SPECTRAL MEASURES IN CLASSES OF FRÉCHET SPACES

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Abstract

A detailed investigation is made of the canonical atomic spectral measure defined in such Fréchet spaces as the Köthe echelon sequence spaces and the sequence spaces $c_0$, as well as the (non-atomic) "natural" spectral measures in such Fréchet spaces of measurable functions as the space of locally p-th power integrable functions on $\mathbb{R}$ and $L_p$, on $[0,1]$. Of particular interest are questions concerned with the range of the spectral measure, whether or not it has finite variation (for certain operator topologies), the Radon-Nikodým property of the underlying spaces involved and, most importantly, does the spectral measure admit unbounded integrable functions?

1 Introduction and preliminaries

The theory of Boolean algebras of projections/spectral measures in Banach spaces was initiated by W. Bade, N. Dunford and others, [12], and is by now well understood. In contrast, there is a distinct lack of concrete, non-trivial examples in the non-normable setting, even within the class of Fréchet (locally convex) spaces. An attempt to rectify this (to some extent) can be found in [5]. The aim of this paper is twofold. Firstly, we wish to summarize the main results of [5] and secondly, to expand on these results and elaborate further on some closely related topics. In order to do so, we begin with some general notation and definitions, so that the questions (some answers) and examples can be properly formulated.

Let $X$ be a locally convex Hausdorff space (briefly, lcHs) and $L(X)$ denote the space of all continuous linear operators from $X$ to itself. The space $L(X)$ is denoted by $L_1(X)$ (resp. $L_2(X)$) when it is equipped with the topology $\tau_1$ (resp. $\tau_2$) of uniform convergence.

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convergence on all finite (resp. bounded) subsets of $X$. A function $m \colon \Sigma \to Y$ (with $Y$ a lcfs) is called a vector measure if it is $\sigma$-additive; here $\Sigma$ is a $\sigma$-algebra of subsets of some non-empty set $\Omega$. If $Y$ coincides with $L_0(X)$ or $L_0(X)$, for some lcfs $X$, then $m$ is called an operator-valued measure ($\sigma$-bounded). An operator-valued measure $P : \Sigma \to L_0(X)$ is a spectral measure if it is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$ for all $E, F \in \Sigma$) and $P(\emptyset) = 0$, the identity operator on $X$. This is an extension of the notion of the resolution of the identity of a normal operator in Hilbert space. An operator-valued measure $Q : \Sigma \to L_0(X)$ is called boundedly $\sigma$-additive (or $\sigma$-countably additive) if it is $\sigma$-additive as an $L_0(X)$-valued measure. For $X$ a Banach space and $P : \Sigma \to L_0(X)$ a spectral measure, the bounded $\sigma$-additivity of $P$ can only occur in trivial cases, due to the fact that $\|\beta \| \geq 1$ for every non-zero projection $R \in L(X)$. In the setting of non-countable $X$ the situation can be quite different, [18], [23].

We will be exclusively interested in certain Fréchet spaces $X$. Since $L_0(X)$ is then quasi-complete, it follows from general vector measure theory that the range of any $L_0(X)$-valued operator measure is relatively weakly compact subset of $L_0(X)$, [14, IV Theorem 6.1], that is, relatively compact for the weak operator topology. In particular, the barreledness of $X$ then ensures that the range is always an epimorphismous subset of $L_0(X)$. Since $L_0(X)$ is also quasi-complete, [15, §39 Theorem 6.5], the range of any $L_0(X)$-valued operator measure is relatively weakly compact in $L_0(X)$. For $L_0(X)$-valued spectral measures, their range is always an (equicontinuous) Banach $\sigma$-complete Boolean algebra of projections, [18, Lemma 3.1]. The range is a Banach complete Boolean algebra of projections if and only if it is a closed subset of $L_0(X)$, [18, Proposition 3.4]. For instance, if $X$ is separable or there exists a cyclic vector, then this is always the case, [18, Proposition 3.4]. If the spectral measure happens to be purely atomic with countably many atoms, then its range is actually a compact subset of $L_0(X)$; apply Theorem 10 of [13] in the quasi-complete lcfs $E := L_0(X)$.

Let $Y$ be a lcfs with topology determined by a family of continuous seminorms $\mathcal{A}$. Let $Y_q = Y_G^{\alpha}((0))$ be the quotient normed space determined by $q \in \mathcal{A}$ and $Y_Y$ denote its Banach space completion. The norm in $Y_q$ is denoted by $\| \cdot \|_q$ and the canonical quotient map $Y \to Y_q = Y_G^{\alpha}((0))$ is denoted by $\rho_q$. Of course, for elements $y \in Y \to Y_q$, we have $\| y \|_q = \varphi(y)$. Given any $Y$-valued vector measure $m$ defined on a measurable space $(\Omega, \Sigma)$, the continuity of $\rho_q$ ensures that $m_q := \rho_q \circ m$ is a vector measure on $\Sigma$ with values in $Y_q$. For each $q \in \mathcal{A}$, the definition of the variation measure $m_q : \Sigma \to [0, \infty]$ of the Banach-space-valued measure $m_q$, see [10, pp 3-9]. The variation $m_q$ is called finite if $m_q((\Omega)) < \infty$. We say that the vector measure $m$ has finite variation if $m_q$ has finite variation for every $q \in \mathcal{A}$.

The only lcfs $Y$ relevant to this paper will be $X \otimes L_0(X)$, where $X$ is a Fréchet space. In this case, if $\rho_q \circ \alpha_n$ is any sequence of continuous seminorms determining the topology of $X$, then the topology $\tau_\alpha$ of $L_0(X)$, respectively $\tau_q$ of $L_0(X)$, is determined by the family of seminorms

$$\rho_q \circ \alpha_n : T \to \rho_q(T_{\alpha_n}) \quad T \in L_0(X),$$

for all $x \in X$ and $\alpha_n \in \mathcal{A}$, respectively.

$$\rho_q : T \to \sup_{\alpha_n} \rho_q(T_{\alpha_n}) \quad T \in L_0(X),$$

for all $x \in X$ and $\alpha_n \in \mathcal{A}$, respectively.

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for all bounded sets $B \subseteq X$ and $n \in \mathbb{N}$.

Let $Q$ be any $L_1(X)$-valued spectral measure defined on some measurable space $(\Omega, \Sigma)$. A $\Sigma$-measurable function $f : \Omega \to C$ is called $Q$-integrable if there exists an operator $f \cdot dQ \in L(X)$, necessarily unique, such that $f$ is integrable with respect to each complex measure $(Q_\omega, x') : E \mapsto \{Q(E|x, x') \mid E \in \Sigma, \text{ for } x \in X \text{ and } x' \in X'\}$ and

$$\left\{ \int f dQ \right\}_{x, x'} = \int f d(Q_\omega, x'), \quad x \in X, x' \in X'.$$

The operator $f \cdot dQ \to Q(E)(f \cdot dQ)$ then satisfies, for each $E \in \Sigma$,

$$\left\{ \int f dQ \right\}_{x, x'} = \int f d(Q_\omega, x'), \quad x \in X, x' \in X'.$$

The space of all $Q$-integrable functions is denoted by $L^0(Q)$. An element $f \in L^0(Q)$ is called $Q$-null if $f \cdot dQ = 0$. Each continuous seminorm $\|f\|_0$ of the form (1) determines a seminorm $\|f\|_0^0(Q)$ in $L^0(Q)$ via the formula

$$\|f\|_0^0(Q) : f \mapsto \sup_{x \in X} \|f\|_0^0 \left( \int f dQ \right)|x|, \quad f \in L^0(Q).$$

The family of seminorms $\{\|f\|_0^0(Q) : n \in \mathbb{N}, x \in X\}$ makes $L^0(Q)$ into a lsc. The quotient space of $L^0(Q)$, modulo the space of all $Q$-null functions, is a klsc which is denoted by $L^0(Q)$. A $\Sigma$, measurable function $f : \Omega \to C$ is called $Q$-essentially bounded if

$$\|f\|_0 = \inf \{\|f\| : \omega \in E \mid E \in \Sigma, P(E) = 1\} < \infty.$$

The Banach algebra of all (equivalence classes of) $Q$-essentially bounded functions is denoted by $L^\infty(Q)$. Since $L_1(X)$ is quasicomplete, we have $L^\infty(Q) \subseteq L^0(Q)$, [14, p.29], with a continuous inclusion.

Some basic questions concerning a particular spectral measure $Q : \Sigma \to L_1(X)$, with $X$ a particular Fréchet space, are the following ones.

**Q1.** What can be said about the range $Q(\Sigma) = \{Q(E) : E \in \Sigma, P(E) = 1\}$ as a subset of $L_1(X)$?

**Q2.** Does $Q$ have finite variation in $L_1(X)$?

**Q3.** Is $Q$ boundary $\sigma$-additive and, in the case when it is, does $Q$ have finite variation in $L_1(X)$?

**Q4.** In relation to $Q2$ and $Q3$ what is the connection with the Radon-Nikodým property of $X$, $L_1(X)$ and/or $L_0(X)$?

**Q5.** What locally convex space properties does $L^0(Q)$ possess and is the containment $L^\infty(Q) \subseteq L^0(Q)$ strict or not (it is always an equality when $X$ is a Banach space)?

For $X$ a Banach space, the answers are essentially known and can be found in [12,26], and the references therein. For $X$ a non-normable Fréchet space, only partial answers are known in general and, as alluded to above, there is a distinct lack of good examples available. The aim here is to discuss the results and examples of [5] in relation
to the above questions and to elaborate further. It is to be expected (and is indeed the case) that both the geometric and analytic properties of the Fréchet space $X$ play a crucial role as do individual features of the particular spectral measure $Q$ involved.

To proceed further we need to introduce various (particular) Fréchet spaces. Let $\Gamma$ denote either $\mathbb{R}$ or $\mathbb{N}$ or any infinite subset of these. An increasing sequence $A = (a_n)_{n \in \mathbb{N}}$ of strictly positive functions $a_n : \Gamma \to [0, \infty]$ is called a Köthe matrix on $\Gamma$, where by increasing we mean $0 < a_n(i) \leq a_{n+1}(i)$ for all $i \in \Gamma$ and $n \in \mathbb{N}$. Of course, each $a_n \in C^0$, for $n \in \mathbb{N}$. Elements $x \in C^0$ are denoted by $x = (x_i)$. Interpreting elements of $C^0$ as functions on $\Gamma$, it is clear what the notation such as $a_n x$, for $n \in \mathbb{N}$ and $x \in C^0$, and $a_n/a_m$, for $m, n \in \mathbb{N}$, means. To each $p \in [1, \infty]$ is associated the linear space

$$
\lambda_p(A) := \{ x \in C^0 : \sup_{i \in \Gamma} a_n(i) |x_i|^p \}^{1/p} < \infty,
$$

for all $n \in \mathbb{N}$. We also require the linear space

$$
\lambda_0(A) := \{ x \in C^0 : a_n x \in \lambda_0(\Gamma), \text{ for all } n \in \mathbb{N} \},
$$

equipped with the seminorms $\lambda_0(A)(x) := \sup_{n \in \mathbb{N}} a_n(i) |x_i|$, for each $n \in \mathbb{N}$. The spaces $\lambda_p(A)$ for $p \in [0, 1]$, are called Köthe echelon spaces of order (of $p$) they are all separable Fréchet spaces relative to the increasing sequence of seminorms $q_0(A) \leq q_1(A) \leq \ldots$. For the general theory of such spaces we refer to [1],[2],[16], for example. We recall a particular class of Köthe matrices $A$, the so-called Köthe-Greischoff spaces briefly, $K$-matrices. Here $\Gamma = \mathbb{N}$ and $a_n : \Gamma \to C$, for $n \in \mathbb{N}$, must satisfy:

$$
\begin{align*}
& a_n(i, j) = 1, \text{ for all } j, n \in \mathbb{N} \text{ and } i > n. \quad \text{(KG-1)} \\
& \sup_{j \in \mathbb{N}} a_n(i, j) = \infty, \text{ for all } n \in \mathbb{N}. \quad \text{(KG-2)} \\
& a_n(i, j) = a_n(j, i), \text{ for all } i, j \in \mathbb{N} \text{ and all } p, q \geq 1. \quad \text{(KG-3)}
\end{align*}
$$

The original KG-matrix corresponds to

$$
\begin{align*}
& a_n(i, j) = \left\{ \begin{array}{ll}
1 & \text{for } i \leq n \text{ and } j \leq n \\
1 & \text{for } i > n \text{ and } j > n,
\end{array} \right. \quad \text{for } n \in \mathbb{N}.
\end{align*}
$$

Another class of Fréchet spaces of relevance consists of the spaces $\mathbb{R}^0 := \lambda_0(\mathbb{R})$, for $p \in [1, \infty]$. Each one is a separable Fréchet space when equipped with the sequence of seminorms given by $q_k(x) := \left( \sum_{i=1}^{\infty} |x_i|^{2k} \right)^{1/2k}$, for $x \in \mathbb{R}^0$, where $\beta(k) := p + \frac{1}{k}$ for $k \in \mathbb{N}$. This class of spaces has been thoroughly investigated in [17].

All of the above sequence spaces are contained in the Fréchet space $\omega = C^0$, consisting of all $C$-functions on $\mathbb{N}$ equipped with the product topology.

For each $p \in [1, \infty]$, let $L^p(\mathbb{R})$ denote the space all (equivalence classes of) Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $q_p(f) := \left( \int_{\mathbb{R}} |f(t)|^p \, dt \right)^{1/p} < \infty$ for every $n \in \mathbb{N}$. Each space $L^p(\mathbb{R})$ is a separable Fréchet space when equipped with the sequence of seminorms $q_p \leq q_1 \leq \ldots$. 102
Finally, we recall the separable Fréchet spaces $L_p := \bigcap_{0 < r < \infty} L^r([0, 1]),$ for $p \in (1, \infty),$ equipped with the semi-norm $\|f\|_p := \|f\|_p.$

- For every $f \in L_p,$ and any increasing sequence $1 \leq \beta(m) \uparrow p$ as $m \to \infty,$ we have, with continuous inclusions, that $L^\beta([0, 1]) \to L_p \to L^\gamma([0, 1])$ for every $1 \leq \gamma < p.$

Some relevant spectral measures in the above Fréchet spaces are as follows. If $\lambda$ denotes one of the sequence spaces $\lambda_p(A),$ for $p \in (0, 1] \cup [1, \infty),$ or one of the sequence spaces $c^0,$ for $p \in [1, \infty),$ then the set function given by

$$P(E) : x \mapsto E_{\lambda_x},$$

for $E \in \mathcal{B}^p,$ defines a spectral measure $P : \mathcal{B}^p \to L_\lambda(\lambda);,$ it is called the canonical spectral measure in $\lambda,$ [19].

For each $p \in [1, \infty),$ the set function given by

$$\tilde{P}(E) : f \mapsto \int_{E_{\lambda_x}} f,$$

for $E \in \mathcal{B}(\lambda)$ (the $\sigma$-algebra of Borel subsets of $\lambda),$ defines a spectral measure $\tilde{P} : \mathcal{B}(\lambda) \to \mathcal{L}_\lambda(\lambda).$ Similarly, for each $p \in [1, \infty),$ the set function given by

$$\tilde{P}(E) : f \mapsto \int_{E_{\lambda_x}} f,$$

for $E \in \mathcal{B}(\lambda)$ (the $\sigma$-algebra of Borel subsets of $\lambda),$ defines a spectral measure $\tilde{P} : \mathcal{B}(\lambda) \to \mathcal{L}_\lambda(\lambda).$ The canonical spectral measures $P$ given by (4) are all purely atomic with countably many atoms (indeed, $P(E) = \sum_{x \in \lambda} P(E_{\lambda_x})$ for $E \in \mathcal{B}^p),$ whereas the spectral measures $\tilde{P}$ and $\tilde{P}$ (as given by (5) and (6)) have no atoms.

2 Q1: The range of spectral measures

For the spectral measures (4), (5), and (6), as described in the appropriate Fréchet space, we conclude from the earlier (general) remarks of Section 1, together with the fact that all the Fréchet spaces involved are separable, that their ranges $P(\mathcal{B}^p), \tilde{P}(\mathcal{B}(\lambda))$ and $\tilde{P}(\lambda)$ are all Bade complete Boolean algebras of projections.

It was noted in [8, Proposition 2.10] that every spectral measure $P$ (as given by (4)) has a cyclic vector. The same is true of the spectral measures $\tilde{P}$ (resp. $\tilde{P}$), as given by (5) (resp. (6)); the constant function 1 on $\lambda$ (resp. on $[0, 1]$) is a cyclic vector (as the $\mathcal{B}(\lambda)$-simple (resp. $\mathcal{B}(\lambda)$-simple) functions are dense in each space $\mathcal{L}_\lambda(\lambda)$ (resp. $L_\lambda.$)) Since each $P$ (as given by (4)) has countably many atoms, it follows from earlier remarks that $P(\mathcal{B}^p)$ is a relatively compact (even compact) subset of $L_\lambda(\lambda).$ This is not the case for $\tilde{P}$ and $\tilde{P}$ as given by (5) and (6). For, if $\tilde{P}(\mathcal{B}(\lambda))$ was relatively compact in $L_\lambda(\lambda),$ then continuity of the map $T \mapsto T_{\lambda_x}$ from $L_\lambda(\mathcal{L}_\lambda(\lambda))$ into $L_\lambda(\lambda)$ would imply that $\{\tilde{P}(E)_{\lambda_x} : E \in \mathcal{B}(\lambda)\}$ is relatively compact in $L_\lambda(\lambda).$ But, the relative topology of $L_\lambda(\lambda)$ in $\tilde{P}(\mathcal{B}(\lambda)) \subseteq L_\lambda(\lambda)$ is precisely that of the Banach space $L_\lambda([0, 1]).$ Since $L_\lambda([0, 1]) \subseteq L_\lambda([0, 1]),$ the set $\{x \mapsto E \in \mathcal{B}(\lambda)\}$ would be relatively compact in $L_\lambda([0, 1]),$ which is not the case, [10, Example 2, p.61]. So,
Proposition 1 (i) The canonical spectral measure $P: X \to L_1{\mathcal{B}}{\mathcal{E}}$, as given by (4), fails to have finite variation for every $p \in [1, \infty)$.  
(ii) The spectral measure $P: {\mathcal{B}}{\mathcal{E}} \to L_1{\mathcal{L}_p}(\mathbb{R})$, as given by (5), fails to have finite variation for every $\nu \in [1, \infty)$ but does have finite variation in $L_1{\mathcal{L}_p}(\mathbb{R})$.  
(iii) The spectral measure $P: {\mathcal{B}}{\mathcal{E}} \to L_1{\mathcal{L}_p}(\mathbb{R})$, as given by (6), fails to have finite variation for every $p \in (1, \infty)$.  

Proof. Part (ii) is Proposition 4.7 of [6].

(ii) Let $p = 1$. Consider a typical seminorm (see (1)) determining the topology of $L_1{\mathcal{L}_p}(\mathbb{R})$, say $q(T) = \int_{\mathbb{R}}|f(t)|X_{\nu}(t)\,dt$, $T \in L_1{\mathcal{L}_p}(\mathbb{R})$ for some fixed (but arbitrary) $\nu \in N$ and $f \in L_1{\mathcal{L}_p}(\mathbb{R})$. Let $\{E(m)\}_{m \in \mathbb{N}}$ be any Borel partition of $\mathbb{R}$. Then

$$\sum_{m \in \mathbb{N}} q\left(\int_{m}^{m+1} X_{\nu}(t)\,dt\right) \leq \sum_{m \in \mathbb{N}} \int_{m}^{m+1} |f(t)| \,dt$$

and so, $\int_{\mathbb{R}}|f(t)|X_{\nu}(t)\,dt < \infty$. Accordingly, $P$ has finite variation in $L_1{\mathcal{L}_p}(\mathbb{R})$.  

For $p > 1$, let $\nu = 1$ and $f = X_{\lambda,t}$. Then

$$q(T) = \int_{\lambda}^{t} |X_{\nu}(t)\,dt|^{1/p} = \lambda^{1/p} - t^{1/p}$$

is a continuous seminorm of the form (i). For the partition $E(m) = [\frac{m-1}{k}, \frac{m}{k}]$, with $1 \leq m \leq k$, and $E(0) = \mathbb{R} \setminus [0, 1)$, we have

$$\sum_{m \in \mathbb{N}} q\left(\int_{e^m/m}^{e^{m+1}/m+1} X_{\nu}(t)\,dt\right)^{1/p} \leq \sum_{m \in \mathbb{N}} k^{1/p} = k^{1/p},$$

where $\frac{1}{\nu} + \frac{1}{p} = 1$. Since $k^{1/p} \to \infty$ as $k \to \infty$, it follows that $\int_{\lambda}^{t} |f(t)|X_{\nu}(t)\,dt < \infty$ and hence, $P$ does not have finite variation in $L_1{\mathcal{L}_p}(\mathbb{R})$.  

(iii) For any fixed (but arbitrary) $\nu \in (1, \infty)$ and with $f = 1$, we see from (1) that

$$q(T) = \int_{\lambda}^{t} |X_{\nu}(t)\,dt|^{1/p} = \lambda^{1/p} - t^{1/p},$$

for $T \in L_1{\mathcal{L}_p}(\mathbb{R})$, is a continuous seminorm in $L_1{\mathcal{L}_p}(\mathbb{R})$.  

For the partition $\{E(m)\}_{m \in \mathbb{N}}$, of $[0, 1]$ as given in the proof of part (ii) we have, by an
The finite variation of the canonical spectral measure in Köthe echelon spaces can also be precisely described. For $\tau$, the complete answer is given by the following result, [5, Proposition 4.2].

Proposition 2 Let $A$ be a Köthe matrix.

(i) Let $p \in (0, 1)$, then the spectral canonical measure $P : 2^A \rightarrow L_1(\lambda_2(A))$, as given by (4), has finite variation if and only if $\lambda_2(A)$ is nuclear.

(ii) The canonical spectral measure $P : 2^A \rightarrow L_1(\lambda_2(A))$ always has finite variation.

Characterizations of when $\lambda_2(A)$ is nuclear are known, [16, Proposition 28.16]. As noted in Remark 4.3 of [6], the space $\lambda_2(A)$ is not always nuclear (e.g. for any $\mathcal{K}$-matrix this is the case).

As indicated by Proposition 2, nuclearity plays a fundamental role for this class of examples. This will again be seen in relation to Q3. Accordingly, the following result is relevant; it was stated (without proof) in [5, Proposition 4.8]. For the sake of completeness we now include a proof.

Proposition 3 For a Fréchet space $X$ the following assertions are equivalent.

(i) $X$ is nuclear.

(ii) $L_0(X)$ is nuclear.

(iii) $L_2(X)$ is nuclear.

Proof. (i) $\Rightarrow$ (iii). Since Fréchet spaces are dual nuclear (i.e. their strong dual is nuclear), [21, p.78], it follows from [21, Proposition 5.5.3] that $L_2(X)$ is nuclear.

(iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) follow from the facts that $X$ is a closed subspace of both $L_0(X)$ and of $L_2(X)$ and that a subspace of a nuclear space is again nuclear, [21, Proposition 5.1.1].

(i) $\Rightarrow$ (ii). Fix $x_1, \ldots, x_t \in X$ and an absolutely convex neighbourhood $U$ of 0 in $X$. Choose elements $u_1, \ldots, u_\infty \in X^*$ such that $\langle x_i, u_\rangle = \delta_j$, for $1 \leq i, j \leq t$. Since $X$ is nuclear, there is an absolutely convex neighbourhood $V$ of 0 in $X$ and a sequence $\{b_n\}_{n=1}^\infty \subset X$ satisfying $\sum_{n=1}^\infty \|\varphi_n(b_n)\|_V < \infty$ (where $V^n \subset X^*$ is the polar of $V$ and $\varphi_n(x) := \sup_{y \in V} \langle x, y \rangle$) such that

$$\|y, b\|_{V^n} \leq \varphi_n(y) \leq \sum_{k=1}^\infty \|y, b_k\|, \quad y \in X.$$  \hfill (7)

for all $b \in U^n$, [21, Proposition 4.1.4]. Here $\varphi_n$ denotes the plurisubharmonic functional of $U^n$.

For $1 \leq m \leq t$ and $n \in \mathbb{N}$, define a linear functional $A_{m,n} : L_2(X) \rightarrow \mathbb{C}$ by $T \mapsto \langle T \cdot x_m, b_n \rangle$ for $T \in L_2(X)$. It is immediate from (1) that each $A_{m,n}$ is $\tau$-continuous, that is, $A_{m,n} \in (L_1(X))'$. For the absolutely convex neighbourhood of 0 in $L_1(X)$ given by

$$W := \{S \in L_2(X) : S \cdot x_m \in V \text{ for all } 1 \leq m \leq t \},$$

we have, for each $1 \leq m \leq t$ and $n \in \mathbb{N}$, that

$$\varphi_{\tau} (A_{m,n}) = \sup_{S \in W} \|S \cdot x_m, b_n\| \leq \sup_{\|y\|_V} \|y, b_n\| = \varphi_{\tau} (b_n).$$

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Accordingly,
\[ \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho_{km}(A_{m}) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \rho_{km}(h_{k}) < \infty. \]

Define
\[ \tilde{W} = \{ T \in L(X) : T_{T} a \in U \text{ for all } 1 \leq m \leq t \}, \]
which is also an absolutely convex neighbourhood of 0 in \( L_{0}(X) \), and let \( q_{0} \) be its Minkowski function. It follows from (7) that
\[ q_{0}(T) = \max_{m} \sup_{k} \{ |T_{T} a_{m}| \} \leq \max_{m} \sum_{k} |T_{T} a_{m} h_{k}| \]
\[ = \max_{m} \sum_{k} |(T, A_{m} h_{k})| \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |(T, A_{n} h_{k})| \]
for every \( T \in L(X) \). By [21, Proposition 4.1.4] applied in the lefts \( L_{0}(X) \), it follows that \( L_{0}(X) \) is nuclear.

\[ \square \]

4 Q3: Bounded \( \sigma \)-additivity and finite \( \gamma \)-variation

Concerning the bounded \( \sigma \)-additivity of \( \rho_{km} L(X) \)-valued operator measures we have the following general result. It is formulated in [5, Proposition 3.1(i)] for sequence spaces, but the same proof applies here.

Proposition 4 Let \( X \) be a Fréchet Montel space. Then every \( L_{0}(X) \)-valued operator measure is bounded \( \sigma \)-additive in \( L_{0}(X) \).

For the canonical spectral measure in Köthe echelon spaces the previous proposition is optimal, as seen by the following result; see [5, Corollary 3.2(ii)].

Proposition 5 For some (all) \( p \in [0,1) \) and any Köthe matrix \( A \), the canonical spectral measure \( P : \mathcal{B}(\mathcal{B}) \rightarrow L_{0}(L_{0}(A)) \) is bounded \( \sigma \)-additive in \( L_{0}(L_{0}(A)) \) if and only if \( L_{0}(A) \) is a Montel space (equivalently, \( L_{0}(A) \) is reflexive).

Characterizations of when \( L_{0}(A) \) is Montel are known, [16, p.329 & p.334]. Moreover, for any K2-matrix \( A \) it follows that \( P \) fails to be bounded \( \sigma \)-additive in \( L_{0}(L_{0}(A)) \) for every \( p \in [0,1) \); see Corollary 3.2(ii) of [5].

Proposition 6 (i) For every \( p \in [1,\infty) \), the canonical spectral measure \( P : \mathcal{B}(\mathcal{B}) \rightarrow L_{0}(L_{0}(\mathcal{B})) \), fails to be bounded \( \sigma \)-additive in \( L_{0}(L_{0}(\mathcal{B})) \).

(ii) For every \( p \in [1,\infty) \), the spectral measure \( \mathcal{B}(\mathcal{B}) \rightarrow L_{0}(L_{0}(\mathcal{B})) \), as given by (i), fails to be bounded \( \sigma \)-additive in \( L_{0}(L_{0}(\mathcal{B})) \).

(iii) For every \( p \in [1,\infty) \), the spectral measure \( \mathcal{B}(\mathcal{B}) \rightarrow L_{0}(L_{0}(\mathcal{B})) \), as given by (i), is bounded \( \sigma \)-additive in \( L_{0}(L_{0}(\mathcal{B})) \).
Proof. (i) is Corollary 3.2(i) of [5].

(ii) Let \( B = \{ f \in \mathcal{L}_b^p(\mathbb{R}) : f = 0 \text{ in } \mathbb{R} \setminus [-1, 1] \} \) and \( \phi^*_B(f) \leq 1 \), where \( \phi^*_B(f) = \left( \frac{1}{L} \int |f(t)|^p \, dt \right)^{1/p} \) for \( f \in \mathcal{L}_b^p(\mathbb{R}) \), is a continuous seminorm on \( \mathcal{L}_b^p(\mathbb{R}) \). Then \( B \) is a bounded subset of \( \mathcal{L}_b^p(\mathbb{R}) \) because \( \phi^*_B \) coincides with \( \phi^*_B \) on \( B \), for every \( n \in \mathbb{N} \). For every Borel subset \( E \subseteq [-1, 1] \) of positive Lebesgue measure, the continuous seminorm \( \eta \) on \( \mathcal{L}_b^p(\mathbb{R}) \) - see (2) - given by \( \eta(T) = \sup_{f \in B} \phi^*_B(Tf) \), for \( T \in \mathcal{L}(\mathcal{L}_b^p(\mathbb{R})) \), satisfies
\[
\left( \mathbb{P}(\mathcal{E}) \right) - \sup_{f \in B} \left( \int \eta_\mathcal{E}(t) |f(t)|^p \, dt \right)^{1/p} = 1.
\]

Accordingly, \( \mathbb{P} \) cannot be bounded \( \sigma \)-additve in \( \mathcal{L}_b^p(\mathbb{R}) \).

(iii) A typical seminorm for \( \eta \) (see (2)) is given by
\[
\eta(T) = \sup_{f \in B} \| Tf \|_{L^p}, \quad T \in \mathcal{L}(\mathcal{L}_b^p),
\]
where \( B \subseteq \mathcal{L}_b^p \), is any bounded set, \( r \in [1, p) \) is arbitrary and \( \| \cdot \|_r \) denotes the usual norm in the Banach space \( L^r([0, 1]) \). So, fix any such \( B \) and \( r \). Let \( t \in (r, p) \) be arbitrary and choose \( s \geq 1 \) according to \( \frac{1}{r} = \frac{1}{s} + \frac{1}{p} \). Since \( B \) is bounded in \( L^r([0, 1]) \), the generalized Hölder inequality (e.g. [11, p.527]) implies, for each \( E \in \mathfrak{B} \), that
\[
\left( \mathbb{P}(\mathcal{E}) \right) = \sup_{f \in B} \| x \|_{L^r} \leq \sup_{f \in B} \| f \|_{L^r(\lambda)(E)^{1/s}} = K_E(\mathcal{E})^{1/r}
\]
for some constant \( K_E \geq 0 \) (with \( \lambda \) denoting Lebesgue measure in \([0, 1]\) ). It follows that \( q(\mathbb{P}(\mathcal{E}(m))) \rightarrow 0 \) as \( m \rightarrow \infty \) whenever \( \{ E(m) \}_{m \geq 1} \subseteq \mathfrak{B} \) is decreasing to \( \emptyset \). This precisely the bounded \( \sigma \)-additivity of \( \mathbb{P} \) in \( \mathcal{L}_b^p(\mathbb{R}) \).

As noted in Section 1, for \( p \in (1, \infty) \), the Fréchet spaces \( \mathcal{L}_p \) are not Montel. So, part (iii) of Proposition 6 does not follow from Proposition 4.

Some of the spectral measures \( \mathcal{F} \) and \( \mathbb{P} \) (as given by (4) and (6)) are bounded \( \sigma \)-additive. Accordingly, one can ask if these spectral measures have finite \( \eta \)-variation. A general result in this direction is the following one. [5, Proposition 4.11).

Proposition 7 Let \( \mathbb{F} \) be a nuclear Fréchet space. Every \( L_1(\mathbb{F}) \)-valued operator measure is necessarily bounded \( \sigma \)-additive and has finite variation in both \( L_1(\mathbb{F}) \) and \( L_2(\mathbb{F}) \).

Since a bounded \( \sigma \)-additive \( L_1(\mathbb{F}) \)-valued operator measure with finite \( \eta \)-variation also has finite \( \eta \)-variation in \( L_2(\mathbb{F}) \), we conclude from Proposition 1(ii) and Proposition 6(iii) that the spectral measures \( \mathcal{F} : \mathfrak{B} \rightarrow L_1(\mathbb{F}) \), for \( p \in (1, \infty) \), all fail to have finite \( \eta \)-variation.

The question of finite \( \eta \)-variation for the bounded \( \sigma \)-additive canonical spectral measures \( \mathcal{P} : 2^\mathfrak{B} \rightarrow L_1(\mathcal{A}(\mathbb{K})) \), with \( \mathcal{A}(\mathbb{K}) \) Montel (see Proposition 5), is answered by the following result; see Corollary 4.4 and Proposition 4.5 of [5].

Proposition 8 Let \( \mathcal{A} \) be a Kiefer measure, \( p \in [0, 1 \cup (1, \infty) \) and \( \mathcal{A}(\mathbb{K}) \) be Montel. Then the bounded \( \sigma \)-additive canonical spectral measure \( \mathcal{P} : 2^\mathfrak{B} \rightarrow L_1(\mathcal{A}(\mathbb{K})) \) has finite variation if and only if \( \mathcal{A}(\mathbb{K}) \) is nuclear.

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5 Q4: The Radon-Nikodym property

Many spectral measures \( Q : \Sigma \to L_p(X) \) have finite \( \tau \)-variation and are absolutely continuous with respect to some finite measure \( \mu : \Sigma \to [0,1] \). The question arises of whether or not there exists a \( \mu \)-integrable function \( G : \Omega \to L(X) \) such that \( Q(E) = \int_E G \, d\mu \) for \( E \in \Sigma \). Measures \( Q \) which have such a density \( G \) often have special features not exhibited by general spectral measures (e.g., relatively \( \tau \)-compact range). In this section we investigate the Radon-Nikodym property (briefly, RNP) for \( X \)-valued vector measures and \( L(X) \)-valued operator measures, with special emphasis on our list of particular examples.

Let \((\Omega,\mathcal{F},\mu)\) be a probability measure space and \( X \) be a Fréchet space. A function \( G : \Omega \to X \) is called strongly \( \mu \)-measurable if there exists a sequence \( G_n : \Omega \to X \), for \( n \in \mathbb{N} \), of \( \Sigma \)-simple functions such that \( G(w) = \lim_{n\to\infty} G_n(w) \) in \( X \), for \( \mu \)-a.e. \( w \in \Omega \). The obvious definition of "Bochner \( \mu \)-integrable functions" is the one suggested by the Banach space case. Namely, a strongly \( \mu \)-measurable function \( G : \Omega \to X \) is called Bochner \( \mu \)-integrable if \( \int_\Omega \|G\| \, d\mu < \infty \), for each \( k \in \mathbb{N} \), where \( \{G_n\}_{n=1}^\infty \) are continuous seminorms determining the topology of \( X \). This is equivalent to the definition in [27, p.262]. Namely, a function \( G : \Omega \to X \) is Bochner \( \mu \)-integrable if there exist \( \Sigma \)-simple functions \( G_n : \Omega \to X \), for \( n \in \mathbb{N} \), such that \( G(w) = \lim_{n\to\infty} G_n(w) \) in \( X \), for \( \mu \)-a.e. \( w \in \Omega \) and

\[
\lim_{n\to\infty} \int_G (G_n - G_{n+1}) \, d\mu = 0, \quad k \in \mathbb{N},
\]

see [20, Lemma 2.5]. The "integral over \( E \)" can then be defined by

\[
\int_E G \, d\mu = \lim_{n\to\infty} \int_E G_n \, d\mu, \quad E \in \Sigma,
\]

using the completeness of \( X \) and the obvious definition of \( \int_E G_n \, d\mu \), for \( n \in \mathbb{N} \). This is independent of the sequence \( \{G_n\}_{n=1}^\infty \). The indefinite integral \( G : \Omega \to X \) is defined by \( E \mapsto \int_E G \, d\mu \), for \( E \in \Sigma \), is then a vector measure of finite variation, [20, Section 2.8].

A vector measure \( m : \Sigma \to X \) is called absolutely continuous with respect to \( \mu \) (denoted by \( m \ll \mu \)) if \( m(E) = 0 \) whenever \( E \in \Sigma \) satisfies \( \mu(E) = 0 \). Given a vector measure \( m : \Sigma \to X \) of finite variation, if there exists a Bochner \( \mu \)-integrable function \( G : \Omega \to X \) such that \( m = G \cdot \mu \), then \( G \) is called the Radon-Nikodym derivative of \( m \) with respect to \( \mu \). The Fréchet space \( X \) has the RNP if for every (complete) probability measure space \((\Omega,\mathcal{F},\mu)\) each vector measure \( m : \Sigma \to X \) satisfying \( m \ll \mu \) has an \( X \)-valued Radon-Nikodym derivative with respect to \( \mu \). It is known that every reflexive Fréchet space has the RNP, a result credited to D.P. Lewis; see [7, Corollary 3.1]. For further relevant references we refer to [3], [4], [7], [8], [28], which also apply to more general left-spaces \( X \) (not necessarily metrizable) with the various definitions given above adapted in a natural way (the number of seminorms needed may no longer be countable). An immediate application to our list of examples is the following fact.

Proposition 9 Each of the separable, reflexive Fréchet spaces \( \lambda(A) \), for any \( 1 \leq p < \infty \) and Köthe matrix \( A \), has the RNP. The same is true of the separable, reflexive

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spaces $\theta^p$, for $1 \leq p < \infty$, the separable, reflexive spaces $L^p_p(\mathbb{R})$, for $1 < p < \infty$, and the separable, reflexive spaces $L^p_p$, for $1 < p < \infty$.

Examples of non-reflexive Köthe echelon spaces with the RNP will be obtained by using the following three observations.

**Proposition 10** Let $(X_k)_{k=1}^\infty$ be a sequence of Banach spaces with the RNP. Then the countable product $X = \prod_{k=1}^\infty X_k$ is a Fréchet space with the RNP.

**Proof.** Let $||| \cdot |||$ denote the norm of $X_k$, for $k \in \mathbb{N}$, and $T_k : X \to X_k$ be the canonical projection onto the $k$-th coordinate. Then

$$q_n(z) = \sum_{k=1}^n |||T_kz|||,$$

for $n \in \mathbb{N}$, is a fundamental, increasing sequence of seminorms determining the topology of $X$.

Let $m : \Sigma \to X$ be any vector measure with finite variation on a complete probability measure space $(\Omega, \Sigma, \mu)$ such that $m \ll \mu$. For each $k \in \mathbb{N}$, the vector measure $T_k \circ m : \Sigma \to X_k$ is absolutely continuous with respect to $\mu$. Accordingly, there exists a Radon-Nikodým derivative $G_k : \Omega \to X_k$ and a sequence of $\Sigma$-simple functions $G_k^* : \Omega \to X_k$, for $n \in \mathbb{N}$, satisfying

$$\lim_{n \to \infty} G_k^*(w) = G_k(w), \quad w \notin A_k \text{ for some } \mu\text{-null set } A_k \in \Sigma,$$

and

$$\int_\Sigma |||G_k^*(\cdot)|||d\mu < \infty,$$

and

$$\int_\Omega (T_k \circ m)(E)d\mu < \infty, \quad E \in \Sigma.$$

Define $G : \Omega \to X$ by $G(w) = (G_1(w), G_2(w), \ldots)$, for $w \in \Omega$, and $G^{(n)} : \Omega \to X$ by $G^{(n)}(w) = (G_1(w), G_2(w), \ldots, G_n(w), 0, 0, \ldots)$, for $w \in \Omega$ and $n \in \mathbb{N}$. Then $(G^{(n)})_n$, is a sequence of $\Sigma$-simple functions with the property that $G^{(\infty)} \to G$ pointwise $\mu$-a.e. on $\Omega$ (as $\mu(\bigcup_{k=1}^\infty A_k) = 0$). Accordingly, $G$ is strongly $\mu$-measurable. Moreover, (2.4) yields

$$\int_\Sigma (G_k \circ m)(w)d\mu = \sum_{n=1}^\infty \int_\Omega |||G_n^*(\cdot)|||d\mu < \infty, \quad n \in \mathbb{N},$$

that is, $G$ is Bochner $\mu$-integrable.

To see that $m = G \cdot \mu$ it suffices to show that equality holds in each coordinate. But, for $k \in \mathbb{N}$, we have by (3.4) that $T_k \circ m = G_k \cdot \mu$ and, by definition of $G$ and $T_k$, that $T_k \left(\int_\Omega G\ d\mu\right) = \int_\Omega G_k\ d\mu$ for each $E \in \Sigma$.

**Remark.** An obvious modification of the previous proof shows that the countable product of Fréchet spaces with the RNP also has the RNP.

The following result is recorded in [4, Proposition 3.4] but, with a rather unclear reference to a result in [28]. For the sake of completeness we include a proof.

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Proposition 13. Let $Y$ be a separable Fréchet space with the RNP. If $X$ is a closed subspace of $Y$, then also $X$ has the RNP.

Proof. Let $m : \Sigma \to X$ be any vector measure with finite variation on a complete probability measure space $(\Omega, \Sigma, \mu)$ such that $m \ll \mu$. If $j : X \to Y$ denotes the canonical inclusion, then $j \circ m : \Sigma \to Y$ is a vector measure with $(j \circ m) \ll \mu$. By hypothesis there is a Bochner $\mu$-integrable function $G : \Omega \to Y$ satisfying $(j \circ m)(E) = \int_{E} G \, d\mu$ for $E \in \Sigma$. Accordingly, $\int_{\Omega} G \, d\mu = m(E) \in X$, for each $E \in \Sigma$, with $G : \Omega \to Y$ strongly $\mu$-measurable. Since $Y$ is a Suslin space, it follows that $G(w) \in X$ for $\mu$-a.e. $w \in \Omega$ [28, VIII p.68]. Since $X$ is also Suslin (being a separable Fréchet space), to check that $G : \Omega \to Y$ is strongly $\mu$-measurable it suffices to verify that $w \mapsto (G(w), x')$ is $\Sigma$-measurable on $\Omega$, for each $x' \in X'$ [28, Theorem 1]. This follows easily from the strong $\mu$-measurability of $G$ (in $Y$) and the Hahn-Banach theorem. Accordingly, $G$ is an $X$-valued Radon-Nikodým derivative for $m$ relative to $\mu$.

As a simple application of Proposition 11 we show that $L_{1}^\mu(\mathbb{R})$ does not have the RNP. Indeed, let $Y = L_{1}^\mu(\mathbb{R})$ and $X = \{ f \in L_{1}^\mu(\mathbb{R}) : f = 0 \text{ in } [1, \infty[ \}$, in which case $X$ is a closed subspace of $Y$. Since $X$ is isomorphic to the Banach space $L_{1}^{\mu}([-1, 1])$ and this space fails to have the RNP, [10, p.219], Proposition 11 implies that $Y$ also fails to have the RNP.

Proposition 12. Let $X$ be a Fréchet space which is the projective limit $\mathcal{X} = \varprojlim X_n$ of separable Banach spaces $\{ X_n \}_{n=1}^{\infty}$ with the RNP. Then also $X$ has the RNP.

Proof. $X$ is isomorphic to a closed subspace of $\prod_{n=1}^{\infty} X_n$, and so the conclusion follows from Proposition 10 and Proposition 11.

We can now decide about the Köthe echelon spaces to which Proposition 9 does not apply.

Proposition 13. Let $A$ be a Köthe matrix.

(i) $\lambda(A)$ always has the RNP.

(ii) $\lambda(A)$ has the RNP if and only if it is Montel.

Proof. (i) Note that $\lambda(A)$ is Montel, then it is necessarily reflexive and hence, as noted earlier, must possess the RNP.

Conversely, suppose that $\lambda(A)$ is not Montel. According to Theorem 23.9(6) and Theorem 27.15 of [16] there exists $n \in N$ and an infinite set $J \subseteq N$ such that, for every $k > n$, there is $a_k > 0$ with the property that $a_k a_n(j) \leq a_n a_n(j)$ for all $j \in J$. Then, for each $x \in \lambda(A)$, we have $\| x \|_{\lambda(A)} \leq a_k a_k(x)$ for all $k > n$. Hence, $\| x \|_{\lambda(A)}$ determines the relative topology in $\lambda(A)$ inherited from $\lambda(A)$, from which it follows that $\lambda(A)$ is isomorphic to the Banach space $c_0$. Since $\lambda(A)$ is separable and $c_0$ does not have the RNP, [10, p.219], it follows from Proposition 11 that $\lambda(A)$ fails to have the RNP.
Proposition 14 Let X be a Fréchet space.

(i) If X is Montel, then \( L_0(X) \) has the RNP.

(ii) If X is nuclear, then both \( L_0(X) \) and \( L_1(X) \) have the RNP.

(iii) If either \( L_0(X) \) or \( L_1(X) \) has the RNP, then \( X \) has the RNP.

Proof. (i) In a Fréchet Montel space \( X \), the topology \( \tau_{tv} \) in \( (L(X), \Omega) \) of uniform convergence on the precompact subsets of \( X \). Since \( L_0(X) \) has the RNP, [1], p.339, we are done.

(ii) Since \( X \) is Montel, the RNP for \( L_0(X) \) follows from part (i).

Suppose that \( m : \Omega \to L_0(X) \) is an operator measure with finite \( \tau_{tv} \)-variation on a complete probability measure space \( (\Omega, \Sigma, \mu) \) such that \( m < \mu \). By Proposition 4 the measure \( m \) is boundedly \( \sigma \)-additive and by Proposition 7 it has finite \( \tau_{tv} \)-variation in \( L_0(X) \). Since \( L_0(X) \) has the RNP, \( m \) has an \( L_0(X) \)-valued Radon-Nikodým derivative relative to \( \mu \), which, clearly, is then also a strongly \( \mu \)-measurable \( L_0(X) \)-valued Radon-Nikodým derivative.

(iii) Since \( X \) is a complemented subspace of both \( L_0(X) \) and \( L_1(X) \), [9], it suffices to establish the following.

Claim. Let \( Y \) be a quasimcomplete LCS with the RNP and \( Z \) be a complemented subspace of \( Y \). Then \( Z \) also has the RNP.

To establish the Claim, let \( R \in L(Y) \) be a projection onto \( Z \). Let \( m : \Sigma \to Z \) be any vector measure with finite variation on a complete probability measure space \( (\Omega, \Sigma, \mu) \) such that \( m < \mu \). By hypothesis there is a Bochner \( \mu \)-integrable function \( G : \Omega \to Y \) satisfying \( m = G \cdot \mu \). Then \( R \circ G : \Omega \to Z \) is also Bochner \( \mu \)-integrable and satisfies \( m = R \circ m = (R \circ G) \cdot \mu \). This completes the proof of the Claim and hence, also of (iii).

Remark. Proposition 14 shows that if \( X \) fails to have the RNP, then so do both \( L_0(X) \) and \( L_1(X) \). For example, it was noted after Proposition 11 that \( L_0(\mathbb{R}) \) fails to possess the RNP. According to Proposition 14, neither \( L_0(\mathbb{N}) \) nor \( L_0(\mathbb{R}) \) has the RNP.

Of course, it can also happen that \( X \) does have the RNP but, \( L_0(X) \) or \( L_1(X) \) fail to have it. For instance, the Hilbert space \( X = \ell^2 \) has the RNP [10, p.218], but \( L_0(X) \) fails to possess the RNP. Indeed, \( L_0(\ell^2) \) is precisely the Banach space \( L(\ell^2) \) equipped with its operator norm topology \( |T| := \sup \|T(x)\|, \) for \( T \in L(\ell^2) \). For each \( \xi \in \ell^m \) define \( M_\xi \in L(\ell^2) \) to be the multiplication operator \( M_\xi : x \mapsto \langle \xi \rangle \langle x \rangle \), for each \( x = (x_k) \in \ell^2 \). Then \( \|M_\xi\| = \|\xi\|_\infty \), and so \( \xi \mapsto M_\xi \) is an isomorphism of the Banach space \( \ell^m \) into \( L(\ell^2) \). Since \( \ell^m \) is an injective Banach space, it follows that \( L(\ell^2) \) contains a complemented copy of \( \ell^m \). But, \( \ell^m \) fails to have the RNP, [10, p.219], and so we can conclude from the Claim in the proof of Proposition 14(iii) that \( L_0(X) = L(\ell^2) \) also fails to have the RNP.

6 Q5: Integrable functions for a spectral measure

Given a Fréchet space \( X \) and a spectral measure \( \Sigma : \Omega \to L_0(X) \), necessarily equicon- tinuous, an important space is \( \mathbb{R}^1(\Sigma) \). Much is known about this space. For instance,
If \( Q \) is a closed spectral measure (i.e. its range is a Baire complete Boolean algebra of projections, [18, Proposition 3.5]), then the \( \mathcal{L}^0(\mathcal{Q}) \) is a complete space and the integration map \( f \mapsto \int f \, d\mathcal{Q} \) is a bicontinuous isomorphism of the locally convex algebra \( \mathcal{L}^0(\mathcal{Q}) \) onto the closed subalgebra of \( L_b(\mathcal{X}) \) generated by the range \( \mathcal{Q}(\mathcal{E}) = \{ E \in \Sigma \} \). [18, Proposition 3.16]. Or, if the \( \sigma \)-algebra \( \Sigma \) is countably generated, then the \( \mathcal{L}^0(\mathcal{Q}) \) is separable, [24, Proposition 2]. All of the examples in this paper are closed spectral measures (as \( \mathcal{X} \) is separable; see Section 2) and are based on countably generated \( \sigma \)-algebras (i.e. \( \mathbb{R}^n \), \( \mathfrak{B}(\mathbb{R}) \) and \( \mathfrak{B} \)). Accordingly, the comments just made just make are applicable. But, what about the actual elements of \( \mathcal{L}^0(\mathcal{Q}) \)?

If \( \mathcal{X} \) is a Banach space, then it is known that \( \mathcal{L}^0(\mathcal{Q}) = \mathcal{L}^0(\mathcal{Q}) \) as vector spaces, [12, XVIII Theorem 2.11c)]. For non-normal Fréchet spaces \( \mathcal{X} \) the situation can be rather different. Although particular spectral measures \( Q \) are known for which \( \mathcal{L}^0(\mathcal{Q}) = \mathcal{L}^0(\mathcal{Q}) \), [5], [23], there are also plenty of examples where the containment \( \mathcal{L}^0(\mathcal{Q}) \subseteq \mathcal{L}^0(\mathcal{Q}) \) is proper. To describe which functions actually belong to \( \mathcal{L}^0(\mathcal{Q}) \) is, in general, rather difficult. However, for the canonical spectral measure \( P \) in Köthe echelon spaces it is possible to give a rather elegant description of \( \mathcal{L}^0(\mathcal{Q}) \) according to the following result, [5, Proposition 5.1], which is based on earlier work in [19].

**Proposition 15** Let \( \lambda \) denote one of the sequence spaces \( \lambda_0(\mathcal{X}) \), for any \( \lambda \)-Köthe matrix \( A \) and \( p \in [0,1] \cup [1,\infty) \), or \( p^* \) with \( p \in [1,\infty) \), and let \( P : \mathcal{L}^0(\mathcal{Q}) \to L_b(\mathcal{X}) \) be the canonical spectral measure, as given by (4). Then a function \( \varphi \in \mathbb{C}^\mathcal{X} \) belongs to \( \mathcal{L}^0(\mathcal{Q}) \) if and only if it satisfies \( \lambda_0^p \subseteq \lambda \), where \( \lambda_0^p := \{ x^p : x \in \mathcal{X} \} \). Moreover, \( \int \varphi \, dP \in L_b(\lambda) \) is precisely the multiplication operator \( M_{\varphi} : x \mapsto x \varphi \), for \( x \in \lambda \).

This result can be used to describe precisely when the containment \( \mathcal{L}^p(\mathcal{Q}) \subseteq \mathcal{L}^0(\mathcal{Q}) \) is proper or is an equality.

**Proposition 16** (i) The canonical spectral measure \( P : \mathcal{L}^0(\mathcal{Q}) \to L_b(\mathcal{X}) \) satisfies \( \mathcal{L}^p(\mathcal{Q}) = \mathcal{L}^p(\mathcal{P}) = \mathcal{L}^p(\mathcal{Q}) \) for every \( p \in [0,1] \cup [1,\infty) \). (ii) Let \( \mathcal{P} \) be a Köthe matrix and \( P : \mathcal{L}^0(\mathcal{Q}) \to L_b(\mathcal{X}) \) be the canonical spectral measure. Then the containment \( \mathcal{L}^p(\mathcal{Q}) \subseteq \mathcal{L}^p(\mathcal{P}) \) is proper if and only if there exists an infinite subset \( J \subseteq \mathbb{N} \) such that the sectional subspace \( \lambda_0(J^*) \subseteq \lambda \) is Schwartz. In particular:

(a) If \( p \in [0,1] \cup [1,\infty) \) is such that \( \lambda_0^p \) satisfies the density condition and is non-normal, then the inclusion \( \mathcal{L}^p(\mathcal{Q}) \subseteq \mathcal{L}^p(\mathcal{P}) \) is proper.

(b) If \( A \) is any \( KG \)-matrix, then \( \mathcal{L}^p(\mathcal{Q}) = \mathcal{L}^p(\mathcal{Q}) \) for every \( p \in [0,1] \cup [1,\infty) \).

The previous result is a combination of Corollary 5.6 and Propositions 5.2, 5.5 and 5.7 of [5]. The criteria of (ii) is quite effective since \( \lambda_0^p \) is known exactly when \( \lambda_0^p \) is Schwartz. [16, Proposition 27.16].

The following results, for \( \Omega = [0,\infty) \) and \( p = 1 \), occurs in [22, Lemma 8]. The proof given can be easily adapted to the case \( p \geq 1 \) and \( \Omega = \mathbb{R} \).

**Proposition 17** Let \( p \in [1,\infty) \) and \( \mathcal{P} : \mathcal{L}^0(\mathcal{Q}) \to L_b(\mathcal{X}) \) be the spectral measure given by (5). Then \( \mathcal{L}^p(\mathcal{P}) = \mathcal{L}_b^p(\mathcal{Q}) \) as vector spaces. Moreover, \( \int \varphi \, d\mathcal{P} : \mathcal{P} \to \mathcal{L}^p(\mathcal{P}) \), for \( \varphi \in \mathcal{L}^p(\mathcal{Q}) \), is the operator of multiplication by \( \varphi \), for each \( \varphi \in \mathcal{L}^p(\mathcal{P}) \). In particular, the inclusion \( \mathcal{L}^p(\mathcal{Q}) \subseteq \mathcal{L}^p(\mathcal{P}) \) is proper.

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To identify $\mathcal{L}^p(\tilde{\Omega})$, we first require some preliminaries.

So, given an $p \in (1, \infty)$ and let $\tilde{\Omega} : \tilde{\Omega} \to L_p(\tilde{\Omega})$ denote the measure space given by (6). Given a Borel measurable function $\varphi : [0, 1] \to \mathbb{C}$ define the vector space

$$D_p(\tilde{\Omega}) = \left\{ f \in L_p(\tilde{\Omega}) : \varphi f \in L_p(\tilde{\Omega}) \right\} = \left\{ f \in L_p(\tilde{\Omega}) : \| \varphi f \|_p < \infty, \forall \varphi \in [1, p] \right\},$$

where $\| \cdot \|_p$ denotes the norm of the Banach space $L_p([0, 1])$. Define the multiplication operator $M_\varphi : D_p(\tilde{\Omega}) \to L_p(\tilde{\Omega})$, by $f \mapsto \varphi f$, for each $f \in D_p(\tilde{\Omega})$.

**Lemma.** The linear operator $M_\varphi : D_p(\tilde{\Omega}) \to L_p(\tilde{\Omega})$ is closed.

**Proof.** Let $(f_n)_{n=1}^\infty \subseteq D_p(\tilde{\Omega})$ be a sequence such that $f_n \to f$ in $L_p(\tilde{\Omega})$, as $n \to \infty$ (for some $f \in L_p(\tilde{\Omega})$), and $M_\varphi f_n \to g$ in $L_p(\tilde{\Omega})$, as $n \to \infty$ (for some $g \in L_p(\tilde{\Omega})$). Then also $f_n \to f$ and $\varphi f_n \to g$ in the Banach space $L_p([0, 1])$, as $n \to \infty$. Accordingly, there exists a subsequence $\varphi f_{n(k)} \to g$ a.e. on $[0, 1]$, in which case also $f_{n(k)} \to f$ in $L_p([0, 1])$. There then exists an increasing subsequence $(n(k))_k$ of $(n(k))_k$ such that both $f_{n(k)} \to f$ and $\varphi f_{n(k)} \to g$ a.e. as $k \to \infty$, pointwise a.e. on $[0, 1]$. Then also $\varphi f_{n(k)} \to \varphi f$ pointwise a.e. on $[0, 1]$ as $k \to \infty$. We conclude that $\varphi f = g$ in $L_p(\tilde{\Omega})$, and so $f \in D_p(\tilde{\Omega})$, with $g = M_\varphi f$. Hence, $M_\varphi$ is a closed operator.

We can now describe $\mathcal{L}^p(\tilde{\Omega})$; part (ii) should be compared with Proposition 15.

**Proposition 18** Let $p \in (1, \infty)$ and $\tilde{\Omega} : \tilde{\Omega} \to L_p(\tilde{\Omega})$ be the spectral measure given by (6).

(i) A Borel measurable function $\varphi : [0, 1] \to \mathbb{C}$ belongs to $\mathcal{L}^1(\tilde{\Omega})$ if and only if $D_p(\tilde{\Omega}) = L_p(\tilde{\Omega})$. In this case

$$\int_{[0, 1]} \varphi f \, d\tilde{\Omega} = M_\varphi f.$$

(ii) As a vector space

$$\mathcal{L}^p(\tilde{\Omega}) = \bigcap_{1 \leq q < p} L^q([0, 1]).$$

In particular, the inclusion $\mathcal{L}^\infty(\tilde{\Omega}) \subseteq \mathcal{L}^1(\tilde{\Omega})$ is proper.

**Proof.** Let $\Omega = [0, 1]$. Note that the $p$-null sets are precisely the null sets for Lebesgue measure $\lambda : \tilde{\Omega} \to (0, 1]$. If $\psi = \sum_{i=1}^n \lambda(A_i)$ is any $\mathcal{S}$-simple function, then it follows from (6) that $M_\psi = \int_{[0, 1]} \psi f \, d\tilde{\Omega}$. Suppose that $\psi \in L^\infty([0, 1]) = \mathcal{L}^1(\tilde{\Omega})$. Choose a sequence $(\lambda_i)_i$ of $\mathcal{S}$-simple functions satisfying $|\lambda_i| \leq |\psi|$ and $\|\lambda_i - \psi\|_\infty \to 0$ as $n \to \infty$. For $f \in L_p(\tilde{\Omega})$, and any $\psi \in [1, p]$, we have

$$\|\psi f - \lambda_i f\|_\psi \leq \|\psi - \lambda_i\|_\psi \|f\|_\psi \to 0, \quad n \to \infty.$$ 

Accordingly, $M_\psi \to M_\lambda$ in $L_p(\tilde{\Omega})$, as $n \to \infty$. Since $|\lambda_i| \leq |\psi| \in \mathcal{L}^1(\tilde{\Omega})$, the Dominated Convergence Theorem for vector measures applied to $\tilde{\Omega}$ in the quasicomplete $\lambda$-Hilbert space $L_p(\tilde{\Omega})$, [15, p.30], yields $M_\lambda = \int_{[0, 1]} \psi f \, d\tilde{\Omega} = \int_{[0, 1]} \psi f \, d\tilde{\Omega}$ in $L_p(\tilde{\Omega})$, as $n \to \infty$. Hence, $\int_{[0, 1]} \psi \, d\tilde{\Omega} = M_\psi$ for all $\psi \in \mathcal{L}^\infty(\tilde{\Omega})$.
Suppose now that \( \varphi \in L^0(\tilde{\mathbf{P}}) \) is arbitrary. Define \( \psi_n := \psi_\mathbb{R}[n] \), where \( E(n) := \{ -\langle 0, n \rangle \} \) for each \( n \in \mathbb{N} \). Then \( \psi_n \rightarrow \varphi \) pointwise \( \mathbb{P} \)-a.e. on \([0, 1]\), each \( \psi_n \in L^0(\tilde{\mathbf{P}}) \) and \( |\psi_n| \leq |\varphi| \in L^0(\tilde{\mathbf{P}}) \) for all \( n \in \mathbb{N} \). Again by the Dominated Convergence Theorem, we conclude that

\[
\lim_{n \to \infty} M_n = \lim_{n \to \infty} \int_0^1 \psi_n \, d\tilde{\mathbf{P}} = \int_0^1 \varphi \, d\tilde{\mathbf{P}},
\]

(9)

in \( L_2(\mathbb{P}_{-}) \). For \( f := 1 \in L_{-} \), we deduce that \( \psi_n \rightarrow \int_0^1 \varphi \, d\tilde{\mathbf{P}} \) \( 1 \) in \( L_{-} \) as \( n \to \infty \). An “a.e. argument” as in the proof of the previous Lemma implies that \( \varphi = \int_0^1 \varphi \, d\tilde{\mathbf{P}} \) \( 1 \). In particular, \( \varphi \in L_{-} \). Using the fact that \( \varphi \in L_{-} \), it is routine to check that \( U^0(0, 1) \subseteq D_\alpha(M_{\alpha}) \). So, we have thus far established that

\[
\varphi \in L^0(\tilde{\mathbf{P}}) \implies \varphi \in L_{-} \quad \text{and} \quad D_\alpha(M_{\alpha}) \quad \text{is dense in} \quad L_{-}.
\]

(10)

Still with \( \varphi \in L^0(\tilde{\mathbf{P}}) \), fix \( f \in D_\alpha(M_{\alpha}) \). Since \( \varphi \in L^0(\tilde{\mathbf{P}}) \), it is clear that \( \varphi \) is also integrable for the \( L_{-} \)-valued vector measure \( \tilde{\mathbf{P}} f : E \rightarrow \tilde{\mathbf{P}}(E) f \), for \( E \in \mathcal{B} \), in the usual sense, [14, Chapter II, §3]. Indeed, the integrals are given by

\[
\int_E \varphi \, d(\tilde{\mathbf{P}} f) = \left( \int_0^1 \varphi \, d\tilde{\mathbf{P}} \right) \tilde{\mathbf{P}}(E) f, \quad E \in \Sigma.
\]

Since \( |\psi_n| \leq |\varphi| \in L^0(\tilde{\mathbf{P}}) \) with \( \psi_n \rightarrow \varphi \) \( \mathbb{P} \)-a.e. \( \left( \tilde{\mathbf{P}} f \right) \) on \( \Omega \), it follows from the Dominated Convergence Theorem applied to the vector measure \( \tilde{\mathbf{P}} f : \mathcal{B} \rightarrow \mathbb{R} \), that

\[
\lim_{n \to \infty} \int_0^1 \psi_n \, d\tilde{\mathbf{P}} f = \int_0^1 \varphi \, d\tilde{\mathbf{P}} f = \left( \int_0^1 \varphi \, d\tilde{\mathbf{P}} \right) f, \quad f \in L_1(\mathbb{P}_{-}).
\]

in \( L_2(\mathbb{P}_{-}) \). But, for each \( n \in \mathbb{N} \), we also have

\[
\int_0^1 \psi_n \, d\tilde{\mathbf{P}} f = \left( \int_0^1 \psi_n \, d\tilde{\mathbf{P}} \right) f = M_{\psi_n} f = \psi_n f = \psi f,
\]

where \( f_n = f |_{\mathcal{F}_{\psi_n}} \). Accordingly, \( \psi_n \rightarrow \int_0^1 \varphi \, d\tilde{\mathbf{P}} \) \( 1 \) in \( L_{-} \) as \( n \to \infty \). Since \( \int_0^1 \varphi \, d\tilde{\mathbf{P}} f \) we deduce from \( f \in D_\alpha(M_{\alpha}) \) that \( \int_0^1 \psi_n \, d\tilde{\mathbf{P}} f \subseteq D_\alpha(M_{\alpha}) \) and, moreover, that \( \int_0^1 \psi_n \, d\tilde{\mathbf{P}} f \) in \( L_{-} \) (as \( E(n) \delta 1 \)). Using the closeness of \( M_{\alpha} \) (see the previous Lemma) we conclude that \( \left( \int_0^1 \psi \, d\tilde{\mathbf{P}} f \right) = M_{\psi} f \). So, we have established:

\[
\varphi \in L^0(\tilde{\mathbf{P}}) \implies \text{the restriction } \left( \int_0^1 \psi_n \, d\tilde{\mathbf{P}} \right) |_{D_\alpha(M_{\alpha})} = M_{\psi}.
\]

(11)

This fact and the density of \( D_\alpha(M_{\alpha}) \) – see (10) – imply that \( D_\alpha(M_{\alpha}) = L_{-} \) and \( M_{\psi} = \int_0^1 \varphi \, d\tilde{\mathbf{P}} \).

Conversely, suppose that \( \psi : [0, 1] \rightarrow \mathbb{C} \) is a Borel measurable function for which \( D_\alpha(M_{\alpha}) = L_{-} \). By the previous Lemma and the Closed Graph Theorem we conclude that \( M_{\psi} \in L(\mathbb{P}_{-}) \). If we can show that

\[
\varphi \in L^0(\tilde{\mathbf{P}} f, g)
\]

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and 
\[ \langle M_{\lambda} f, g \rangle = \int_{\Omega} \rho \ d(P f, g), \]  
(12) 
for every \( f \in L_{\rho_p} \) and \( g \in (L_{\rho_{p'}})' \), then, according to the definition of \( \hat{P} \)-integrability (see 1)), the function \( \rho \in L^{2}(\hat{P}) \) and \( \int_{\Omega} \rho \ d\hat{P} = M_{\lambda} \). Observe if \( f \in L_{\rho_p} \) and \( g \in (L_{\rho_{p'}})' = \bigcup_{\rho_{1}, \rho_{2}, \ldots, \rho_{n}} L^{2}(0, 1) \), where \( \frac{1}{\rho} + \frac{1}{\rho'} = 1 \), then \( (P f, g) \) is the complex measure 
\[ E \mapsto \langle (P(E)f), g \rangle = \int_{E} f g \ d\lambda, \ E \in \mathcal{B}, \]  
(13) 
with \( f g \in L^{1}(0, 1) \). Accordingly, 
\[ \int_{0}^{1} \left| \int_{0}^{1} f \ d\lambda \right| \ d\lambda \leq 1 \]  
because \( g \in (L_{\rho_{p'}})' \) and \( f \in L_{\rho_p} \) (by hypothesis of \( \mu_{P}(\mathcal{M}) \) = \( L_{\rho_{p}} \) ). This establishes (11). To verify (12), observe that 
\[ \langle M_{\lambda} f, g \rangle = \langle \rho * f, g \rangle = \int_{0}^{1} \rho * f \ d\lambda = \int_{0}^{1} \rho \ d(P f, g), \]  
(13) 
where the last equality uses (12). So, \( \rho \in L^{2}(\hat{P}) \) and the proof of (i) is complete.

(ii) Denote the right-hand-side of (8) by \( A \). Suppose \( \varphi \in A \). Let \( f \in L_{\rho_p} \) and \( 1 \leq r < p \). Choose \( t \in [r, p] \) arbitrarily and let \( n > 0 \) satisfy \( \frac{1}{t} + \frac{1}{p} = 1 \). Then \( \frac{1}{n} = \frac{1}{t} - \frac{1}{p} < 1 \) and so \( \lambda > 1 \). Hence, \( \varphi \in L^{n}(0, 1) \) and so, by the generalized Hölder inequality, \( \| \varphi f \|_{L_{\rho_p}} \leq \| \varphi \|_{L^{n}(0, 1)} \| f \|_{L^{n}(0, 1)} \). This shows that \( \lambda \mathcal{L}_{\rho_p} \subseteq L_{\rho_{p}} \) and so, by part (i), we conclude that \( \rho \in L^{2}(\hat{P}) \).

Suppose now that \( \varphi \in L^{2}(\hat{P}) \). Let \( q \in [1, \infty) \) and choose \( k \in N \) such that \( k > q \). Since \( L^{2}(\hat{P}) \) is an algebra under pointwise multiplication, [18, pp.12-13], also \( \varphi^{k} \in L^{2}(\hat{P}) \). By part (i) it follows that \( \varphi^{k} \mathcal{L}_{\rho_p} \subseteq L_{\rho_{p}} \) and hence, \( \varphi^{k} \mathcal{L}_{\rho_p} \subseteq L^{q}(0, 1) \). So, \( \varphi^{k} \in L^{q}(0, 1) \) from which we deduce that 
\[ \| \varphi \|_{L^{q}(0, 1)} \leq \| \varphi^{k} \|_{L^{q}(0, 1)}^{1/k} \| \varphi \|_{L^{n}(0, 1)} \leq \| \varphi \|_{L^{n}(0, 1)}^{1/k}, \]  
that is, \( \varphi \in L^{q}(0, 1) \). Since \( q \in [1, \infty) \) is arbitrary, it follows that \( \varphi \in A \). This establishes that \( A = L^{2}(\hat{P}) \).

To show that \( L^{\infty}(\hat{P}) \subseteq L^{2}(\hat{P}) \) is a proper inclusion, let \( \{ F(n) \}_{n=1}^{\infty} \) be any pairwise disjoint sequence of sets in \( \mathcal{B} \) satisfying \( \lambda(F(n)) = e^{-n} \), for \( n \in N \). Then \( \varphi := \sum_{n=1}^{\infty} 1_{F(n)} \) is surely not in \( L^{2}(0, 1) = L^{2}(\hat{P}) \), however, for any \( q \in [1, \infty) \) we have 
\[ \| \varphi \|_{L^{q}(0, 1)} \leq \sum_{n=1}^{\infty} e^{-n} \| \varphi \|_{L^{q}(0, 1)}^{1/k}, \]  
showing that \( \varphi \in L^{q}(0, 1) \). Accordingly, \( A \in L^{2}(\hat{P}) \).

Remark. It is interesting to note, for the spectral measures \( \hat{P} : \mathcal{B}(\mathbb{R}) \to L_{\mathbb{P}}(L_{\rho_p}^{2}(\mathbb{R})) \) as given by (5) for each \( p \in [1, \infty) \), that the space \( L^{\infty}(\hat{P}) \) is independent of \( p \). The same is true for the spectral measures \( \hat{P} : \mathcal{B} \to L_{\mathbb{P}}(L_{\rho_p}^{2}(\mathbb{R})) \) as given by (6) for each \( p \in (1, \infty) \).

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