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Multifractal functions: Recent advances and open problems

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Abstract: We raise several questions related to the pointwise regularity of functions and their multifractal analysis.

1 Introduction

The present paper can be seen as a sequence of [25], which was written in 1996 and listed open problems related with the multifractal analysis of functions. The subject at that time was in full bloom, driven by an explosion of applications: All possible kinds of signals were submitted to a "multifractal analysis", with apparent success. Indeed, as we shall see, the criterium of multifractality for a signal amounts to check that a certain curve, obtained from the data, is *not* linear, a criterium which is particularly easy to check in practice. Several scientists challenged these results and asked for a scientific interpretation of these numerical computations which would be backed by mathematical theorems. Though multifractal analysis of *measures* had largely developed at that time, very few mathematical results were available for functions, and the most simple and natural questions were open. The motivation of [25] was to list these questions, organize them as what could be described as a "research program" and hopefully attract students to a very promising area. And, indeed, the subject widely developed, largely due to talented students. It evolved widely since then, and its foundations have been largely renewed. Here are two examples of these changes:

Multifractal analysis is based on the notion of Hölder pointwise regularity; its purpose is to determine the dimensions of the sets of points with a given regularity. All mathematical results concerning this type of regularity assume that the function considered has some uniform Hölder regularity; this prevented the introduction of these tools in image analysis, since images always present discontinuities, as a result of the occlusion phenomenon (some objects are partly hidden behind others). Recently, it was found that a slight weakening of the notion of Hölder pointwise regularity, the T^p_α regularity, allows to derive

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similar results for which the uniform Hölder regularity assumption is no more needed, thus allowing for discontinuous functions (see Section 2.4). This example shows how in some settings, the basic definitions of the tools which are used had to be changed.

Another example is supplied by the derivation of the multifractal formalism: The dimensions of the sets with a given Hölder regularity cannot be computed directly in applications; they have to be deduced from global, averaged quantities. The quantities used were based on the Sobolev or Besov regularity of the function. This had strange consequences: The domain of validity of the formulas thus obtained were partial, and these failures could be witnessed, but not really explained, untill it was understood that these quantities should not be based on Sobolev spaces, but on other new function spaces of a different kind. This alternative point of view shed a new light on all the problems concerning the validity of the multifractal formalism.

As these radical changes took place, most problems raised in [25] became either solved or obsolete; now, the mathematical foundations of the subject are firm, and the purpose of the present paper is to give an overview of these recent deep changes in the field and of the many new open problems which have thus been uncovererd.

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2 Pointwise regularity

2.1 The Hölder exponent

Historically, the first definition of pointwise regularity introduced was the *Hölder* regularity.

Definition 1. *Let $x_0 \in \mathbb{R}^d$ and let $\alpha \geq 0$. A locally bounded function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exists a constant $C > 0$ and a polynomial P of degree less than α and such that, in a neighbourhood of x_0,*

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{1}$$

The Hölder exponent of f at x_0 is

$$h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

A first natural question is to determine which functions $h(x)$ are Hölder exponents. The answer may depend on the global regularity assumptions made on f. Since (1) implies that f is bounded in a neighbourhood of x_0, we do not restrict the problem by assuming that $f \in L^\infty(\mathbb{R}^d)$. On the other hand, the Hölder exponents of *continuous* functions are known to be exactly the functions which are lim inf of a sequence of continuous functions, see [14, 23] (we will say that such a function is LIC). However, the general problem of determining which functions are Hölder exponents of bounded functions remains open. Note that several classes of functions which have a dense set of discontinuities have been studied in the one-dimensional case ($d = 1$) and also led to LIC Hölder exponents. One example is given by the sample paths of Lévy processes, see [26]; another example is supplied by *Hecke's functions* for $s > 2$, see [28]; these functions are defined as follows: Let

$$\left.\begin{array}{lll} \{x\} & = x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ & = 0 & \text{otherwise} \end{array}\right\} \tag{2}$$

($\{x\}$ is the "sawtooth function"); Hecke's functions are

$$\mathcal{H}_s(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^s}. \tag{3}$$

Keeping these examples in mind, the general problem of finding the class of Hölder exponents, which we raised in the L^∞ setting, can be specified by making additional assumptions on f (listed here in increasing order of requirements):

- f has only discontinuities of the first kind,
- f is a *pure jump* function, i.e. f can be written as the uniform limit of a series $\sum f_n(x)$ where each f_n is of the form $f_n = a_n H(x - b_n) + c_n$, and $H(x)$ is the Heaviside function,
- f is the sum of a normally convergent *Davenport series* $\sum_{n=1}^{\infty} a_n\{nx\}$, with $a_n \in l^1$ (and $\{x\}$ is defined by (2)).

The following lemma can always be applied in these more precise settings, see [28].

Lemma 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded function having discontinuities of the first kind on a dense set of points. If $x_0 \in \mathbb{R}$ then, let $\mathcal{R} = (r_n)_{n\in\mathbb{N}}$ be a sequence of discontinuities of f converging to x_0; denote by s_n the jump of f at r_n. Then*

$$h_f(x_0) \leq \inf_{\mathcal{R}} \liminf_{n\to\infty} \left(\frac{\log|s_n|}{\log|r_n - x_0|} \right), \tag{4}$$

where the first infimum is taken on all sequences of discontinuities $\mathcal{R}$ that converge to x_0.

Note that it is very easy to construct counterexamples where (4) is a strict inequality, at least for pure jump functions: For instance, let f be the even function defined on $(-1,1)$ by

$$\forall n \geq 1, \quad \text{on} \quad \left[\frac{1}{n+1}, \frac{1}{n}\right), \quad f(x) = \frac{1}{\sqrt{n}}; \tag{5}$$

then $h_f(0) = 1/2$, whereas (4) only yields the upper bound 3/2. Beyond this simple example, classes of pure jumps functions where the Hölder exponent is given by (4) at some points but not everywhere, have been constructed by J. Barral and S. Seuret in [4]. (These examples are actually measures constructed as series of Dirac masses at the rationals, but their distribution functions yield the required examples.) Under which condition the Hölder irregularity at every point comes only from the individual contribution of each jump (in which case (4) is an equality), and not their accumulation, as in (5)?

2.2 Stochastic processes

If f is a stochastic process (or a random field, in the several dimensional case), then its the Hölder exponent also is a stochastic process; actually, in some cases, it is a deterministic function; a standard example is supplied by the Brownian motion $B(x)$ where

$$a.s.\ \forall x, \ \ h_B(x) = \frac{1}{2}, \tag{6}$$

or, more generally, by the fractional Brownian motion of order H ($0 < H < 1$) for which a.s. $\forall x$. $h_{B_H}(x) = H$. Such functions for which the Hölder exponent is a constant are called *monohölder functions*. In the general case, the standard measurability problems which are met in the theory of stochastic processes reappear in this context: Which hypotheses on a process X_t imply that the process $h_X(t)$ is indeed measurable? Is the mapping $X \rightarrow h_X$ measurable?

For other processes, problems are posed by the positions of quantifiers in (6); the comparison between the Brownian motion and a Lévy process of upper index β without Brownian part is enlightening: The sample paths of such a Lévy process satisfy $\forall t$, a.s. $h_X(t) = 1/\beta$, see [8]; but the $a.s.$ and the quantifier $\forall t$ cannot be commuted here, since the Hölder exponent is random, and takes all values in $[0, 1/\beta]$ on sets of vanishing measure, see [26]; thus there is no almost sure Hölder exponent valid $\forall t$. FBMs and Lévy processes share a common characteristic: The a.e. Hölder exponent is deterministic. Does this property hold under general, weak asssumptions on the process? Note that it cannot be true in general: Consider for instance the (very artificial!) following process: Let θ be a Bernouilli variable ($\mathbb{P}(\theta = 1) = \mathbb{P}(\theta = 0) = 1/2$), and let B_1 and B_2 be two Fractional Brownian Motions of different indices. Let us further assume that θ, B_1 and B_2 are independent. The process X_t is defined by:

$$\text{If} \quad \theta = 0, \ \text{then} \ \ X_t = B_1(t); \qquad \text{else} \ \ X_t = B_2(t); \tag{7}$$

The a.e. Hölder exponent of this process is random. Under which assumption is the Hölder exponent of a process almost surely a deterministic function? The case of Gaussian processes is particularly important; if X_t is Gaussian, then $\forall t$, a.s. $H_X(t)$ is deterministic (this is a straightforward consequence of general results concerning the 0-1 law for the oscillation of Gaussian processes, see [35] Section 7 Theorem 1). However, one cannot commute the $\forall x$ and the a.s. in this statement, as we now show.

We first need to recall the notion of pointwise regularity which is attached with a probability measure μ defined on $\mathbb{R}^d$. The *lower pointwise dimension* of μ at x_0 is

$$\mathcal{D}_\mu(x_0) = \liminf_{r \to 0} \left(\frac{\log \mu(B(x_0, r))}{\log r} \right), \tag{8}$$

where $B(x_0, r)$ denotes the ball of center x_0 and radius r. For measures, this quantity plays the role played by the Hölder exponent for functions; the spectrum of singularities of a measure is defined in the same way as for functions: It is the Hausdorff dimension of the level sets of $\mathcal{D}_\mu$. Let now μ be a deterministic probability measure supported by $\mathbb{R}^+$, and let f be its distribution function $f(x) = \mu([0, t])$. The *Brownian motion in time* μ is the Markov Gaussian process

$$B_\mu(t) = B(f(t)) \tag{9}$$

where $B(t)$ is a standard Brownian motion, see [38]. Uniform regularity results on the Hölder regularity of $B(t)$ imply that

$$a.s.\ \forall t,\ \ h_{B_\mu}(t) = \frac{1}{2}\mathcal{D}_\mu(t). \tag{10}$$

Thus a Gaussian Markov process can have essentially any deterministic Hölder exponent. Is it possible to construct Gaussian processes with *random* Hölder exponent?

Finally, let us recall what is implied by the stationary increments assumption; if $X_{t+t_0} - X_{t_0}$ has the same law as $X_t - X_0$, then it follows that the law of $h_X(t)$ is invariant by translation; therefore if X_t has an a.e. deterministic Hölder exponent, then this Hölder exponent is a.e. a constant.

2.3 Wavelet characterizations

Signals and images are more and more often stored trough their coefficients on orthonormal wavelet bases. Therefore deriving regularity properties directly from the wavelet coefficients is important. We use smooth, localized wavelets $\psi^{(i)}$ (where (i) takes $2^d - 1$ values) such that the $2^{dj/2}\psi^{(i)}(2^j x - k)$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, form an orthonormal basis of $L^2(\mathbb{R}^d)$, see [39]. Therefore any L^2 function f can be written $f(x) = \sum_{j,k} c_{j,k}^{(i)}\, \psi^{(i)}(2^j x - k)$ where $c_{j,k}^{(i)} = 2^{dj} \int f(x)\, \psi(2^j x - k)\, dx$. (We choose an L^∞ normalisation for the wavelets.) We will use an alternative indexing by the dyadic cubes. Since i takes $2^d - 1$ values, we can assume that it takes values in $\{0,1\}^d - (0,\ldots,0)$; we will denote by λ the cube

$$\lambda \ (= \lambda(i,j,k)) \ = \frac{k}{2^j} + \frac{i}{2^{j+1}} + \left[0, \frac{1}{2^{j+1}}\right)^d, \quad c_\lambda = c_{j,k}^{(i)}, \quad \psi_\lambda(x) = \psi^{(i)}(2^j x - k).$$

The wavelet ψ_λ is essentially localized near the cube λ.

If $f \in L^\infty$, then $|c_\lambda| \leq 2^j \int |f(x)||\psi(2^j x - k)|dx \leq C \parallel \psi \parallel_{L^1} \parallel f \parallel_{L^\infty}$; the $d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|$ are thus finite; they are reffered to as the *wavelet leaders*, see [27]. We denote by $\lambda_j(x_0)$ the dyadic cube of side 2^{-j} containing x_0 and

$$d_j(x_0) = \sup_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|,$$

where $3\lambda_j(x_0)$ denotes the cube of same center as $\lambda_j(x_0)$, but three times wider. The wavelet characterization of the Hölder exponent requires a regularity hypothesis which is slightly stronger than continuity: A function f is said to be *uniform Hölder* if $\exists \epsilon > 0$ such that $f \in C^\epsilon(\mathbb{R}^d)$, i.e. $\exists C > 0$ such that $\forall x, y \in \mathbb{R}^d$, $|f(x) - f(y)| \leq C|x-y|^\epsilon$. The following theorem is a restatement of a result of [22] and allows to characterize the pointwise regularity by a decay condition of the $d_j(x_0)$ when $j \to +\infty$.

Theorem 1. *Let $\alpha > 0$. If f is $C^\alpha(x_0)$, then there exists $C > 0$ such that*

$$\forall j \geq 0, \qquad d_j(x_0) \leq C 2^{-\alpha j}. \tag{11}$$

Conversely, if (11) holds and if f is uniform Hölder, then $\exists C > 0$ and a polynomial P satisfying $deg(P) < \alpha$ and such that, in a neighbourhood of x_0,

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \log(1/|x - x_0|).$$

In several applications, directional regularity plays an important role (analysis and synthesis of clouds, of bones, or more generally, in medical image processing, see [1]).

Suppose that $d > 1$ and let θ be a vector of modulus 1. A function f belongs to $C^\alpha_\theta(x_0)$ if the one-dimensional function $t \to f(x_0 + \theta t)$ belongs to $C^\alpha(0)$. One cannot expect directional regularity to be characterized by conditions bearing on the moduli of the wavelet coefficients. However it might become the case if one uses extensions of wavelet bases which can be "elongated" in particular directions. Such systems include for instance the "ridgelets" introduced by E. Candes and D. Donoho, see [12], or the "bandelets" introduced by S. Mallat, see [36]. Such characterizations might be a first step towards constructing functions with a prescribed directional regularity. Actually, prescribing directional regularity at one point is no problem; for instance, let g be a continuous positive function defined on the unit sphere S^{d-1}; write any point $x \in \mathbb{R}^d$ in polar coordinates $x = r\theta$, with $\theta \in S^{d-1}$; then the function

$f(x) = r^{g(\theta)}$ has the directional smoothness $g(\theta)$ at the origin. Is it possible to impose arbitrary directional smoothness at different points, or should these smoothnesses satisfy some compatibility conditions? Constructing stochastic processes with rather flexible directional properties is also an important issue in modelling (clouds, medical imaging,...), see [1, 9].

2.4 Weakenings of the Hölder regularity condition

Substitutes of the pointwise Hölder regularity condition were first introduced by Calderón and Zygmund in 1961, see [11].

Definition 2. *Let $p \geq 1$; a function $f \in L^p_{loc}(\mathbb{R}^d)$ belongs to $T^p_\alpha(x_0)$ if there exist $R > 0$ and a polynomial P, such that $deg(P) < \alpha$, satisfying*

$$\forall r \leq R \quad \left(\frac{1}{r^d} \int_{B(x,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C r^\alpha. \tag{12}$$

The p-exponent of f at x_0 is

$$u^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}. \tag{13}$$

Remarks: The usual Hölder condition $C^\alpha(x_0)$ corresponds to $p = \infty$; it follows that the ∞-exponent is the usual Hölder exponent. If f belongs to $C^\alpha(x_0)$, then, $\forall p \geq 1$, f belongs to $T^p_\alpha(x_0)$; more generally, if $p' < p$, then $T^p_\alpha(x_0) \hookrightarrow T^{p'}_\alpha(x_0)$. The p-exponent can be characterized by conditions on the wavelet coefficients, with the help of the *local square function* of f at x_0: $S_f(j, x_0)(x) = \left(\sum_{\lambda \in 3\lambda_j(x_0)} |c_\lambda|^2 1_\lambda(x) \right)^{1/2}$. The following result is proved in [29].

Theorem 2. *Let $p \in (1, \infty)$ and $u > -d/p$; if $f \in T^p_u(x_0)$, then $\exists C \geq 0$ such that $\forall j \geq 0$,*

$$\| S_f(j, x_0) \|_p \leq C 2^{-j(u+d/p)}. \tag{14}$$

Conversely, if (14) holds and if $\alpha \notin \mathbb{N}$, then $f \in T^p_u(x_0)$.

Since the converse part of Theorem 2 does not require a uniform regularity assumption, it follows that using the $T^p_u(x_0)$ conditions allows to deal with discontinuous functions; a requirement which, as previously mentioned, is mandatory in image analysis.

Of course, all problems that we raised for the the Hölder exponent can be posed exactly in the same way concerning the p-exponent. We won't list them here a second time. On the other hand, an important problem which is specific to this setting is to understand the relationships between the different p-exponents of a given, locally bounded function f. For instance, are there global assumptions on f which allow for discontinuities but imply that these exponents are independent of p (at least for

$p < \infty$)? What of characteristic functions? The p-exponent of characteristic functions can take any nonnegative value; therefore, it is interesting to determine which functions are the p-exponent of a characteristic function.

The $T^p_\alpha(x_0)$ condition expresses the fact that the pointwise Hölder regularity conditions holds "on the average" (in the L^p sense). Another possible generalization can be obtained by imposing that it holds "outside a small set", see [10].

Definition 3. *If $E \subset \mathbb{R}^d$ is measurable, denote by $Meas(E)$ its Lebesgue measure. The lower density of E at x is*

$$d(E,x) = \liminf_{h\to 0} \left(\frac{Meas(E \cap B(x,r))}{Meas(B(x,r))} \right).$$

The function f has an approximate *Hölder regularity α at x_0 (we write $f \in C^\alpha_{ap}(x_0)$) if there exist a set E of vanishing lower density at x_0, a constant $C > 0$ and a polynomial P of degree less than α such that*

$$\forall x \notin E, \quad |f(x) - P(x - x_0)| \le C|x - x_0|^\alpha. \tag{15}$$

Approximate continuity or differentiability can be defined in a similar manner. Approximate regularity and T^p_α conditions have beeen widely used to prove results of *pointwise* regularity or irregularity for functions having a certain global regularity (e.g. are continuous, or belong to a Sobolev space) in situations where Hölder regularity is too strong to be expected.

If $f \in T^p_\alpha(x)$, then f has an approximate Hölder regularity α at x_0. However, one cannot recover any regularity, or any estimate on the wavelet coefficients of f, from its approximate Hölder regularity, because we miss information on how f behaves on the set E which has been excluded. However, T^p_α regularity can be recovered from approximate regularity under two types of additional conditions: If an a priori global regularity assumption is available and if the control on the size of the "bad set" E is stronger than in the definition of approximate regularity. The control needed is given by an assumption on the lower pointwise dimension at x_0 of the measure μ of density $d\mu = 1_E dx$. Recall that the lower pointwise dimension of a measure μ is given by (8); by extension, the *lower pointwise dimension of a set E* at x_0 is the lower pointwise dimension of the measure μ of density $d\mu = 1_E dx$.

The lower pointwise dimension of a set E at a point x is always bounded from below by d. We can now give a simple example, motivated by the mathematical modelling of images, where this setting is pertinent. Images are grey-level, so that it is natural to model them by bounded functions; this will be our a priori global assumption. The following result is straightforward.

Lemma 2. *Let f be a bounded function; if there exist a set E of lower pointwise dimension at x_0 larger than $\delta = d + \alpha p$ and a polynomial P of degree less than α such that (15) holds, then $f \in T^p_\alpha(x_0)$.*

It would be interesting to extend this lemma and study functions which satisfy near x_0 the following type of conditions (which depend on parameters that can be tuned arbitrarily):

- A $T^p_\alpha(x_0)$ condition outside a "small" set E of given lower pointwise dimension at x_0.
- A uniform regularity assumption of Besov or Hölder type.

2.5 The abstract function space setting

The condition $f \in C^\alpha(x_0)$ can be rewritten as follows: There exists a polynomial P and constants $C, R0$ such that

$$\forall r \leq R \quad \| (f-P)1_{B(x_0,r)} \|_\infty \leq Cr^\alpha. \tag{16}$$

Therefore, it can be interpreted as a local condition bearing on the L^∞ norm. It can be generalized simply by replacing the L^∞ norm by another one. Note that the $T^p_u(x_0)$ condition can clearly be rewritten $\| (f-P)1_{B(x_0,r)} \|_p \leq Cr^{u+d/p}$, so that it is of this form.

Let E be a vector space of distributions eventually defined modulo P_N, the vector space of polynomials of degree less than N and satisfying $\mathcal{S} \hookrightarrow E \hookrightarrow \mathcal{S}'$; we assume that E is endowed with a semi-norm, which becomes a norm on E/P_N. Thus, if B is a ball of $\mathbb{R}^d$, we note

$$\| f \|_{E,B} = \inf_{f=g \text{ on } B} \| g \|_E; \tag{17}$$

recall that the notation $f = g$ on B means that the distributions f and g coincide on B, i.e.

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^d), \quad supp(\varphi) \subset B \Longrightarrow \langle f|\varphi\rangle = \langle g|\varphi\rangle.$$

Note that (17) may be defined even when f does not belong to E; when (17) is well defined, we say that f belongs *locally* to E.

Definition 4. *If E is a space of distributions satisfying the above assumptions, we call two-microlocal space of order α associated with E the space C^α_E defined by the condition*

$$\exists P \text{ polynomial } \exists R, C > 0, \forall r \leq R \quad \| f - P \|_{E,B} \leq Cr^\alpha.$$

Another generalization of pointwise regularity in an abstract Banach space framework was proposed by Y. Meyer in Chapter 1.2 of [40]. What are the relationships between both points of view?

Note that the two-microlocal spaces associated with L^∞ is $C^\alpha(x_0)$. Yves Meyer showed that, $E = \dot{C}^\beta(\mathbb{R}^d)$, then the corresponding two-microlocal spaces are the spaces $C^{\alpha+\beta,-\alpha}(x_0)$ which were previously introduced by Jean-Michel Bony.

In the general case, the pointwise exponent of f at x_0 with respect to E is

$$h_f^E(x_0) = \sup\{\alpha : \ f \in C_E^{\alpha}(x_0)\}.$$

When $E = L^\infty$ (respectively $E = L^p$), we obtain the Hölder exponent (respectively the p-exponent shifted by d/p). Does the set $\{h_f^E : \ f \in E_{loc}\}$ depend on E? We are now interested in the wavelet characterization of such conditions. We use the following notions.

Definition 5. *Let E be a Banach space; a sequence e_n is an unconditional basis of E if the two following conditions are satisfied:*

- *For any $f \in E$, there exists a unique sequence c_n such that the partial sums $\sum_{n \leq N} c_n e_n$ converge to f in E.*
- *There exists $C > 0$ such that, for any sequence ϵ_n satisfying $|\epsilon_n| \leq 1$, and for any sequence c_n,*

$$\| \sum c_n \epsilon_n e_n \|_E \leq C \| \sum c_n \epsilon_n e_n \|_E .$$

If E is not separable (but is the dual of a separable space F), then E cannot have an unconditional basis, and one uses instead *weak-unconditional bases*, introduced by Y. Meyer, where the first condition is replaced by the two weaker statements:

- $\exists C > 0$ such that, $\forall f \in E$, there exists a unique sequence c_n such that the partial sums $\sum_{n \leq N} c_n e_n$ converge to f in the weak-star topology, and $\| \sum c_n e_n \|_E \leq \| f \|_E$;

and the second statement of Definition 5 is kept.

If the wavelets form an unconditional basis (or a weak-unconditional basis) of E, one easily deduces a condition satisfied by the elements of $C_E^{\alpha}(x_0)$; indeed the condition of membership to E under the form $A^E(\{|c_\lambda|\}_{\lambda \in \Lambda}) \leq C$, we denote by $A_j^E(x_0)$ the condition $A^E(\{|d_\lambda|\}_{\lambda \in \Lambda})$ where the sequence d_λ is defined by $d_\lambda = c_\lambda$ if $\lambda \subset 3\lambda_j(x_0)$ and $d_\lambda = 0$ else. Clearly, if $f \in C_E^{\alpha}(x_0)$, then

$$\forall j \geq 0, \quad A_j^E(x_0) \leq C 2^{-\alpha j}. \tag{18}$$

Under which additional conditions is the converse true?

3 Spectrum of singularities

It often happens that one is not so much interested by the exact value of the Hölder exponent at a given point as by a more qualitative information: Which nonnegative numbers H are values attained by the Hölder exponent ? What is the size of the *isohölder sets* $E_H = \{x : \ h_f(x) = H\}$? Usually, this size is measured by the Hausdorff dimension, the definition of which we now recall.

Definition 6. *Let $A \subset \mathbb{R}^d$. If $\varepsilon > 0$ and $\delta \in [0, d]$, we denote*

$$M_\varepsilon^\delta = \inf_R \left(\sum_i |A_i|^\delta \right),$$

where R is an ϵ-covering of A, i.e. a covering of A by bounded sets $\{A_i\}_{i\in\mathbb{N}}$ of diameters $|A_i| \le \varepsilon$ (the infimum is therefore taken on all ϵ-coverings). For any $\delta \in [0, d]$, the δ-dimensional Hausdorff measure of A is $mes_\delta(A) = \lim_{\epsilon \to 0} M_\epsilon^\delta$. Let $A \subset \mathbb{R}^d$; then there exists $\delta_0 \in [0, d]$ such that

$$\forall \delta < \delta_0, \quad mes_\delta(A) = +\infty \quad \text{and} \quad \forall \delta > \delta_0, \quad mes_\delta(A) = 0.$$

This critical δ_0 is called the Hausdorff dimension of A.

Definition 7. *Let f be a locally bounded function $f : \mathbb{R} \to \mathbb{R}$. The spectrum of singularities of f (denoted by $d_f(H)$) is the Hausdorff dimension of E_H. (By convention, $dim(\emptyset) = -\infty$.)*

To perform the multifractal analysis of a function means to determine its spectrum of singularities. The terminology "multifractal" refers to the fact that we have to deal with an infinite number of (potentially) fractal sets, the $(E_H)_{H \in \mathbb{R}^+}$; however, there is no precise mathematical definition of what a *multifractal function* is; a common view is that a function is multifractal if the support of d_f, which is $\{H : E_H \neq \emptyset\}$, contains at least one interval $[a, b]$ with $a < b$. By contrast, a function f is called *monofractal* if $Supp(d_f)$ is reduced to one point. A simple example of monofractal function is given by the devil's staircase associated with the triadic Cantor set, see [17].

The basic problems that we raised concerning the Hölder exponent can also be raised for the spectrum of singularities, for which much less is known. For instance, even if f is assumed to be uniform Hölder, we do not know the most general form taken by a spectrum of singularities; very partial results can be found in [24].

Another collection of problems is raised by the relationships between the smoothness of the Hölder exponent and the possible shapes of the spectrums of singularities; for the sake of simplicity, assume in the following that $d = 1$. If $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then its level sets E_H will either be empty, or of dimension 0 (as a consequence, for instance, of Theorem 10 in Chapter 6 of [33]) therefore leading to poor possibilities for the spectrums of singularities:

$$\left. \begin{aligned} d_f(H) &= 0 && \text{on } [a, b] \\ &= -\infty && \text{else,} \end{aligned} \right\} \tag{19}$$

where $[a, b]$ is the range of f. If h_f belongs to $C^\beta(\mathbb{R})$ for $0 \le \beta < 1$, then the same theorem implies that $\forall H$, $d_f(H) \le 1 - \beta$. Is this the only a priori condition satisfied by the spectrum of functions whose

Hölder exponent belongs to a Hölder class? It would be interesting to obtain precise results relating the exact smoothness of h_f with the possible corresponding spectra.

Two generic results (in the sense of Baire categories) are interesting in this context, see [34] and references therein:

- Quasi every continuous bounded function satisfies: $\forall y \in Im(f),\ \ dim(f^{-1})(y) = 0$.
- Quasi every approximately continuous bounded function satisfies: $\forall y \in Im(f),\ \ dim(f^{-1})(y) = 1$.

If one picks such functions as Hölder exponents then the first case leads to the same spectra as (19). The second case would lead to "full spectra" (satisfying $d_f(H) = 1$ on the range of h_f) but only after checking that approximately continuous functions are admissible Hölder exponents.

It often happens that spectra of singularities have concave shapes. It would be important to determine general conditions under which this property holds; indeed, the multifractal formalism, which will be described in the next section, is expected to yield the spectrum but, by construction, it can, at best, only yield its concave hull (since it is obtained as a Legendre transform). Therefore knowing a priori that the spectrum of a function is concave would be an additional guarantee that the multifractal formalism can be applied.

Another common shape adopted by spectra is a straight line; for instance, up to now, all examples of functions having a dense set of discontinuities of the first kind and for which the Hölder exponent is given *at every point* by (4) have a spectrum of the form

$$d_f(H) = Ah \quad \text{if } h \in [0, 1/A], \qquad \text{and} \quad d_f(H) = -\infty \quad \text{else;}$$

Is this result true in all generality? Or under an additional equidistribution hypothesis of the jumps? Or for normally convergent Davenport series? A very explicit example that should be checked is the case of Hecke's functions (3) when $s \in (1,2)$, for which the spectrum is only partly known, see [28].

3.1 Recent generalizations

One defines the *p-spectrum* of a function as the Hausdorff dimension of the level sets of the p-exponent. Considering the p-spectrum rather that the usual spectrum of singularities is often the right setting when one considers discontinuous functions; indeed, no a-priori upper bound for the spectrum of singularities is available for discontinuous functions; but upper bounds for the p-spectrum are available without any uniform regularity assumption, see [29]. We will see a consequence of this remark in Section 3.5. All the problems that we mentioned concerning the spectra of singularities can also be considered for the p-spectrum. It would be important to understand the relationships between the different p-spectra of a given, locally bounded function f. For instance, are there global assumptions on f which imply that these p-spectra are independent of p (at least for $p < \infty$)? The special case where f is a characteristic function 1_Ω is of particular importance. Another relevant problem is to determine which functions are the p-spectrum of a characteristic function.

We can define a spectrum of singularities in the abstract setting which we introduced in Section 2.5. As expected, the E-spectrum, which is denoted as $d^E(H)$ is simply defined by

$$d^E(H) = dim(\{x : h_f^E(x) = H\}). \tag{20}$$

We do not know if changing the function space E can alter the class of possible pointwise exponents, or of possible spectra. We would also like to understand which functions have a spectrum which changes a lot when the underlying space E is changed (we saw that it is the case for characteristic functions) and which functions have a "robust" spectrum with respect to changes of E. How can one interpret such differences?

3.2 Stochastic processes

The same measurability problems as for the Hölder exponent can be raised for the spectrum of singularities: Under which conditions is $d_X(H)$ a measurable stochastic process? Is the mapping $X \to d_X$ measurable? However, the situation concerning deterministic spectra is slightly different; indeed, many processes (or random measures) which have a *random* Hölder exponent turn out to have a *deterministic* spectrum. Here again, this property does not hold in all generality; indeed, the counterexample (7) also yields a process with a random spectrum. Nonetheless, it would be interesting to obtain weak general conditions under which a stochastic process has a deterministic spectrum.

The problem of the commutation of the a.s. and the $\forall H$ seems even more pertinent here: In the case where $d_X(H)$ is a deterministic function $D(H)$, one obtains the spectrume of a.e. sample path if one shows that $a.s.\ \forall H, \quad d_X(H) = D(H)$. It is usually much easier to prove that $\forall H \ a.s., \quad d_X(H) = D(H)$; of course, this is not sufficient to obtain the spectrum of almost every sample path; it only yieds the value of $d_X(H)$ for almost every H. Nonetheless, I do not know of any example of stochastic process where one cannot commute the a.s. and the $\forall H$ in this setting. This commutation is therefore probably licit under very weak hypotheses. Finding out which ones would spare a lot of pains to the multifractalists' community! A more conventional way to tackle this problem is to start with a standard stochastic process as Höldere exponent; here is a simple example: Let $B(t)$ be a Brownian motion, $a > 0$, and $B^a(t) = \sup(a, B(t))$; let X_t be a random process of Hölder exponent $B^a(t)$; then

$$\text{a.s.} \quad \forall H > a, \quad dim(E_H) = 1/2, \tag{21}$$

as a consequence of uniform dimension estimates on the level sets of the Brownian motion, see [43]. However, in most cases, such results are not uniform and only give, for each y, the a.s. dimension of the set $\{t : X_t = y\}$. For which processes is the uniform formulation of the result valid? Note that we are led back here to an old problem concerning the fractal properties of the level sets of stochastic processes, which was already raised by S. J. Taylor in 1973, see [47].

3.3 Additional properties of Isohölder sets

In practice, most examples of deterministic functions, or of stochastic processes, which turn out to be multifractal, share an important additional "universality" property: If Ω is an open nonempty set, denote by $d_f(H, \Omega)$ the spectrum of singularities of f restricted to Ω, i.e.

$$d_f(H, \Omega) = dim(\{x \in \Omega : \ h_f(x) = H\}).$$

A multifractal function is *universal* if $d_f(H, \Omega)$ is independent of Ω. It would be important to determine what is the most general spectrum that can be taken by a universal function. The juxtaposition argument of [24] cannot be applied in this setting so that no general result is available. One way could be to construct generic functions in some function space settings and obtain universality properties as a consequence of genericity. Many stochastic processes which have been studied a.s. have universal sample paths; is it the consequence of a simple probabilistic property? (One should however keep in mind the particularly simple counterexample supplied by Poisson processes.)

Additional properties of the isohölder sets, and related sets, would be interesting to determine. In the multifractal analysis of measures (in which one determines the dimensions of the sets where a measure has a given pointwise dimension), packing dimensions are also determined. In the setting of functions, very few results have been obtained for packing dimensions (we will mention a few in the following).

Very often, the sets $F_h = \{x : \ h_f(x) \leq H\}$ have a remarkable property which was discovered and studied by K. Falconer: They are *sets with large intersection*, see [18]. It would be important to understand why the sets F_H so often have this very remarkable property.

3.4 Spectrum of composed functions

In Section 2.2, we considered the Brownian motion in time μ, where μ is a measure carried by $\mathbb{R}^+$. More generaly, one can start with a Fractional Brownian motion $B^H(t)$, and change its time through the distribution function f of a multifractal measure μ. If μ is deterministic, then one obtains Gaussian processes with a deterministic Hölder exponent $B^H_\mu(t) = B^H(f(t))$. If μ is random, then the process needs not be Gaussian any longer. These models have been used as models in finance, see [38] and references therein. In this case, (10) still holds, but the factor 1/2 is replaced by the order H of the Fractional Brownian motion. Thus, the spectrum of singularities of B_μ^H is given by $d_{B_\mu^H}(h) = d_{\mathcal{D}_\mu}(h/H)$.

These models can be complicated by composing a multifractal function (or process) $g(t)$ with a multifractal time-change $f(t)$. In this case there is no general formula for the spectrum of singularities of $g \circ f$ in terms of the spectra of f and g, because it depends on how the singularities of the two functions combine. Important subcases are supplied by subordinator time changes (which are increasing processes with independent and stationary increments, see [5, 6]); for instance, if f is a subordinator and g an arbitrary Lévy process then an explicit formula for the characteristic function of the composed process is available, from which one can derive its lower exponent, and thus its spectrum, see [5, 6]. Several other examples have been worked out by R. Riedi in [44]. However the understanding of the multifractal properties of composed functions is now at its very beginnings.

3.5 Solutions of partial differential equations

Another problem brings us back to the very initial motivation of multifractal analysis: Recall that it was introduced in order to determine if the velocity of turbulent flows is multifractal. Therefore an important mathematical problem is to determine if some PDEs develop multifractal solutions. Right now, to our knowledge, there is just one such result: Consider the non-viscid Burgers equation in one space dimension

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0, \quad u(x,0) = u_0(x) \quad (x \in \mathbb{R},\ t \geq 0),$$

which is understood in the limit of vanishing viscosity; assume that the initial condition $u_0(x)$ is a Brownian motion; then the solution at time $T > 0$ is a Lévy process without Brownian part of lower index 1/2, see [7]; therefore Burgers equation is an example of a PDE which, for some *monohölder* initial data, develops Hölder singularities in such a way that the solution becomes multifractal, see [26]. This beautiful result is frustrating because of its singleness; as soon as we slightly change the setting, nothing is known; if we stick to Burgers equation, very natural question are: What happens if the Brownian motion is replaced by a fractional Brownian motion? (There is numerical evidence that solutions are still multifractal, see [46], but no mathematical result is available.) One could also consider deterministic initial values; the Weierstrass functions come to mind as the natural substitutes of the Brownian motion. One could also investigate generic results in this context: For instance, let us assume that the initial value is a "generic" function in $C^s(\mathbb{R})$ for an $s \in (0,1)$ (generic being defined in the sense of Baire's categories, or in some of its extensions, see [13, 19, 21]); then, is it true that the solution of Burgers equation at time $T > 0$ is generically multifractal? Note however that Burgers equation develops discontinuous solutions (shocks) even if the initial condition is smooth. Therefore, in this case, determining the p-spectrum may be easier than determining the Hölder spectrum. The following a priori Besov regularity estimates were obtained by R. Devore and B. Lucier in [15, 16]: Assume that $u_0 \in L^1$, and that $supp(u_0) \subset [0,1]$, and let α be an arbitrary positive number and $q = 1/(\alpha+1)$; if $u_0 \in B_q^{\alpha,q}$ then $\forall t > 0$, $u(.,t) \in B_q^{\alpha,q}$. Note however that no conservation law can hold in a Sobolev or Besov space embedded in $C(\mathbb{R})$; therefore no upper bound for the spectrum of singularities can be deduced from conservation laws; by contrast, it is the case if the spectrum of singularities is replaced by the p-spectrum, see [31].

Nothing is known either for other PDEs; the most straightforward generalization is the 3-dimensional Burgers equation which is important because it models the evolution of the repartition of the mass in the universe, see [48]. Here again, there is strong numerical evidence that monohölder initial data (Brownian fields) develop multifractal solution, but no mathematical results are available.

3.6 Other functions

The following remarkable trigonometric series

$$\mathcal{R}(x) = \sum_{n=1}^{\infty} \frac{\sin(\pi n^2 x)}{n^2}$$

has been proposed by Riemann as a possible example of continous nowhere differentiable function. J. Gerver proved in 1970 that it is differentiable at some rational points. Since then it has been shown to be a multifractal function, see [32] and references therein for many additional fascinating properties of this function. The trigonometric series $\mathcal{R}$ can be generalized in 2 dimensions by

$$\mathcal{R}_\alpha(x) = \sum \frac{1}{(n^2+m^2)^\alpha} \sin(\pi n^2 x + m^2 y).$$

Multifractal properties of this function have been investigated by H. Oppenheim, see [41]. However its spectrum of singularities is not completely known yet. It would be important to investigate simple generalizations of this Riemann's function. For instance, let $P(n)$ be a polynomial with integer coefficients and $Q(n)$ be an arbitrary polynomial, $P(n)$ and $Q(n)$ being of degree at least 2. What can be said of the trigonometric series

$$\sum{}^{*} \frac{\sin(\pi P(n) x)}{Q(n)}$$

(where $\sum^*$ means that the sum is taken only on the frequencies n which are not a rooot of Q)? It seems that, apart from Riemann's function, the only results available concern points of differentiablity. Other trigonometric series (where $P(n)$ and $Q(n)$ are arithmetic functions) are considererd in analytic number theory; their multifractal properties should also be investigated.

Another intriguing function for which there is strong evidence that it is multifractal is Minkowski's function $?(x)$, which is a strictly increasing function that maps $[0,1]$ on $[0,1]$ and can be defined as follows, see [37, 45]: If $x \in [0,1]$ is developed as a regular continued fraction

$$x = [0; a_1, a_2, \dots, a_n, \dots], \quad \text{then } ?(x) = \frac{1}{2^{a_1-1}} - \frac{1}{2^{a_1+a_2-1}} + \frac{1}{2^{a_1+a_2+a_3-1}} - \dots$$

4 The multifractal formalisms

4.1 Derivation of the formalism

Multifractal formalisms are meant to allow a derivation of the spectrum of singularities from global quantities that should, in practice, be computable on signals. Such formulas were first derived by Parisi and Frisch, see [42]. Afterwards, Arneodo and his collaborators proposed a reformulation using the continuous wavelet transform, see [3]. Let us present the standard derivation in the context of wavelet bases.

If f is compactly supported and if $p \neq 0$, then the *wavelet structure function* of f is

$$\Sigma_f(p,j) = 2^{-j} \sum_{\lambda \in \Lambda_j} |c_\lambda|^p .$$

The *wavelet scaling function* of f is defined, for $p \neq 0$ by

$$\eta_f(p) = \liminf_{j \to +\infty} \left(\frac{\log\left(\Sigma_f(p,j)\right)}{\log(2^{-j})} \right).$$

In the past, Theorem 1 has loosely been interpreted as stating that, if the Hölder exponent of f is H, then the wavelet coefficient of f corresponding to $\lambda = \lambda_j(x_0)$ will have size $|c_\lambda| \sim 2^{-Hj}$ (it is the case for *cusp-like* singularities which behave like $A + B|x - x_0|^\alpha$ near x_0) under this assumption, the points of Hölder exponent H brings a contribution $\sim 2^{-j} 2^{-Hpj} 2^{d_f(H)j}$ to $\Sigma_f(p,j)$; since $\Sigma_f(p,j) \sim 2^{-\eta_f(p)j}$, a standard steepest descent argument yields

$$\eta_f(p) = \inf_H (d - d_f(H) + Hp).$$

If the spectrum is concave, then this Legendre transform can be inverted and

$$d_f(H) = \inf_{p \neq 0} (d - \eta_f(p) + Hp), \tag{22}$$

which is the standard statement of the multifractal formalism used for wavelet-based computations.

However, there are also counterexamples to the assumption that $|c_\lambda| \sim 2^{-h_f(x_0)j}$ for $\lambda = \lambda_j(x_0)$; the most famous ones being *chirps* of the form $|x - x_0|^\alpha \sin\left(1/|x - x_0|^\beta\right)$, with $\alpha, \beta > 0$ for which, if

$\lambda = \lambda_j(x_0)$ then $c_\lambda = o(2^{-\gamma j})$ $\forall \gamma > 0$, see [2, 30]. Thus a safer way to derive the multifractal formalism (since it does not make any assumption on the nature of the Hölder singularities) is to base the structure function on the wavelet leaders instead of the wavelet coefficients. Therefore, we define, for $p \neq 0$, the *leaders'structure function* and the corresponding *leaders'scaling function* as, respectively

$$\Delta_f(p,j) = 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p, \quad \text{and} \quad \tau_f(p) = \liminf_{j \to +\infty} \left(\frac{\log(\Delta_f(p,j))}{\log(2^{-j})} \right).$$

The same argument as above yields a multifractal formalism based on the wavelet leaders

$$d_f(H) = \inf_{p \neq 0} (d - \tau_f(p) + Hp). \tag{23}$$

The heuristic argument used in the derivation of the multifractal formalism is backed by several mathematical results: For instance, one can prove that, for any uniform Hölder function,

$$d_f(H) \leq \inf_{p \in \mathbb{R}} (d - \tau_f(p) + Hp), \tag{24}$$

see [27] (there exists a similar bound using $\eta_f(p)$, but it is far from being as sharp, and, in particular, the infimum there does not bear on all values of p see [24]); (23) can be shown to hold in many situations where (22) is wrong; apart from cases where chirps come up, a striking example in dimension 1 is supplied by the FBM of order α where (22) yields a wrong spectrum

$$\begin{aligned} d(H) &= 1 - H + \alpha \quad \text{if } H \in [\alpha, \alpha+1], \\ &= -\infty \quad \text{else;} \end{aligned}$$

whereas (23) yields a.s. the right spectrum

$$\begin{aligned} d(H) &= 1 \quad \text{if } H = \alpha, \\ &= -\infty \quad \text{else.} \end{aligned}$$

It would be interesting to check this difference on other multifractal processes such as Lévy processes, random wavelet series and more general Gaussian processes.

The use of d_λ is reminiscent of the *wavelet maxima method* initially introduced by S. Mallat and used by A. Arneodo E. Bacry and J.-F. Muzy in the context of multifractal analysis, see [3]: One computes the continuous wavelet tranform of f

$$C_f(a,b) = a \int f(t) \psi\left(\frac{t-b}{a}\right) dt$$

which is a function defined in the upper half plane $\{(a,b) : a > 0, \ b \in \mathbb{R}\}$. For each scale a, one spots the local maxima of the functions $b \to |C_f(a,b)|$ and the partition function is based on the values of the continuous wavelet transform at these local maxima. The main differences between these two methods are the following: In the wavelet maxima method, the spacing between the local maxima need not be of the order of magnitude of the scale a or even be regularly spaced; therefore, the scaling function thus obtained can differ from $\eta_f(p)$ (see [24] for counterexamples). It follows that no mathematical result similar to (24) is expected to hold for the wavelet maxima method. However the wavelet maxima method has been successfully used in many situations, see [1, 3]. Obtaining mathematical results concerning the validity of the multifractal formalism in this setting is an important challenge. Note that, if one is interested in the p-spectrum, using Theorem 2, the multifractal formalism can be based on the $\| S_f(j, x_0) \|_p$ instead of d_λ. More generally, if (18) is shown to be a characterization of the condition $C^\alpha_E(x_0)$, then the multifractal formalism can be based on the $A^E_j(x_0)$ in order to recover the E-spectrum. Upper bounds for spectra can be recoverd by the same argument as in the Hölder exponent setting, see [27] for instance; all other questions concerning the spectrum of singularitie can be raised in this setting.

4.2 Stochastic processes

The multifractal formalism can be tested on arbitrary functions and, in particular on sample paths of stochastic processes (or random fields for $d > 1$). However, in this case, a variant is often used; indeed the quantities $\Sigma_f(p, J)$ and $\Delta_f(p, J)$ can be replaced by $\mathbb{E}(|c_\lambda|^p)$ and $\mathbb{E}(|d_\lambda|^p)$, assuming of course that these quantities do not depend on the choice of $\lambda \in \Lambda_J$, which is expected to be true under some ergodicity assumption on the stochastic process considered. Of course, a multifractal formalism based on such quantities yields a deterministic spectrum, so that their use is connected with the problem already mentioned of understanding when the spectrum of singularities of a process is deterministic. What are the relationships between the multifractal formalism based on expectations and the standard one based on sample paths? Can one obtain, in the context of expectations, the same results as above (upper bounds for the spectra, conditions of validity for the multifractal formalism).

5 Generic results

Multifractal formalisms are not expected to hold without additional assumptions, since many types of counterexamples are easily constructed. However, a key question is to determine if their validity is a consequence of additional properties of the function considered (and therefore is quite exceptional) or if it is a "generic" property which holds for "most" functions in some sense.

Up to now, the mathematical results mostly pointed in the first direction; the validity of the multifractal formalisms was shown to hold for specific models; in most situations, the additional properties were a kind of selfsimilarity (being understood in a very loose way: It could be exact or approximate, deterministic or stochastic). Recently, the second path started to be explored and several generic results of validity of the multifractal formalisms were obtained. In order to explain in which settings, we first have to give the "function space interpretation" for the scaling function.

Definition 8. *Let $s > 0$ and $p > 0$; A function $f : \mathbb{R}^d \to \mathbb{R}$ belongs to $\mathcal{O}^s_p(\mathbb{R}^d)$ if and only if f belongs to L^p (or to the real Hardy space H^p when $p \leq 1$) and if $\exists c$ such that*

$$\forall j \geq 0, \quad 2^{-dj} \sum_{\lambda \in \Lambda_j} |d_\lambda|^p \leq C 2^{-spj}. \tag{25}$$

When $p > 0$, the information given by the scaling function $\eta(p)$ for $p > 0$ implies that $\forall \epsilon > 0$, $\forall p > 0$, $f \in \mathcal{O}^{(\eta(p)-\epsilon)/p,p}$, therefore f belongs to the function space $\cap_{p,\epsilon>0} \mathcal{O}^{(\eta(p)-\epsilon)/p,p}$. Since this function space can be written as a countable intersection of quasi-Banach spaces, it is a complete quasi-metric vector space. Several notions of genericity have been introduced in such spaces and the problem of the generic validity of the multifractal formalism can be raised for each of these settings. The first notion of genericity that was investigated (both historically and in this context) is supplied by *Baire categories*. Recall that Baire's category theorem states that, if E is a complete metric (or quasi-metric) space, every countable intersection of open dense sets is dense. If a property $\mathcal{P}(x)$ holds (at least) on a countable intersection of open dense sets, it is said to hold *quasi-surely*. Note that a quasi-sure property $\mathcal{P}$ does not necessarily hold on a large set; indeed, in $\mathbb{R}^d$ a quasi-sure property may hold only on a set of vanishing measure (and even of vanishing dimension). In $\mathbb{R}^d$, the notion of "Lebesgue almost everywhere" offers a much more natural framework for generic properties; indeed, the generic sets are obtained as the complements of the measure-zero sets, for a "canonical" measure which is both σ-finite and shift-invariant: The Lebesgue measure. Therefore, a natural question is to wonder if such a measure also exists in an infinite dimensional Banach spaces. Unfortunately, the answer is negative: There does not exist a σ-finite translation-invariant measure in any infinite dimensional normed space. However, this remark does not kill any hope for an infinite-dimensional extension of the notion of translation-invariant "Lebesgue measure zero"; indeed, let us consider the following standard characterization of the Lebesgue measure, see [13, 21].

Lemma 3. *In $\mathbb{R}^d$, a Borel set S has Lebesgue measure zero if and only if there exists a compactly supported probability measure μ such that $\forall x \in \mathbb{R}^d \quad \mu(x + S) = 0$.*

The characterization of the sets of vanishing Lebesgue measure supplied by Lemma 3 does not refer explicitely to the Lebesgue measure; therefore it can be turned into a definition in infinite dimension spaces; the sets thus defined are called *Haar-null*, and the notion of genericity that it yields is called *prevalence*. The following definition was introduced by J. Christenssen [13].

Definition 9. *Let E be a complete metric space. A Borel set $A \subset E$ is* Haar-null *if there exists a compactly supported probability measure μ on E such that*

$$\forall x \in \mathbb{R}^d \quad \mu(x+A) = 0. \tag{26}$$

A subset A of E is Haar-null *if it is included in a Haar-null Borel set. The complement of a Haar-null set is called a* prevalent *set. If (26) holds, the measure μ is said to be transverse to A. Almost every element of E satisfies a property $\mathcal{P}$ if the set of elements satisfying $\mathcal{P}$ is a prevalent set; equivalently, we will say that $\mathcal{P}$ holds almost everywhere in E.*

The basic properties of prevalence and several applications are detailed in [13, 21] and we refer to them for additional information. The following result describes the prevalent Hölder regularity of functions of $\mathcal{O}_p^s$ when $s > d/p$ (which coincides which the space $B_p^{s,\infty}$ in this case).

Theorem 3. *Let $s > d/p$; the Hölder exponent of almost every function f of the space $\mathcal{O}_p^s$ takes values in $[s - d/p, s]$ and $\forall H \in [s - d/p, s]$, $d_f(H) = Hp - sp + d$; furthermore, for almost every x, $h_f(x) = s$. Let x_0 be an arbitrary given point in $\mathbb{R}^d$; then, for almost every function in $\mathcal{O}_p^s$, $h_f(x_0) = s - d/p$.*

Several extensions of this result have been proved, see [19]. However no result exists for "true" oscillation spaces (i.e. when they do not coincide with Besov spaces) or, a fortiori for intersections of oscillation spaces. Therefore, the problem of the generic validity of the multifractal formalism for $p > 0$ remains open, either in the Baire setting or in the prevalence setting. Of course the same problems can be raised for the p-spectrum, or in the more abstract setting of (20).

One drawback of prevalence is that it is adapted only to a vector space setting (or, at least, a group setting). Sometimes, the natural setting is not a vector space: One would like generic properties for characteristic functions; in that case, the same problems as above can be raised, but only for Baire genericity. In worse cases, the natural formulation of the problem does not supply a topology; here is an example: Can one obtain generic results of multifractality if the a priori information available is the scaling function *including the negative values of p* ?

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