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PARTIAL DIFFERENTIAL OPERATORS MODULO SMOOTH FUNCTIONS

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ABSTRACT. For a constant coefficient linear partial differential operator $P(D)$ on $\mathscr{D}'(\Omega)$ we provide new characterizations when

$$\mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega),\ (u,f) \mapsto P(D)u + f$$

is surjective and when it has a continuous linear right inverse. Both results are in the spirit of a celebrated result of Meise, Taylor, and Vogt who characterized right invertibility of $P(D)$ on $\mathscr{D}'(\Omega)$ by properties of fundamental solutions.

1. INTRODUCTION

Already in 1962, L. Hörmander [7] characterized the surjectivity of $P(D)$: $\mathscr{D}'(\Omega)/\mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)/\mathscr{E}(\Omega)$ by a condition called P-convexity for singular supports:

> For each compact set $K \subset \Omega$ there is another compact set $M \subset \Omega$ such that each $u \in \mathscr{E}'(\Omega)$ with $P(D)^t(u) \in \mathscr{E}(\Omega \setminus K)$ belongs to $\mathscr{E}(\Omega \setminus M)$

where $P(D)^t = P(-D)$ is the transposed operator. To prove sufficiency, Hörmander used a complicated ad hoc argument to find a seminorm to which the Hahn-Banach theorem applies, necessity of the convexity condition was proved by a delicate construction. In [5] we gave a new proof using abstract results for (LF)-spaces, and in [6] we extended the characterization to the more general setting of convolution operators on spaces of ultradistributions.

The aim of section 2 is to provide a characterization in the space $\mathscr{D}'(\Omega)$ which involves the operator itself instead of its transposed, namely by the existence of fundamental solutions which are differentiable up to a fixed order in large sets. This is in the same spirit as Meise, Taylor, and Vogt's [10, 11] characterization of the existence of a continuous linear right inverse for $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ where differentiability must be replaced by vanishing.

The proof is based on a variant of a classical result of Palamodov and Retakh about projective spectra of (LB)-spaces.

In section 3 we show that the same condition with "differentiable up to a fixed order" replaced by "infinitely differentiable" characterizes the existence of a continuous linear right inverse of the operator $\mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega), (u,f) \mapsto P(D)u + f$.

In view of the similarity of both characterizations and since existence of right inverses on $\mathscr{D}'(\Omega)$ and $\mathscr{E}(\Omega)$ are the same conditions by [10, 11] (and by [1] both are even equivalent to the existence of an operator R : $\mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$ with $P(D) \circ R = id$) it might have been hoped at least for convex sets Ω or $\Omega = \mathbb{R}^N$ that the operator above always has a right inverse. Using results of Hörmander about the propagation of regularity, we show in section 4 that this is not the case. The Laplace operators in two variables considered as an operator on $\mathscr{D}'(\mathbb{R}^3)$ does not have a right inverse modulo smooth functions. In particular this shows that having an inverse mod $\mathscr{E}$ does not only depend on the principal part of the operator.

Throughout this article $P(D)$ denotes a (non-zero) linear partial differential operator on $\mathbb{R}^N$ with constant coefficients. For an open set $\Omega \subseteq \mathbb{R}^N$ we abuse standard notation and denote the operator

$$\mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega),\ (u, f) \mapsto P(D)u + f$$

by $P(D) + id$ and we call $P(D)$ surjective or right invertible mod $\mathscr{E}$ on Ω if $P(D) + id$ is surjective or has a continuous linear right inverse, respectively. $(\Omega_n)_{n\in\mathbb{N}}$ will always denote an open and relatively compact exhaustion of Ω with $\overline{\Omega}_n \subset \Omega_{n+1}$ and – but this is just for convenience – we assume Ω_n to be regular enough so that the possible definitions of $\mathscr{C}^k(\overline{\Omega}_n)$ – as Whitney jets, restrictions of $\mathscr{C}^k(\mathbb{R}^N)$-functions to $\overline{\Omega}_n$, or $\mathscr{C}^k$-functions on Ω_n where all partial derivatives up to order k extend continuously to $\overline{\Omega}_n$ – all coincide. That this is immaterial up to more or less notational changes for the proofs in due to the fact that (just by multiplying with cut-off functions) the three corresponding sequences $\mathscr{C}^k(\overline{\Omega}_n)$ define equivalent projective spectra of Banach spaces with the Fréchet space $\mathscr{C}^k(\Omega)$ as projective limit.

2. Surjectivity modulo smooth functions

Theorem 1. *$P(D)$ is surjective mod $\mathscr{E}$ on an open set Ω if and only if*

$$\forall\, n \in \mathbb{N} \quad \exists\, m \geq n \quad \forall\, k \geq m,\, \xi \notin \overline{\Omega}_m \quad \exists\, E_\xi \in \mathscr{D}'(\mathbb{R}^N)$$

$$P(D)E_\xi = \delta_\xi \text{ in } \Omega_k \text{ and } E_\xi \in \mathscr{C}^k(\Omega_n).$$

As already mentioned, if the requirement $E_\xi \in \mathscr{C}^k(\Omega_n)$ is replaced by $E_\xi = 0$ in Ω_n one obtains the characterization for $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ having a continuous linear right inverse, which is due to Meise, Taylor, and Vogt [10, 11]. A similar condition (where distributions are replaced by hyperfunctions and differentiability by an appropriate analyticity condition) was obtained by Langenbruch [9] as a characterization of surjectivity of $P(D)$ on the space of real analytic functions.

The proof of the theorem will use a version of a classical result of Palamodov and Retakh for projective spectra $\mathscr{X} = (X_n, \varrho_m^n)$ of (LB)-spaces, i.e., $\varrho_m^n : X_m \to X_n$ for $n \leq m$ are continuous linear "spectral" maps with $\varrho_m^n \circ \varrho_k^m = \varrho_k^n$ for $n \leq m \leq k$ and $\varrho_n^n = id$. The projective limit $\operatorname{Proj} \mathscr{X} = \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho_m^n x_m = x_n\}$ is endowed with the relative topology of the product. The vector space $\operatorname{Proj}^1 \mathscr{X}$ is defined as the cokernel of the map $\psi : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n,\ (x_n)_{n\in\mathbb{N}} \mapsto (x_n - \varrho_{n+1}^n x_{n+1})$, i.e. $\operatorname{Proj}^1 \mathscr{X} = \prod_{n\in\mathbb{N}} X_n / \operatorname{im} \psi$. This ad-hoc definition indeed coincides with the derivative of the functor Proj (acting on the category of projective spectra where morphisms $T = (T_n)_{n\in\mathbb{N}} : (X_n, \varrho_m^n) \to (Y_n, \sigma_m^n)$ are sequences of continuous linear maps commuting with the spectral maps) as defined by Palamodov

[12, 13]. The main point is that for a morphism $T : \mathscr{Y} \to \mathscr{Z}$ with surjective components $T_n : Y_n \to Z_n$ the projective limit $(y_n)_{n\in\mathbb{N}} \mapsto (T_n y_n)_{n\in\mathbb{N}}$ is surjective whenever the kernel spectrum $\mathscr{X} = (\ker T_n, \varrho_m^n)$ satisfies $\mathrm{Proj}^1 \mathscr{X} = 0$. If $\mathrm{Proj}^1 \mathscr{Y} = 0$ holds, this is also necessary for surjectivity.

All this can be seen in [16] where it is also shown that instead of surjectivity of the components $T_n : Y_n \to Z_n$ it is enough to require $\sigma_m^n(Z_m) \subseteq T_n(Y_n)$ for some $m \geq n$. To prepare the proof of theorem 1, let us describe the spectra to which we will apply the abstract theory. Since we are aiming at surjectivity of $T = P(D) + id : \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$ we have to take $Y_n = \mathscr{D}'(\overline{\Omega}_n) \times \mathscr{C}^n(\overline{\Omega}_n)$ and $Z_n = \mathscr{D}'(\overline{\Omega}_n)$ with restrictions as spectral maps. The condition $\sigma_{n+1}^n(Z_{n+1}) \subseteq T_n(Y_n)$ is easily verified by cutting off and forming the convolution with a fundamental solution of $P(D)$. Since $\mathrm{Proj}^1\mathscr{Y} = 0$ (this follows e.g. from a particular case of the proposition below), we get:

$$P(D) \text{ is surjective mod } \mathscr{E} \text{ on } \Omega \text{ if and only if } \mathrm{Proj}^1 \mathscr{X} = 0,$$

where $\mathscr{X} = (X_n, \varrho_m^n)$ with $X_n = \{(F, f) \in \mathscr{D}'(\overline{\Omega}_n) \times \mathscr{C}^n(\overline{\Omega}_n) : P(D)F = f\}$ and restrictions as spectral maps. $\mathrm{Proj}^1 \mathscr{X}$ will be evaluated by using the following version of the Palamodov-Retakh theorem ([13, theorem 5.4] and [14], see also [16, theorem 3.2.9]) where we call a spectrum $\mathscr{X}$ reduced if $\forall\, n \in \mathbb{N} \quad \exists\, m \geq n \quad \forall\, k \geq n$ the closures of $\varrho_m^n X_m$ and $\varrho_k^n X_k$ in X_n coincide.

Proposition 2. *For a reduced projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ of (LB)-spaces assume that there are Banach spaces $Y_n \subseteq X_n$ with continuous inclusions such that*

$$(\alpha) \quad \varrho_m^n Y_m \subset Y_n \text{ for all } n \leq m \text{ and}$$

$$(\beta) \quad \forall\, n \in \mathbb{N} \quad \exists\, m \geq n \quad \forall\, k \geq m \quad \varrho_m^n X_m \subseteq \varrho_k^n X_k + Y_n.$$

Then $\mathrm{Proj}^1 \mathscr{X} = 0$ *holds.*

Let us remark that if Y_n are replaced by the unit balls B_n of these Banach spaces one obtains the classical Palamodov-Retakh theorem. Then (β) implies reducedness and $(\alpha), (\beta)$ constitute in fact a characterization of $\mathrm{Proj}^1 \mathscr{X} = 0$. Moreover, one can replace (β) by the a priori stronger condition $\varrho_m^n X_m \subseteq \varrho^n \mathrm{Proj}\, \mathscr{X} + B_n$ (which characterizes $\mathrm{Proj}^1 \mathscr{X} = 0$ also in absence of (α) by [16, 3.2.16]).

The proof of the proposition requires an application of Grothendieck's factorization theorem to obtain a condition like (P_2) in [2] or (P_3) in [16, 3.2.17] which then reduces the proposition to the classical Palamodov-Retakh result as in [2, lemma 5] or [16, 3.2.18].

If, in our situation, we endow X_n with the relative topology of $\mathscr{D}'(\overline{\Omega}_n) \times \mathscr{C}^n(\overline{\Omega}_n)$, reducedness of the spectrum is easy to prove. Indeed, for $(F, f) \in X_{n+1}$ we take $\psi \in \mathscr{D}(\Omega_{n+1})$ with $\psi = 1$ near Ω_n (this means $\psi(\xi) = 1$ for all ξ in a neighbourhood of $\overline{\Omega}_n$) and an approximate identity e_r (i.e., $e_r(\xi) = r^N \chi(r\xi)$ for some $\chi \in \mathscr{D}(\mathbb{R}^N)$ with $\int \chi(\xi) d\xi = 1$), and we set $F_r := e_r * \psi F$ and $f_r = P(D)F_r$ to obtain $(F_r, f_r) \in \mathrm{Proj}\mathscr{X}$ and $(F_r, f_r)|_{\Omega_n} \to (F, f)$ in X_n for $r \to \infty$.

To apply the proposition as stated we would need X_n to be (LB)-spaces which is not evident since closed subspaces of (LB)-spaces may fail to be

(LB)-spaces. However, we can take the associated (LB)-space topology on X_n and the argument above shows that the spectrum remains reduced for these stronger topologies (a different argument would be, that $\mathscr{D}'(\Omega)\times\mathscr{E}(\Omega)$ has an equivalent spectrum of (LS)-spaces, i.e., (LB)-spaces with compact inclusions, and since this class is indeed stable with respect to closed subspaces, $(X_n)_n$ is equivalent to an (LB)-spectrum, which is enough since the condition in the proposition is invariant under passing to equivalent spectra). To prove the suffiency part of theorem 1 it is therefore enough to verify the condition of the proposition for X_n as above.

The first step is to replace the Dirac distributions by general ones:

Lemma 3. *The condition in theorem 1 implies the following one:*

$$\forall\, n\in\mathbb{N}\quad \exists\, m\geq n\quad \forall\, k\geq m,\ F\in\mathscr{D}'(\Omega)\ \text{with}\ F|_{\Omega_m}=0$$

$$\exists\,(E,e)\in\mathscr{D}'(\mathbb{R}^N)\times\mathscr{C}^k(\mathbb{R}^N)\ \text{with}\ E|_{\Omega_n}=0\ \text{and}\ P(D)E=F+e\ \text{in}\ \Omega_k.$$

Proof. We first note that for Dirac distributions this is the same condition as in theorem 1 (up to replacing n by $n+1$ and k by k plus the degree of P, we just take $e=P(D)(\psi E_\xi)$ with a cut-off function ψ).

For $n\in\mathbb{N}$ we now take $m\in\mathbb{N}$ from theorem 1 according to $n+1$. For $k\geq m$ we set $\varepsilon=\min\{\operatorname{dist}(\overline{\Omega}_n,\Omega_{n+1}^c),\operatorname{dist}(\overline{\Omega}_k,\Omega_{k+1}^c)\}$, cover $\overline{\Omega}_k\setminus\Omega_m$ by finitely many balls $B(\xi_j,\varepsilon)$, and choose a subordinated partition of unity (φ_j), i.e. $\varphi_j\in\mathscr{D}(B(\xi_j,\varepsilon))$ satisfy $\sum\varphi_j=1$ on $\overline{\Omega}_k\setminus\Omega_m$.

For $F\in\mathscr{D}'(\Omega)$ vanishing in Ω_m and each j the distribution φ_jF has compact support in $B(\xi_j,\varepsilon)$ and therefore some finite order $\alpha_j\in\mathbb{N}$. Now we choose $E_j\in\mathscr{D}'(\mathbb{R}^N)$ vanishing in Ω_{n+1} and $e_j\in\mathscr{C}^{k+\alpha_j}(\mathbb{R}^N)$ with $P(D)E_j=\delta_{\xi_j}+e_j$ in Ω_{k+1}.

We set $E=\sum\delta_{-\xi_j}*E_j*\varphi_jF\in\mathscr{D}'(\mathbb{R}^N)$ which vanishes in Ω_n and $e=\sum\delta_{-\xi_j}*e_j*\varphi_jF\in\mathscr{C}^k(\mathbb{R}^N)$ (by [8, 4.2.3]) and obtain

$$\begin{aligned}P(D)E &= \sum\delta_{-\xi_j}*P(D)E_j*\varphi_jF=\sum\varphi_jF+\sum\delta_{-\xi_j}*e_j*\varphi_jF\\ &= F+e\ \text{in}\ \Omega_k.\end{aligned}$$

□

Proof of theorem 1. Now we can verify the abstract condition of the proposition with $Y_n=\{(F,f)\in X_n : F\in\mathscr{C}(\overline{\Omega}_n)\}$.

We choose a fundamental solution L of $P(D)$ with finite order $r\in\mathbb{N}$, see [8, 7.3.10], and choose for fixed $n\in\mathbb{N}$ some $m\geq\max\{n+1,r\}$ satisfying the condition in lemma 3. For $(F,f)\in X_{m+1}$ and $k\geq m+1$ we have to construct $(G,g)\in X_k$ with $(F,f)-(G,g)|_{\Omega_n}\in Y_n$.

We take $\psi\in\mathscr{D}(\Omega_{m+1})$ with $\psi=1$ near Ω_m and obtain

$$P(D)\psi F=f-P(D)((1-\psi)F)=:f+R$$

where $R\in\mathscr{D}'(\mathbb{R}^N)$ vanishes on Ω_m.

From lemma 3 we get $E\in\mathscr{D}'(\mathbb{R}^N)$ vanishing on Ω_n and $e\in\mathscr{C}^k(\mathbb{R}^N)$ with $P(D)E=R+e$ in Ω_{k+1}.

Extending f to a $\mathscr{C}^m(\mathbb{R}^N)$-function with support in Ω_{k+1} and forming the convolution with L we find $E_0\in\mathscr{C}(\mathbb{R}^N)$ with $P(D)E_0=f$.

Now $G = \psi F - E - E_0$ defines a distribution with $P(D)G = -e$. Hence $(G, -e) \in X_k$ and since $\psi F = F$ and $E = 0$ on Ω_n we get

$$(F, f) - (G, -e)|_{\Omega_n} = (E_0|_{\Omega_n}, f + e|\Omega_n) \in Y_n.$$

By proposition 2 and its preceding remarks this completes the proof of the sufficiency part.

We now prove necessity of the condition for surjectivity mod $\mathscr{E}$ on Ω. In this case we have $\mathrm{Proj}^1 \mathscr{X} = 0$ and obtain from the Retakh-Palamodov theorem

$$\forall\, n \in \mathbb{N} \quad \exists\, m \geq n,\ B \in \mathscr{B}(X_n) \quad \varrho^n_m X_m \subseteq \varrho^n \mathrm{Proj}\mathscr{X} + B.$$

There are $\alpha(n) \in \mathbb{N}$ such that for every $(F, f) \in B$ the order of F is bounded by $\alpha(n)$. We fix $k \geq m$ and $\xi \notin \overline{\Omega}_m$, and choose $r \in \mathbb{N}$ with $2r - \alpha(n) \geq k$.

Let $Q(D)$ be the r-th power of the Laplacian (i.e., $Q(\xi) = -|\xi|^{2r}$) and let T be a fundamental solution of $Q(D)$ which then has singular support equal to $\{0\}$. Since $P(D)$ is surjective mod $\mathscr{E}$ there are $(F, f) \in \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega)$ with $P(D)F = Q(D)\delta_\xi + f \in \mathscr{E}(\Omega_m)$.

Hence there are $G \in \mathscr{D}'(\Omega)$ and $g \in \mathscr{E}(\Omega)$ such that $P(D)G = g$ and $\varrho^n_m(F - G, Q(D)\delta_\xi + f - g) \in B$. We thus obtain that the restriction of $F - G$ to Ω_n has order $\leq \alpha(n)$.

We now take $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ near Ω_{k+1} and get

$$P(D)\left(\psi(F - G)\right) = Q(D)\delta_\xi + f - g + P(D)\left((\psi - 1)(F - G)\right),$$

hence $u = f - g + P(D)\left((\psi - 1)(F - G)\right)$ is a distribution with compact support and at the same time in $\mathscr{E}(\Omega_{k+1})$.

For $H = T * \psi(F - G) \in \mathscr{D}'(\mathbb{R}^N)$ we obtain $P(D)H = \delta_\xi + T * u$ with $T * u \in \mathscr{E}(\Omega_{k+1})$ since the singular support of T is $\{0\}$. Using a fundamental solution of $P(D)$ we finally solve $P(D)e = \chi(T * u)$ with a cut-off function $\chi \in \mathscr{D}(\Omega_{k+1})$ which is equal to 1 near Ω_k and $e \in \mathscr{E}(\mathbb{R}^N)$ and set

$$E_\xi = H - e.$$

This solves $P(D)E_\xi = \delta_\xi$ in Ω_k and since $Q(D)E_\xi = \psi(F - G) - Q(D)e$ has order less than $\alpha(n)$ in Ω_n, E_ξ itself must be $2r - \alpha(n)$ times continuously differentiable there.

□

For some purposes, theorem 1 works equally well as Hörmander's powerful characterization by $P(D)$-convexity for singular supports. For instance, the method of [8, 7.3.8] to change in the Fourier inversion formula $\varphi(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{i\langle x, \xi\rangle} \hat{\varphi}(x)dx$ integration over $\mathbb{R}^N$ to a suitable cycle in $\mathbb{C}^N$ and to use the Paley-Wiener-Schwartz-theorem applies directly to obtain the desired fundamental solutions of theorem 1. In this way one can show that $P(D)$ is always surjective mod $\mathscr{E}$ on convex open sets.

Another consequence of theorem 1 (obtained by rather logical than mathematical reasoning) is that surjectivity mod $\mathscr{E}$ on Ω^1 and on Ω^2 carries over to $\Omega = \Omega^1 \cap \Omega^2$.

In this section we characterize right invertibility of $P(D)$ mod $\mathscr{E}$. The investigations of Meise, Taylor, and Vogt [11] about $P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ start with the observation that $\mathscr{D}'(\Omega) = \operatorname{Proj}\mathscr{D}'(\overline{\Omega}_n)$ ($= \operatorname{Proj}\mathscr{D}'(\Omega_n)$ with equivalent spectra) is strict in the sense that $\varrho^n_m X_m = \varrho^n_k X_k$ for all $k \geq m = m(n)$ and thus the existence of a right inverse implies that (as a quotient) $\ker P(D)$ is strict, too. A surprisingly simple construction showed that strictness is also sufficient.

A different method would be to use the splitting theory for $\mathscr{D}'$ developed by Domański and Vogt [3, 4] which in fact was inspired by [11] and even allows applications to distributional complexes. In our situation, the "model spaces" $\mathscr{D}' \cong (s')^{\mathbb{N}}$ and $\mathscr{E} \cong s^{\mathbb{N}}$ are mixed up and it seems too ambitious to search for a general splitting theory (for instance, already $\operatorname{Ext}^1_{PLS}(\mathscr{E}, \mathscr{D}')$ does not vanish). However, the direct approach of [11] works well after changing from the natural but non-strict representation $\mathscr{E}(\Omega) = \operatorname{Proj}\mathscr{C}^n(\overline{\Omega}_n)$ to the strict spectrum $(\mathscr{E}(\Omega_n))_{n\in\mathbb{N}}$. We therefore consider $Y_n := \mathscr{D}'(\Omega_n) \times \mathscr{E}(\Omega_n)$ and $X_n = \{(F, f) \in Y_n : P(D)F = f\}$.

Theorem 4. *$P(D)$ has a right inverse mod $\mathscr{E}$ on Ω if and only if*

$$\forall n \in \mathbb{N} \quad \exists\, m \geq n \quad \forall\, k \geq m,\ \xi \notin \overline{\Omega}_m \quad \exists\, E_\xi \in \mathscr{D}'(\mathbb{R}^N)$$
$$P(D)E_\xi = \delta_\xi \text{ in } \Omega_k \text{ and } E_\xi \in \mathscr{E}(\Omega_n).$$

Proof. We first prove that invertibility mod $\mathscr{E}$ implies strictness for the kernel spectrum.

Since $(Y_n)_n$ is not a spectrum of (LB)-spaces, this does not already follow from the abstract theory (since there is no result about automatic equivalence of general strongly reduced spectra). Nevertheless, strictness of the kernel spectrum follows from the fact that the continuity of a linear map $R : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega)$ implies that for each $n \in \mathbb{N}$ there is $m \geq n$ such that $R(F)$ vanishes on Ω_n whenever F vanishes on Ω_m. Indeed, since the separable Fréchet space $\mathscr{D}(\overline{\Omega}_n)$ has many total bounded sets B, for the typical 0-neighbourhood $B^\circ \times U$ in $\mathscr{D}'(\Omega) \times \mathscr{E}(\Omega)$ with $U = \{f \in \mathscr{E}(\Omega) : |f^{(\alpha)}(x)| \leq 1 \text{ for all } |\alpha| \leq n \text{ and } x \in \overline{\Omega}_n\}$ the elements (F, f) of $\bigcap_{\varepsilon>0} \varepsilon(B^\circ \times U)$ all vanish on Ω_n. And since each 0-neighbourhood of $\mathscr{D}'(\Omega)$ contains some C° with a bounded set C in some $\mathscr{D}(\overline{\Omega}_m)$, the continuity estimate $R(C^\circ) \subseteq B^\circ \times U$ gives the desired implication.

Let now R be a continuous linear right inverse of $P(D) + id$ and $n \leq m$ as above. We choose $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ near Ω_m. For $(F, f) \in X_m$ we consider $(G, g) := (\psi F, -\psi f) \in \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega)$. From the vanishing of $P(D)G + g$ on Ω_m we conclude that $(H, h) = R(P(D)G + g)$ vanishes on Ω_n and we obtain $(G, -g) - (H, -h) \in \operatorname{Proj}\mathscr{X}$ with restriction to Ω_n equal to the restriction of (F, f).

Let us now assume that $(X_n)_{n\in\mathbb{N}}$ is strict which implies $\operatorname{Proj}^1\mathscr{X} = 0$, and as in Section 2 this gives that $P(D)$ is surjective mod $\mathscr{E}$. Moreover, we have $\varrho^n_m X_m \subseteq \varrho^n \operatorname{Proj}\mathscr{X}$ for some $m > n$. We fix $\xi \notin \overline{\Omega}_m$ and choose $(F, f) \in \mathscr{D}'(\Omega) \times \mathscr{E}(\Omega)$ with $P(D)F - f = \delta_\xi$. Since $(F, f)|_{\Omega_n} \in X_n$ there is $(G, g) \in \operatorname{Proj}\mathscr{X}$ with $(G, g)|_{\Omega_n} = (F, f)|_{\Omega_n}$.

We get $P(D)(F-G) = \delta_\xi + f - g$ and that $F-G$ vanishes in Ω_n. For $k \geq m$ we take $\psi \in \mathscr{D}(\Omega)$ with $\psi = 1$ near Ω_k and form the convolution $h = L * \psi(f-g)$ where L is a fundamental solution of $P(D)$. Then $E_\xi = F - G - h$ has the desired properties.

To prove sufficiency of the condition, we proceed as in the proof of lemma 3 to obtain (with the same quantifiers as there) for each $F \in \mathscr{D}'(\Omega)$ with $F|_{\Omega_m} = 0$ some $(E, e) \in \mathscr{D}'(\mathbb{R}^N) \times \mathscr{E}(\mathbb{R}^N)$ with $E|_{\Omega_n} = 0$ and $P(D)E = R + e$ in Ω_k. The crucial difference now is, that the constructive proof gives a continuous linear operator $T = T(n, m, k) : \mathscr{D}'(\Omega, \Omega_m) \to \mathscr{D}'(\mathbb{R}^N) \times \mathscr{E}(\mathbb{R}^N)$ such that $(E, e) = T(R)$ has these properties for each $R \in \mathscr{D}'(\Omega, \Omega_m) = \{R \in \mathscr{D}'(\Omega) : R|_{\Omega_m} = 0\}$ (in the proof of lemma 3, $e_j \in \mathscr{C}^{s+\alpha_j}$ depend on F via its order, in the present situation, $e_j \in \mathscr{E}$ are independent of F).

By renumbering the elements of the exhaustion we may assume $\Omega_0 = \emptyset$, $m(0) = 0$, and $m(n) = n + 1$ for $n \in \mathbb{N}$, and we abbreviate $T_n = T(n, m(n), n+2)$. Thus, by restricting the images, we obtain continuous linear operators

$$T_n : \mathscr{D}'(\Omega, \Omega_{m(n)}) \to \mathscr{D}'(\Omega, \Omega_n) \times \mathscr{E}(\Omega)$$

such that $S_n := (P(D) + id) \circ T_n$ induces the identity operator in $\mathscr{D}'(\Omega_{n+2})$. Inductively, we set $Q_0 = T_0$ and

$$Q_{n+1}(F) = Q_n(F) + T_{n+1}(F - S_n(F)) \text{ for } F \in \mathscr{D}'(\Omega).$$

Since $F - S_n(F)$ vanishes in Ω_{n+2} this is well-defined, and since $Q_{n+1}(F) - Q_n(F)$ vanishes in Ω_n the limit $Q = \lim Q_n$ defines a continuous linear right inverse of $P(D) + id$. □

The formulation of the third condition in the theorem is chosen to emphasize the similarity with theorem 1. If $P(D)$ is surjective on $\mathscr{E}(\Omega)$, strictness of the kernel spectrum implies the (formally) stronger condition

$$\forall n \in \mathbb{N} \quad \exists\, m \in \mathbb{N} \quad \forall\, \xi \notin \overline{\Omega}_m \quad \exists\, E_\xi \in \mathscr{D}'(\Omega) \text{ with}$$

$$P(D)E_\xi = \delta_\xi \text{ and } E_\xi \in \mathscr{E}(\Omega_n).$$

For $\Omega = \mathbb{R}^N$ yet another formulation is that for each $|\xi| \geq m$ there is a parametrix for $P(D)$ which vanishes in the ball $B(\xi, n)$.

The typical examples of operators with right inverses mod $\mathscr{E}$ are hypoelliptic operators which have fundamental solutions that are $\mathscr{C}^\infty$ outside the origin. Since hypoellipticity at the same time is the typical situation where $P(D)$ does not have a right inverse properly (which was proved by Vogt [15], see also [11, corollary 2.11]), one might guess that each partial differential operator has a right inverse mod $\mathscr{E}$ on convex sets (then the condition of theorem 1 is satisfied and the one in theorem 3 looks only slightly stronger).

In the next section we show that this is not the case even for $\Omega = \mathbb{R}^N$.

4. An example

Example 5. *For $P(x, y, z) = x^2 + y^2$, the operator $P(D)$ is surjective on $\mathscr{E}(\mathbb{R}^3)$ and $\mathscr{D}'(\mathbb{R}^3)$ but it does not have a right inverse mod $\mathscr{E}$ on $\mathbb{R}^3$.*

Since the heat operator has the same principal part as the operator above this shows that having a right inverse mod $\mathscr{E}$ does not only depend on the principal part of the symbol.

To prove non-existence of a right inverse mod $\mathscr{E}$ we use Hörmander's result [8, 11.3.7] about the propagation of regularity, namely that for open convex sets $\Omega_1 \subset \Omega_2$ the following conditions are equivalent:

(1) Every $u \in \mathscr{D}'(\Omega_2) \cap \mathscr{E}(\Omega_1)$ with $P(D)u = 0$ belongs to $\mathscr{E}(\Omega_2)$.

(2) Every hyperplane H with $\sigma_p(H') = 0$ which intersects Ω_2 also intersects Ω_1.

Here, H' denotes the orthogonal complement of H and for a subspace V of $\mathbb{R}^N$ we use Hörmander's [8, section 11.3] definitions

$$\begin{aligned} \sigma_p(V) &= \inf_{t>1} \varliminf_{\xi\to\infty} \widetilde{P}_V(\xi,t)/\widetilde{P}(\xi,t) \text{ with} \\ \widetilde{P}_V(\xi,t) &= \sup\{|P(\xi+\theta)| : \theta \in V, |\theta| \le t\} \text{ and} \\ \widetilde{P}(\xi,t) &= \widetilde{P}_{\mathbb{R}^3}(\xi,t) \sim \sum_\alpha |P^{(\alpha)}(\xi)| t^{|\alpha|} \end{aligned}$$

(where the tilde $\sim$ means that the quotient is contained in $[1/c, c]$ for some $c \ge 1$). For a one dimensional subspace $V = \{\lambda(a,b,c) : \lambda \in \mathbb{R}\}$ and $P(x,y,z) = x^2 + y^2$ as in the example, an elementary calculation yields $\sigma_P(V) \sim a^2 + b^2$, hence a hyperplane H in $\mathbb{R}^3$ satisfies $\sigma_p(H') = 0$ if and only if it is parallel to the (x,y)-plane. Therefore, regularity of zero solutions propagates along such hyperplanes (which is not too surprising, since for fixed $z = z_0$, $P(D)$ "is" the Laplacian).

We assume now, that $P(D)$ has a right inverse mod $\mathscr{E}$ on $\mathbb{R}^3$. According to theorem 3 and the remark after its proof we obtain (by shifting ξ to 0)

$$\forall n \in \mathbb{N} \quad \exists\, m \ge n \quad \forall\, |\xi| \ge m \quad \exists\, E \in \mathscr{D}'(\mathbb{R}^3)$$

$$P(D)E = \delta_0 \text{ and } E \in \mathscr{E}(B(\xi, n))$$

where $B(\xi, n)$ denotes the ball of radius n around ξ. In particular, there is a fundamental solution E with $E \in \mathscr{E}(B((m,0,0),1))$ for some $m \in \mathbb{N}$.

For each point $p = (x,y,z)$ with $|z| < 1$ there is a line L through p and some point of the ball $B((m,0,0),1)$ which is parallel to the x,y-plane and does not contain 0. Applying the propagation of regularity to $\Omega_2 = L + B(0,\varepsilon)$ and $\Omega_1 = \Omega_2 \cap B((m,0,0),1)$ with an appropriate $\varepsilon > 0$ we obtain that E is $\mathscr{C}^\infty$ in the set $\{|z| < 1\} \setminus \{0\}$.

Choosing $\psi \in \mathscr{D}(\{|z| < 1\})$ with $\psi = 1$ in a neighbourhood of 0 we see that $f = P(D)(1-\psi)E = \delta_0 + P(D)(\psi E)$ is a $\mathscr{C}^\infty$-function on $\mathbb{R}^3$ (the right hand side is $\mathscr{C}^\infty$ outside the origin and $P(D)(1-\psi)E$ is $\mathscr{C}^\infty$ at the origin). Since every constant coefficient partial differential operator is surjective on $\mathscr{E}(\mathbb{R}^3)$ we find $g \in \mathscr{E}(\mathbb{R}^3)$ with $P(D)g = f$ and obtain a fundamental solution $F = \psi E + g$ of $P(D)$ which is singular only at 0. This implies that $P(D)$ is hypoelliptic (see e.g. [8, 11.1.1]) which is not true since $P(D)F = 0$ for every distribution which only depends on the z-variable.

Essentially, we know only this single example of a partial differential operator without right inverse mod $\mathscr{E}$ on $\mathbb{R}^N$. A more natural candidate would be the Schrödinger operator $i\partial_t - \Delta_x$ for which we suspect that it does not have a right inverse mod $\mathscr{E}$ on $\mathbb{R}^N$ either.

References

[1] J. Bonet, C. Fernandez, R. Meise, *Operators of solution for convolution equations.* Note di Mat. **17** (1997), 1-12.

[2] R. Braun, D. Vogt, *A sufficient condition for* $\text{Proj}^1 \mathcal{X} = 0$. Michigan Math. J. **44** (1997), 149-156.

[3] P. Domański, D. Vogt, *A splitting theory for the space of distributions.* Studia Math. **140** (2000), 57-77.

[4] P. Domański, D. Vogt, *Distributional complexes split for positive dimensions.* Journal Reine Angew. Math. **522** (2000), 63-79.

[5] L. Frerick, J. Wengenroth, *Surjective convolution operators on spaces of distributions.* Rev. R. Acad. Cien. Seria A. Mat. **97** (2003), 263-272.

[6] L. Frerick, J. Wengenroth, *Convolution equations for ultradifferentiable functions and ultradistributions.* J. Math. Anal. Appl., to appear.

[7] L. Hörmander, *On the range of convolution operators.* Ann. Math. **76** (1962), 148-170.

[8] L. Hörmander, *The Analysis of Linear Partial Differential Operators I and II.* Springer-Verlag, Berlin, 1983.

[9] M. Langenbruch, *Characterization of surjective partial differential operators on spaces of real analytic functions.* Studia Math. **162** (2004), 53-96.

[10] R. Meise, B. A. Taylor, D. Vogt, *Caractérisation des opérateurs linéaires aux dérivées partielles avec coefficients constants sur* $\mathscr{E}(\mathbb{R}^N)$ *admettant un inverse à droite qui est linéaire et continu.* C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 239-242.

[11] R. Meise, B. A. Taylor, D. Vogt, *Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse.* Ann. Inst. Fourier (Grenoble) **40** (1990), 619-655.

[12] V.P. Palamodov, *The projective limit functor in the category of linear topological spaces.* Mat. Sb. **75** (1968), 567-603 (in Russian); English transl.: Math. USSR-Sb. **4** (1968), 529-558.

[13] V.P. Palamodov, *Homological methods in the theory of locally convex spaces.* Uspekhi Mat. Nauk **26**, 3 - 65 (1971) (in Russian); English transl.: Russian Math. Surveys **26**, 1-64 (1971)

[14] V.S. Retakh, *Subspaces of a countable inductive limit.* Dokl. Akad. Nauk SSSR **194**, 1277-1279 (1970) (in Russian); English transl.: Soviet Math. Dokl. **11**, 1384-1386 (1970)

[15] D. Vogt, *On the solvability of* $P(D)g = f$ *for vector valued functions.* RIMS Kokyuroku **508** (1983), 168-182.

[16] J. Wengenroth, *Derived Functors in Functional Analysis.* Lecture Notes in Math. 1850, Springer, Berlin, 2003.

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