *if and only if it is Montel.*

Proof. (i) Note that $\lambda_1(A) = \text{proj}_n \ell^1(a_n)$ with each $\ell^1(a_n) \simeq \ell^1$, for $n \in \mathbb{N}$. Since the Banach space ℓ^1 has the RNP, [10, p.218], the conclusion follows from Proposition 12.

(ii) If $\lambda_0(A)$ is Montel, then it is necessarily reflexive and hence, as noted earlier, must possess the RNP.

Conversely, suppose that $\lambda_0(A)$ is not Montel. According to Theorem 27.9(6) and Theorem 27.15 of [16] there exists $n \in \mathbb{N}$ and an infinite set $J \subseteq \mathbb{N}$ such that, for every $k > n$, there is $\alpha_k > 0$ with the property that $a_k(j) \le \alpha_k a_n(j)$ for all $j \in J$. Then, for each $x \in \lambda_0(J, A) := \{x\chi_J : x \in \lambda_0(A)\}$, we have $q_k^{(0)}(x) \le \alpha_k q_n^{(0)}(x)$ for all $k > n$. Hence, $q_n^{(0)}$ determines the relative topology in $\lambda_0(J, A)$ inherited from $\lambda_0(A)$, from which it follows that $\lambda_0(J, A)$ is isomorphic to the Banach space c_0. Since $\lambda_0(A)$ is separable and c_0 does not have the RNP, [10, p.219], it follows from Proposition 11 that $\lambda_0(A)$ fails to have the RNP. □

Proposition 14 *Let X be a Fréchet space.*

(i) *If X is Montel, then $L_b(X)$ has the* RNP.

(ii) *If X is nuclear, then both $L_s(X)$ and $L_b(X)$ have the* RNP.

(iii) *If either $L_s(X)$ or $L_b(X)$ has the* RNP, *then X has the* RNP.

Proof. (i) In a Fréchet Montel space τ_b coincides with the topology τ_{pc}, in $L(X)$, of uniform convergence on the precompact subsets of X. Since $L_{pc}(X)$ has the RNP, [3, p.339], we are done.

(ii) Since X is Montel, the RNP for $L_b(X)$ follows from part (i).

Suppose that $m : \Sigma \to L_s(X)$ is an operator measure with finite τ_s-variation on a complete probability measure space (Ω, Σ, μ) such that $m \ll \mu$. By Proposition 4 the measure m is boundedly σ-additive and by Proposition 7 it has finite τ_b-variation in $L_b(X)$. Since $L_b(X)$ has the RNP, m has an $L_b(X)$-valued Radon-Nikodým derivative relative to μ which, clearly, is then also a strongly μ-measurable $L_s(X)$-valued Radon-Nikodým derivative.

(iii) Since X is a complemented subspace of both $L_s(X)$ and $L_b(X)$, [9], it suffices to establish the following

Claim. *Let Y be a quasicomplete lcHs with the* RNP *and Z be a complemented subspace of Y. Then Z also has the* RNP.

To establish the Claim, let $R \in L(Y)$ be a projection onto Z. Let $m : \Sigma \to Z$ be any vector measure with finite variation on a complete probability measure space (Ω, Σ, μ) such that $m \ll \mu$. By hypothesis there is a Bochner μ-integrable function $G : \Omega \to Y$ satisfying $m = G \cdot \mu$. Then $R \circ G : \Omega \to Z$ is also Bochner μ-integrable and satisfies $m = R \circ m = (R \circ G) \cdot \mu$. This completes the proof of the Claim and hence, also of (iii). □

Remark. Proposition 14 shows that if X fails to have the RNP, then so do both $L_s(X)$ and $L_b(X)$. For example, it was noted after Proposition 11 that $L^1_{\text{loc}}(\mathbb{R})$ fails to possess the RNP. According to Proposition 14, neither $L_s(L^1_{\text{loc}}(\mathbb{R}))$ nor $L_b(L^1_{\text{loc}}(\mathbb{R}))$ has the RNP.

Of course, it can also happen that X does have the RNP but, $L_s(X)$ or $L_b(X)$ fail to have it. For instance, the Hilbert space $X = \ell^2$ has the RNP, [10, p.218], but $L_b(X)$ fails to possess the RNP. Indeed, $L_b(X)$ is precisely the Banach space $L(\ell^2)$ equipped with its operator norm topology $\|T\| := \sup_{\|x\| \leq 1} \|Tx\|_{\ell^2}$, for $T \in L(\ell^2)$. For each $\xi \in \ell^\infty$ define $M_\xi \in L(\ell^2)$ to be the multiplication operator $M_\xi : x \mapsto (\xi_1 x_1, \xi_2 x_2, \ldots)$, for each $x = (x_n) \in \ell^2$. Then $\|M_\xi\| = \|\xi\|_\infty$ and so $\xi \mapsto M_\xi$ is an isomorphism of the Banach space ℓ^∞ into $L(\ell^2)$. Since ℓ^∞ is an injective Banach space, it follows that $L(\ell^2)$ contains a complemented copy of ℓ^∞. But, ℓ^∞ fails to have the RNP, [10, p.219], and so we can conclude from the Claim in the proof of Proposition 14(iii) that $L_b(X) = L(\ell^2)$ also fails to have the RNP.

6 Q5: Integrable functions for a spectral measure

Given a Fréchet space X and a spectral measure $Q : \Sigma \to L_s(X)$, necessarily equicontinuous, an important space is $\mathscr{L}^1(Q)$. Much is known about this space. For instance, if Q is a *closed spectral measure* (i.e. its range is a Bade complete Boolean algebra of projections, [18, Proposition 3.5]), then the lcHs $\mathcal{L}^1(Q)$ is a *complete* space and the integration map $f \mapsto \int_\Omega f dQ$ is a bicontinuous isomorphism of the locally convex algebra $\mathcal{L}^1(Q)$ onto the closed subalgebra of $L_s(X)$ generated by the range $Q(\Sigma) = \{Q(E) : E \in \Sigma\}$, [18, Proposition 3.16]. Or, if the σ-algebra Σ is countably generated, then the lcHs $\mathcal{L}^1(Q)$ is *separable*, [24, Proposition 2]. All of the examples in this paper are closed spectral measures (as X is separable; see Section 2) and are based on countably generated σ-algebras (i.e. $2^{\mathbb{N}}, \mathscr{B}(\mathbb{R})$ and $\mathscr{B}$). Accordingly, the comments just made are applicable. But, what about the actual elements of $\mathcal{L}^1(Q)$?

If X is a Banach space, then it is known that $\mathcal{L}^1(Q) = \mathcal{L}^\infty(Q)$ as vector spaces, [12, XVIII Theorem 2.11(c)]. For non-normable Fréchet spaces X the situation can be rather different. Although particular spectral measures Q are known for which $\mathcal{L}^1(Q) = \mathcal{L}^\infty(Q)$, [5], [25], there are also plenty of examples where the containment $\mathcal{L}^\infty(Q) \subseteq \mathcal{L}^1(Q)$ is proper. To describe which functions actually belong to $\mathcal{L}^1(Q)$ is, in general, rather difficult. However, for the canonical spectral measure P in Köthe echelon spaces it is possible to give a rather elegant description of $\mathcal{L}^1(P)$ according to the following result, [5, Proposition 5.1], which is based on earlier work in [19].

Proposition 15 *Let λ denote one of the sequence spaces $\lambda_p(A)$, for any Köthe matrix A and $p \in \{0\} \cup [1, \infty)$, or ℓ^{p+} with $p \in [1, \infty)$, and let $P : 2^{\mathbb{N}} \to L_s(\lambda)$ be the canonical spectral measure, as given by (4). Then a function $\varphi \in \mathbb{C}^{\mathbb{N}}$ belongs to $\mathcal{L}^1(P)$ if and only if it satisfies $\lambda\varphi \subseteq \lambda$, where $\lambda\varphi := \{x\varphi : x \in \lambda\}$. Moreover, $\int_{\mathbb{N}} \varphi \, dP \in L_s(\lambda)$ is precisely the multiplication operator $M_\varphi : x \mapsto x\varphi$, for $x \in \lambda$.*

This result can be used to describe precisely when the containment $\mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P)$ is proper or is an equality.

Proposition 16 (i) *The canonical spectral measure $P : 2^{\mathbb{N}} \to L_s(\ell^{p+})$ satisfies $\mathcal{L}^1(P) = \mathcal{L}^\infty(P) = \ell^\infty$ for every $p \in [1, \infty)$.*

(ii) *Let $p \in \{0\} \cup [1, \infty)$, A be a Köthe matrix and $P : 2^{\mathbb{N}} \to L_s(\lambda_p(A))$ be the canonical spectral measure. Then the containment $\mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P)$ is proper if and only if there exists an infinite subset $J \subseteq \mathbb{N}$ such that the sectional subspace $\lambda_p(J, A) = \{x\chi_J : x \in \lambda_p(A)\}$ is Schwartz. In particular:*

(a) *If $p \in \{0\} \cup [1, \infty)$ is such that $\lambda_p(A)$ satisfies the density condition and is non-normable, then the inclusion $\mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P)$ is proper.*

(b) *If A is any KG-matrix, then $\mathcal{L}^1(P) = \mathcal{L}^\infty(P) = \ell^\infty$ for every $p \in \{0\} \cup [1, \infty)$.*

The previous result is a combination of Corollary 5.6 and Propositions 5.2, 5.5 and 5.7 of [5]. The criteria of part (ii) is quite effective since it is known exactly when $\lambda_p(J, A)$ is Schwartz, [16, Proposition 27.10].

The following result, for $\Omega = [0, \infty)$ and $p = 1$, occurs in [22, Lemma 8]. The proof given there can be easily adapted to the case $p \geq 1$ and $\Omega = \mathbb{R}$.

Proposition 17 *Let $p \in [1, \infty)$ and $\widehat{P} : \mathscr{B}(\mathbb{R}) \to L_s(L^p_{\mathrm{loc}}(\mathbb{R}))$ be the spectral measure given by (5). Then $\mathcal{L}^1(\widehat{P}) = L^\infty_{\mathrm{loc}}(\mathbb{R}))$ as vector spaces. Moreover, $\int_{\mathbb{R}} \varphi \, d\widehat{P} : f \mapsto \varphi f$ for $f \in L^p_{\mathrm{loc}}(\mathbb{R})$, is the operator of multiplication by φ, for each $\varphi \in \mathcal{L}^1(\widehat{P})$. In particular, the inclusion $\mathcal{L}^\infty(\widehat{P}) \subseteq \mathcal{L}^1(\widehat{P})$ is proper.*

To identify $\mathscr{L}^1(\widetilde{P})$ requires some preliminaries.

So, fix $p \in (1, \infty)$ and let $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$ denote the spectral measure given by (6). Given a Borel measurable function $\varphi : [0,1] \to \mathbb{C}$ define the vector space

$$D_p(M_\varphi) = \{f \in L_{p-} : \varphi f \in L_{p-}\} = \{f \in L_{p-} : \|\varphi f\|_r < \infty, \forall r \in [1,p)\}$$

where $\|\cdot\|_r$ denotes the norm of the Banach space $L^r([0,1])$. Define the multiplication operator $M_\varphi : D_p(M_\varphi) \to L_{p-}$ by $f \mapsto \varphi f$, for each $f \in D_p(M_\varphi)$.

Lemma. *The linear operator $M_\varphi : D_p(M_\varphi) \to L_{p-}$ is closed .*

Proof. Let $\{f_n\}_{n=1}^\infty \subseteq D_p(M_\varphi)$ be a sequence such that $f_n \to f$ in L_{p-} as $n \to \infty$ (for some $f \in L_{p-}$) and $M_\varphi f_n \to g$ in L_{p-} as $n \to \infty$ (for some $g \in L_{p-}$). Then also $f_n \to f$ and $\varphi f_n \to g$ in the Banach space $L^1([0,1])$ as $n \to \infty$. Accordingly, there exists a subsequence $\varphi f_{n(k)} \to g$ a.e. on $[0,1]$, in which case also $f_{n(k)} \to f$ in $L^1([0,1])$. There then exists an increasing subsequence $\{n(k_i)\}_{i=1}^\infty$ of $\{n(k)\}_{k=1}^\infty$ such that both $f_{n(k_i)} \to f$ and $\varphi f_{n(k_i)} \to g$ as $i \to \infty$, pointwise a.e.on $[0,1]$. Then also $\varphi f_{n(k_i)} \to \varphi f$ pointwise a.e. on $[0,1]$ as $i \to \infty$. We conclude that $\varphi f = g \in L_{p-}$ and so $f \in D_p(M_\varphi)$ with $g = M_\varphi f$. Hence, M_φ is a closed operator. □

We can now describe $\mathscr{L}^1(\widetilde{P})$; part (i) should be compared with Proposition 15.

Proposition 18 *Let $p \in (1,\infty)$ and $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$ be the spectral measure given by* (6).

(i) *A Borel measurable function $\varphi : [0,1] \to \mathbb{C}$ belongs to $\mathscr{L}^1(\widetilde{P})$ if and only if $D_p(M_\varphi) = L_{p-}$, that is, $\varphi L_{p-} \subseteq L_{p-}$. In this case*

$$\int_{[0,1]} \varphi \, d\widetilde{P} = M_\varphi.$$

(ii) *As a vector space*

$$\mathscr{L}^1(\widetilde{P}) = \bigcap_{1 \le q < \infty} L^q([0,1]). \tag{8}$$

In particular, the inclusion $\mathscr{L}^\infty(\widetilde{P}) \subsetneq \mathscr{L}^1(\widetilde{P})$ is proper.

Proof. (i) Let $\Omega = [0,1]$. Note that the $\widetilde{P}$-null sets are precisely the null sets for Lebesgue measure $\lambda : \mathscr{B} \to [0,1]$. If $\psi = \sum_{j=1}^k \alpha_j \chi_{E(j)}$ is any $\mathscr{B}$-simple function, then it follows from (6) that $M_\psi = \int_\Omega \psi \, d\widetilde{P}$. Suppose that $\psi \in L^\infty([0,1]) = \mathscr{L}^\infty(\widetilde{P})$. Choose a sequence $\{\psi_n\}_{n=1}^\infty$ of $\mathscr{B}$-simple functions satisfying $|\psi_n| \le |\psi|$ and $\|\psi_n - \psi\|_\infty \to 0$ as $n \to \infty$. For $f \in L_{p-}$ and any $r \in [1,p)$ we have

$$\|\psi f - \psi_n f\|_r \le \|\psi_n - \psi\|_\infty \|f\|_r \to 0, \qquad n \to \infty.$$

Accordingly, $M_{\psi_n} \to M_\psi$ in $L_s(L_{p-})$ as $n \to \infty$. Since $|\psi_n| \le |\psi| \in \mathscr{L}^1(\widetilde{P})$, the Dominated Convergence Theorem for vector measures applied to $\widetilde{P}$ in the quasicomplete lcHs $L_s(L_{p-})$, [14, p.30], yields $M_{\psi_n} = \int_\Omega \psi_n \, d\widetilde{P} \to \int_\Omega \psi \, d\widetilde{P}$ in $L_s(L_{p-})$ as $n \to \infty$. Hence, $\int_\Omega \psi \, d\widetilde{P} = M_\psi$ for all $\psi \in \mathscr{L}^\infty(\widetilde{P})$.

Suppose now that $\varphi \in \mathscr{L}^1(\widetilde{P})$ is arbitrary. Define $\psi_n := \varphi\chi_{E(n)}$, where $E(n) := |\varphi|^{-1}([0,n])$ for each $n \in \mathbb{N}$. Then $\psi_n \to \varphi$ pointwise $\widetilde{P}$-a.e. on $[0,1]$, each $\psi_n \in \mathscr{L}^\infty(\widetilde{P})$ and $|\psi_n| \leq |\varphi| \in \mathscr{L}^1(\widetilde{P})$ for $n \in \mathbb{N}$. Again by the Dominated Convergence Theorem we conclude that

$$\lim_{n\to\infty} M_{\psi_n} = \lim_{n\to\infty} \int_\Omega \psi_n \, d\widetilde{P} = \int_\Omega \varphi \, d\widetilde{P}, \tag{9}$$

in $L_s(L_{p-})$. For $f := \mathbf{1} \in L_{p-}$ we deduce that $\psi_n \to (\int_\Omega \varphi \, d\widetilde{P})\mathbf{1}$ in L_{p-} as $n \to \infty$. An "a.e. argument" as in the proof of the previous Lemma implies that $\varphi = (\int_\Omega \varphi \, d\widetilde{P})\mathbf{1}$. In particular, $\varphi \in L_{p-}$. Using the fact that $\varphi \in L_{p-}$ it is routine to check that $L^\infty([0,1]) \subseteq D_p(M_\varphi)$. So, we have thus far established that

$$\varphi \in \mathscr{L}^1(\widetilde{P}) \text{ implies } \varphi \in L_{p-} \text{ and } D_p(M_\varphi) \text{ is dense in } L_{p-}. \tag{10}$$

Still with $\varphi \in \mathscr{L}^1(\widetilde{P})$, fix $f \in D_p(M_\varphi)$. Since $\varphi \in \mathscr{L}^1(\widetilde{P})$, it is clear that φ is also integrable for the L_{p-}-valued vector measure $\widetilde{P}f : E \mapsto \widetilde{P}(E)f$, for $E \in \mathscr{B}$, in the usual sense, [14, Chapter II, §2]. Indeed, the integrals are given by

$$\int_E \varphi \, d(\widetilde{P}f) = \left(\int_\Omega \varphi \, d\widetilde{P}\right) \widetilde{P}(E)f, \qquad E \in \Sigma.$$

Since $|\psi_n| \leq |\varphi| \in \mathscr{L}^1(\widetilde{P}f)$ with $\psi_n \to \varphi$ a.e. (for $\widetilde{P}f$) on Ω, it follows from the Dominated Convergence Theorem applied to the vector measure $\widetilde{P}f : \mathscr{B} \to L_{p-}$ that

$$\lim_{n\to\infty} \int_\Omega \psi_n \, d(\widetilde{P}f) = \int_\Omega \varphi \, d(\widetilde{P}f) = \left(\int_\Omega \varphi \, d\widetilde{P}\right) f,$$

in $L_s(L_{p-})$. But, for each $n \in \mathbb{N}$, we also have

$$\int_\Omega \psi_n \, d(\widetilde{P}f) = \left(\int_\Omega \psi_n \, d\widetilde{P}\right) f = M_{\psi_n} f = \psi_n f = \varphi f_n,$$

where $f_n = f\chi_{E(n)}$. Accordingly, $\varphi f_n \to (\int_\Omega \varphi \, d\widetilde{P})f$ in L_{p-} as $n \to \infty$. Since $|f_n| \leq |f|$ we deduce from $f \in D_p(M_\varphi)$ that $\{f_n\}_{n=1}^\infty \subseteq D_p(M_\varphi)$ and, moreover, that $f_n \to f$ in L_{p-} (as $E(n) \uparrow \Omega$). Using the closedness of M_φ (see the previous Lemma) we conclude that $(\int_\Omega \varphi \, d\widetilde{P})f = M_\varphi f$, So, we have established:

$$\varphi \in \mathscr{L}^1(\widetilde{P}) \text{ implies the restriction } \left(\int_\Omega \varphi \, d\widetilde{P}\right)\Big|_{D_p(M_\varphi)} = M_\varphi.$$

This fact and the density of $D_p(M_\varphi)$ – see (10) – imply that $D_p(M_\varphi) = L_{p-}$ and $M_\varphi = \int_\Omega \varphi \, d\widetilde{P}$.

Conversely, suppose that $\varphi : [0,1] \to \mathbb{C}$ is a Borel measurable function for which $D_p(M_\varphi) = L_{p-}$. By the previous Lemma and the Closed Graph Theorem we conclude that $M_\varphi \in L(L_{p-})$. If we can show that

$$\varphi \in L^1(\langle \widetilde{P}f, g\rangle) \tag{11}$$

and

$$\langle M_\varphi f, g\rangle = \int_\Omega \varphi \, d\langle \widetilde{P} f, g\rangle, \tag{12}$$

for every $f \in L_{p-}$ and $g \in (L_{p-})'$ then, according to the definition of $\widetilde{P}$-integrability (see §1), the function $\varphi \in \mathscr{L}^1(\widetilde{P})$ and $\int_\Omega \varphi \, d\widetilde{P} = M_\varphi$. Observe if $f \in L_{p-}$ and $g \in (L_{p-})' = \bigcup_{s>p'} L^s([0,1])$, where $\frac{1}{p} + \frac{1}{p'} = 1$, then $\langle \widetilde{P} f, g\rangle$ is the complex measure

$$E \mapsto \langle \widetilde{P}(E) f, g\rangle = \int_E fg \, d\lambda, \qquad E \in \mathscr{B}, \tag{13}$$

with $fg \in L^1([0,1])$. Accordingly,

$$\int_\Omega |\varphi| \, d|\langle \widetilde{P} f, g\rangle| = \int_\Omega |\varphi| \cdot |fg| \, d\lambda = \int_\Omega |\varphi f| \cdot |g| \, d\lambda < \infty$$

because $g \in (L_{p-})'$ and $\varphi f \in L_{p-}$ (by hypothesis of $D_p(M_\varphi) = L_{p-}$). This establishes (11). To verify (12), observe that

$$\langle M_\varphi f, g\rangle = \langle \varphi f, g\rangle = \int_\Omega \varphi f g \, d\lambda = \int_\Omega \varphi \, d\langle \widetilde{P} f, g\rangle,$$

where the last equality uses (13). So, $\varphi \in \mathscr{L}^1(\widetilde{P})$ and the proof of (i) is complete.

(ii) Denote the right-hand-side of (8) by Λ. Suppose $\varphi \in \Lambda$. Let $f \in L_{p-}$ and $1 \le r < p$. Choose $t \in (r, p)$ arbitrarily and let $s > 0$ satisfy $\frac{1}{t} + \frac{1}{s} = \frac{1}{r}$. Then $\frac{1}{s} = \frac{1}{r} - \frac{1}{t} < \frac{1}{r}$ and so $s > r \ge 1$. Hence, $\varphi \in L^s([0,1])$ and so, by the generalized Hölder inequality, $\|\varphi f\|_r \le \|\varphi\|_s \|f\|_t < \infty$. This shows that $\varphi L_{p-} \subseteq L_{p-}$ and so, by part (i), we conclude that $\varphi \in \mathscr{L}^1(\widetilde{P})$.

Suppose now that $\psi \in \mathscr{L}^1(\widetilde{P})$. Let $q \in [1, \infty)$ and choose $k \in \mathbb{N}$ such that $k > q$. Since $\mathscr{L}^1(\widetilde{P})$ is an algebra under pointwise multiplication, [18, pp.12-13], also $\psi^k \in \mathscr{L}^1(\widetilde{P})$. By part (i) it follows that $\psi^k L_{p-} \subseteq L_{p-}$ and hence, $\psi^k = \psi^k \mathbf{1} \in L_{p-} \subseteq L^1([0,1])$. So, $\psi^k \in L^1([0,1])$ from which we deduce that

$$\|\psi\|_q \le \|\psi\|_k = \left(\|\psi^k\|_1\right)^{1/k} < \infty,$$

that is, $\psi \in L^q([0,1])$. Since $q \in [1, \infty)$ is arbitrary, it follows that $\psi \in \Lambda$. This establishes that $\Lambda = \mathscr{L}^1(\widetilde{P})$.

To show that $\mathscr{L}^\infty(\widetilde{P}) \subseteq \mathscr{L}^1(\widetilde{P})$ is a proper inclusion, let $\{F(n)\}_{n=1}^\infty$ be any pairwise disjoint sequence of sets in $\mathscr{B}$ satisfying $\lambda(F(n)) = e^{-n}$, for $n \in \mathbb{N}$. Then $\varphi := \sum_{n=1}^\infty n \chi_{F(n)}$ is surely not in $L^\infty([0,1]) = \mathscr{L}^\infty(\widetilde{P})$. However, for any $q \in [1, \infty)$ we have $\|\varphi\|_q^q = \sum_{n=1}^\infty n^q e^{-n} < \infty$, showing that $\varphi \in L^q([0,1])$. Accordingly, $\varphi \in \Lambda = \mathscr{L}^1(\widetilde{P})$.
□

Remark. It is interesting to note, for the spectral measures $\widehat{P} : \mathscr{B}(\mathbb{R}) \to L_s(L^p_{\mathrm{loc}}(\mathbb{R}))$ as given by (5) for each $p \in [1, \infty)$, that the space $\mathscr{L}^1(\widehat{P})$ is independent of p. The same is true for the spectral measures $\widetilde{P} : \mathscr{B} \to L_s(L_{p-})$ as given by (6) for each $p \in (1, \infty)$.

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