Bulletin de la Société Royale des Sciences de Liège, Vol. 73, 2-3, 2004, pp. 171-185

KKM PROPERTY IN TOPOLOGICAL SPACES

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ABSTRACT. In this paper we give a brief survey of some recent generalizations of the Fan-KKM theorem. We introduce a new convex structure on a nonempty set M which contains all different concepts of convexity. Some approximate fixed point theorems will be established for the multivalued mapping with S-KKM property on the Φ spaces. We also obtain a generalized Fan matching theorem, a generalized Fan-Browder type theorem, and a new version of Sadovskii's fixed point theorem.

1. INTRODUCTION

In 1929, Kanster-Kuratowski-Mazurkiewicz established the celebrated KKM theorem [37]. The most important result for KKM mappings is the famous Fan-KKM theorem [26], which has been used as a very versatile tool in modern nonlinear analysis and from which many far-reaching extensions have been made. The generalization for the concept of KKM mappings was first introduced by Park [40] and followed by Chang and Zhang [17] and many others (cf. [11] and [45]).
Chang and Yen [13-14] made a systematic study of the class of the KKM mappings. Motivated by their work, Chang et al. [15] introduced the family of multivalued mappings with the S-KKM property.

As shown in [14], the KKM mappings are contained in the S-KKM mappings and generally this inclusion is proper.

Here we introduce the class of the S-KKM mappings for the sets with a Γ-convex structure, a class of convexity which contains all the different concepts of convexity, mainly abstract convex structure [39, 46] and metric spaces. We obtain also some fixed point theorems for the multivalued mappings with S-KKM property on the Φ spaces. Furthermore, we obtain a generalized Fan matching theorem, a generalized version of Fan-Browder's fixed point theorem and a new version of Sadovskii's fixed point theorem in our context.

1991 *Mathematics Subject Classification.* Primary 47H10, 54H25.

Key words and phrases. KKM property, Γ-space, Φ-space, approximate fixed point, metric space, condensing mapping.

Let us introduce the notations used in this paper and recall some basic facts. Let M be a nonempty set; we shall denote by 2^M the family of all subsets of M and by $\langle M\rangle$ the family of all nonempty finite subsets of M.

Suppose that Y and M are two topological spaces and $F : M \to 2^Y$ is a multivalued mapping, the fibers $F^-(y)$ for $y \in Y$ are defined by $F^-(y) = \{x \in M : y \in F(x)\}$.

A multivalued mapping $F : M \to 2^Y$ is said to be:
compact if the closure of its range $F(M)$ is compact in Y;
upper semi-continuous (u.s.c.) if for each closed set $B \subset Y$, $F^-(B) = \{x \in M : F(x) \cap B \neq \varnothing\}$ is closed in M;
lower semi-continuous(l.s.c.) if for each open set $B \subset Y$, $F^-(B) = \{x \in M : F(x) \cap B \neq \varnothing\}$ is open in M;
continuous if it is both u.s.c. and l.s.c.;
closed if its graph $G_r(F) = \{(x, y) \in M \times Y : y \in F(x)\}$ is closed.

A nonempty topological space is *acyclic* if all its reduce homology groups over the rationals vanish.

Kanster-Kuratowski-Mazurkiewicz [37] established the famous KKM theorem which is of great importance in nonlinear analysis. This theorem first appeared in their well known proof of the Brower fixed point theorem.

The KKM Principle 1.1 [37]. *Let D be the set of vertices of Δ_n and $G : D \to 2^{\Delta_n}$ be a KKM map (that is, $coA \subseteq G(A)$ for each $A \subseteq D$) with closed values. Then, $\cap_{z\in D} G(z) \neq \emptyset$.*

The most important result for the KKM mappings is the famous Fan-KKM theorem [26], which is a generalization of the KKM principle in the setting of infinite dimensional spaces. This theorem provides an essential tool to study minimax inequalities.

Fan-KKM Theorem 1.2 [26]. *Let X be a convex subset of a Hausdorff t.v.s. E, $X_0 \subseteq X$. Suppose that $G : X_0 \to 2^X$ is a KKM map. (that is, $coA \subseteq G(A)$ for each $A \in \langle X_0\rangle$) If all the sets $G(x)$ are closed subsets of Y , then the family $\{G(x) : x \in X_0\}$ has the finite intersection property. Moreover, if the value of G at a point $x_0 \in X_0$ is compact, then $\cap_{x\in X_0} G(x) \neq \emptyset$.*

Afterwards, some authors (cf. [4], [16], [18], [20], [45]) improved the Fan theorem by introducing the concept of transfer closedness and relaxing the closedness condition. Let X be a nonempty set, Y a topological space. Then $G : X \to 2^Y$ is said to be *transfer closed-valued* if for any $(x, y) \in X \times Y$ with $y \notin G(x)$, there exists $x' \in X$ such that

$y \notin \text{cl}G(x')$. It is clear that this definition is equivalent to saying that

$$\bigcap_{x\in X} G(x) = \bigcap_{x\in X} \text{cl}G(x).$$

If $A \subseteq X$ and $B \subseteq Y$, then we call $G : A \to 2^B$ transfer closed-valued if the multi-valued mapping $x \to G(x) \bigcap B$ is transfer closed-valued. In the case where $X = Y$ and $A = B$, we call G transfer closed-valued on A. On the other hand, in [11] the authors improved the Fan theorem by assuming the closedness condition only upper finite dimensional subspaces, with some topological pseudomonotone condition. In [18], Chowdhury and Tan, replacing finite dimensional subspaces by polytopes, restated the Brezis-Nirenberg-Stampachia result [11] under weaker assumptions. In [31], Kalmoun gave a refined version of Chowdhury and Tan [18]. In [23], we improved this last refined version based on a work of Ding and Trafdar [20]. More recently we could obtain the following result.

Theorem 1.3 [25]. *Let X be a nonempty convex subset of a Husdorff t.v.s. E. Suppose that G, $F : X \to 2^X$ are two multi-valued mappings such that the following conditions are satisfied:*

(A1) *$F(x) \subseteq G(x)$ for all $x \in X$,*
(A2) *F is a KKM map,*
(A3) *for each $A \in \langle X\rangle$, G is transfer closed-valued on coA,*
(A4) *for each $A \in \langle X\rangle$*

$$cl_X\Big(\bigcap_{x\in coA} G(x)\Big) \cap coA = \Big(\bigcap_{x\in coA} G(x)\Big) \cap coA,$$

(A5) *there is a nonempty compact convex set $B \subseteq X$ such that $cl_X(\bigcap_{x\in B} G(x))$ is compact.*

Then, $\bigcap_{x\in X} G(x) \neq \emptyset$.

On the other hand Chang and Zhang [17] introduced the concept of generalized KKM maps as follows: Let X be a set and Y be a convex subset of a topological space E. A multivalued mapping $G : X \to 2^Y$ is called a *generalized KKM map* if for each subset $A = \{x_0, ..., x_n\}$ of X, there exists a finite subset $B = \{y_0, ..., y_n\}$ of Y, not necessarily all different, such that: $co(\{y_{i_0}, ..., y_{i_j}\}) \subseteq \bigcup_{k=0}^{j} G(x_{i_k})$ for any subset $\{y_{i_0}, ..., y_{i_j}\}$ of B.

Until 1983, all the KKM theorems and related topics were considered and studied in topological vector spaces in the most general framework. In this setting, convexity assumptions play a crucial role in solving this variety of problems. Horvath [28] replacing convex hulls by contractible subsets, gave a purely topological version of the KKM theorem. This has motivated other mathematicians to go into the question for generalized KKM theorems over topological spaces with no linear structure (cf. [7], [13], [16], [41], [43], [22]). The concept of generalized KKM maps for these cases has been obtained by Tan [23], Ding [4], Park-Lee [44], Kirk et al. [35] and more recently in [21].

Let A be a bounded subset of a metric space (M, d). Then $coA = \cap\{B \subseteq M : B$ closed ball , A $\subset B\}$. $\mathcal{A}(M) = \{A \subseteq M : A = co(A)\}$ i.e. A is an intersection of closed balls. In this case we say that A is admissible set in M. A is called subadmissible, if for each $D \in \langle A \rangle$, $co(D) \subseteq A$. Obviously , if A is an admissible subset of M, then A must be subadmissible.

The following definition of metric KKM mappings is given by Khamsi in [32]. Let X be nonempty subset of a metric space M and Y be a topological space. A multivalued mapping $F : X \to 2^Y$ is called a metric KKM mapping if for each $A \in \langle X \rangle$, $co(A) \subset F(A)$.

A metric space (M, d) is called hyperconvex Aronszjan and Panitchpakdi [5], if for any collection of points $\{x_\alpha : \alpha \in I\}$ of M and any collection of nonnegative reals $\{r_\alpha : \alpha \in I\}$ such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for all $\alpha, \beta \in I$, then $\cap_{\alpha \in I} B(x_\alpha, r_\alpha) \neq \emptyset$.

The spaces $(\mathbb{R}^n, \|.\|_\infty)$, l^∞, and L^∞ are concrete examples of Hyperconvex spaces.

Khamsi-KKM Theorem 1.4.[32]. *Let M be a hyperconvex space, $X \subseteq M$, and $G : X \to 2^M$ a KKM-map such that the sets $G(x)$ are closed subsets of M, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

Chang and Yen [13-14] made a systematic study of the class of the KKM mappings: Let X be nonempty convex subset of a topological vector space and Y a topological space. If $G : X \to 2^Y$, $F : X \to 2^Y$ are two multivalued maps such that for any $A \in \langle X \rangle$, $F(co(A)) \subseteq G(A)$, then G is said to be a generalized KKM mapping respect to F. Let $F : X \to 2^Y$ be a multivalued mapping such that if $G : X \to 2^Y$ is a generalized KKM mapping with respect to F, then the family $\{clG(x) : x \in X\}$ has the finite intersection property. In this case we say that F has the KKM property. We define $KKM(X, Y) := \{F : X \to 2^Y : F$ has KKM property$\}$. It is shown that the class $KKM(X, Y)$ contains the admissible class introduced by Park [40], and many other important classes of multivalued mappings [38]. Moreover, Chang and Yen [14] have shown that this inclusion is proper.

Motivated by this work of Chang and Yen [14], Chang et al. [15] introduced the family of multivalued mappings with S-KKM property. Let X be nonempty set, Y be a convex set and Z a topological space. If $S : X \to 2^Y$, $T : Y \to 2^Z$ and $F : X \to 2^Z$ are three multivalued mappings satisfying :

$$T(coS(A)) \subseteq F(A)$$

for each $A \in \langle X \rangle$, then F is called a generalized S-KKM mapping with respect to T. If the multivalued mapping $T : Y \to 2^Z$ satisfies the requirement that for any generalized S-KKM mapping F with respect to T the family $\{clF(x) : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. We define

$$S - KKM(X, Y, Z) := \{T : Y \to 2^Z : T \text{ has S-KKM property}\}.$$

As shown [15], when $X = Y$ and S is the identity mapping I_X, then $S - KKM(X, X, Z) = KKM(X, Z)$ and moreover, $KKM(Y, Z)$ is contained in $S - KKM(X, Y, Z)$ for any $S : X \to 2^Y$ and generally this inclusion is proper.

2. Γ-CONVEX SPACES AND FIXED POINT THEOREMS

In this section we introduce a new convex structure on a nonempty set M which contains all the different concepts of convexity. We define KKM mappings and S-KKM mappings similarly to the case of convex spaces. Some approximate fixed point theorems will be established for the multivalued mapping with S-KKM property on Φ spaces.

Definition 2.1. A Γ-convex space $(M, D; \Gamma)$ consists two nonempty sets M, D and a multivalued mapping $\Gamma : \langle D \rangle \to 2^M$. If $D \subseteq M$ and $X \subset M$, then X is called Γ-convex if for each $A \in \langle D \cap X \rangle$ implies $\Gamma(A) \subset X$.

The following are the main examples of Γ-convex spaces.

Examples 2.2.(a). A family $\mathcal{C}$ of subsets of a set M is an abstract convexity structure for M if $\emptyset$ and M belong to $\mathcal{C}$ and $\mathcal{C}$ closed under arbitrary intersection. This kind of convexity was widely studied (cf. [39] and [46]). For any $X \subseteq M$, a natural definition of the $\mathcal{C}$-hull is $co_{\mathcal{C}}(A) = \bigcap\{B \in \mathcal{C} : A \subseteq B\}$. We say that X is $\mathcal{C}$-convex (or in brief, convex) if X is equal to its $\mathcal{C}$- convex hull. If M has an abstract convexity structure $\mathcal{C}$, then $(M, M; \Gamma)$ is a Γ-convex space where $\Gamma(A) = co_{\mathcal{C}}(A)$ for each $A \in \langle M \rangle$.

(b). Let (M, d) be a pseudo-metric space. If we set $D = M$ and define $\Gamma(A) = co(A)$ for each $A \in \langle M \rangle$, then $(M, M; \Gamma)$ is a Γ-convex space.

(c). There are more examples of Γ-convex spaces namely K-convex structure, hyperconvex spaces, H-spaces, L-spaces, G-convex spaces and mc-spaces. For further information about these structures and spaces, one can refer to Llinares [39] and references therein.

Remark. Our definition of Γ-convexity is similar to the definitions of G-convex spaces introduced by Park [41, 43] and to the L-spaces due to Ben-El-Mechaiekh et al. [9]. The distinction between our definition of Γ-convexity with other kinds of convexity which are mainly mentioned in the part (c) of the above example is that we do not consider the existence of a continuous function from a simplexe to $\Gamma(A)$ for each $A \in \langle D \rangle$.

Motivated by the work of Chang and Yen [14], we define in a similar way the class of the multivalued mappings with KKM property. Let $(M, D; \Gamma)$ be a Γ-convex space and Y a topological space. If $T : M \to 2^Y$ and $F : D \to 2^Y$ are two multivalued mappings such that for any $A \in \langle D \rangle$ $T(\Gamma(A)) \subseteq \bigcup_{x\in A} F(x)$, then F is said to be a generalized KKM mapping with respect to T. A multivalued mapping $T : M \to 2^Y$ is said to have the KKM property if for any KKM map $F : D \to 2^Y$ with respect to T, the family $\{clF(x) : x \in D\}$ has the finite intersection property. We let $\text{KKM}(M, Y) := \{T : M \to 2^Y : T$ has KKM property$\}$.

Similarly, by the work of Chang et al. [15] we introduce the family of multivalued mappings with the S-KKM property as follows. Let X be nonempty set, $(M, D; \Gamma)$ a Γ-convex space, and Y a topological space. If $S : X \to D$, $T : M \to 2^Y$ and $F : X \to 2^Y$ are three multivalued mappings satisfying :

$$T(\Gamma(S(A))) \subseteq \bigcup_{x\in A} F(x)$$

for each $A \in \langle X \rangle$, then F is called a generalized S-KKM mapping with respect to T. If the multivalued mapping $T : M \to 2^Y$ satisfies the requirement that for any generalized S-KKM mapping F with respect to T the family $\{clF(x) : x \in X\}$ has the finite intersection property, then T is said to have the S-KKM property. We define $S\text{-KKM}(X, M, Y) := \{T : M \to 2^Y : T$ has S-KKM property$\}$.

In order to establish the main result of this paper for the mappings with the S-KKM property, we define the Φ maps and Φ-spaces.

Definition 2.3. **(a).** Let $(M, D; \Gamma)$ be a Γ-convex space and Y a topological space. A map $T : Y \to 2^M$ is called a Φ-map if there exists a map $G : Y \to 2^D$ such that

(i) for each $y \in Y$, $A \in \langle(G(y)\rangle$ implies $\Gamma(A) \subseteq T(y)$; and
(ii) $Y = \bigcup\{IntG^{-}(x) : x \in D\}$.

(b). A Γ-convex space $(M, D; \Gamma)$ is called a Φ-space if M is a uniform space and for each entourage V there is a Φ-map $T : M \to 2^M$ such that $Gr(T) \subseteq V$.

The concept of Φ-maps and Φ-spaces are originated from Ben-El-Mechaiekh et al. [8], Horvath [28-29] and motivated by the works of Fan and Browder [12]. These notions also have been studied by Ben-El-Mechaiekh et al. [9], and more recently by Park [41] and Kim and Park [34]. Let $(M, D; \Gamma)$ be a Φ-space and $F : M \to 2^M$. We say that F has an approximate fixed point if for any $U \in \mathcal{U}$ where $\mathcal{U}$ is a basis of the uniform structure of M, there exists an $x \in M$ such that $U[x] \bigcap F(x) \neq \varnothing$.

Theorem 2.4.[3]. *Let $(M, D; \Gamma)$ be a Φ-space and $S : M \to D$ a surjective function. Suppose that $F \in S$-KKM(M, M, M) is compact, then F has an approximate fixed point.*

By the above theorem we obtain the following fixed point theorem.

Corollary 2.5. *Suppose that all of the assumptions of the above theorem hold and F is closed, then F has a fixed point.*

Remark (a). As G-convex spaces are Γ-convex spaces and as by Lemma 2.5 of [24], any better admissible which is upper semicontinuous, compact and closed valued has the KKM property, the above corollary refines the main results of [34, Theorem 4.2] and [41, Theorem 3.3] in our context.

(b). Horvath [30] found that hyperconvex spaces are a particular type of C-spaces, hence they are G-convex spaces. In [24, Lemma 2.7] it has been shown that those multivalued mappings defined on G-convex spaces which are closed, compact and acyclic valued have the KKM property. Hence, the above Corollary improve Theorems 2.1 and 2.2 of Wu et al. [47].

By a similar proof to the one given by Chang et al. [14 , Proposition 2.3(ii)], we can obtain the following lemma.

Lemma 2.6. *Let $(M, D; \Gamma)$ be a Γ-convex space, X a nonempty set and Y, Z topological spaces. Suppose that the maps $S : X \to D$, $T \in S$-KKM(X, M, Y) and $f : Y \to Z$ is continuous, then fT belongs to S-KKM(X, M, Z).*

As a consequence of Corollary 2.5 and Lemma 2.6, we obtain a Schauder type fixed point theorem for Γ-convex spaces.

Corollary 2.7. *Let $(M, D; \Gamma)$ be a Hausdorff Φ-space. Suppose that the identity mapping $I : M \to M$ belongs to $KKM(M, M)$, then any continuous mapping $f : M \to M$ such that $clf(M)$ is compact, has a fixed point.*

In the following we give some examples of metric spaces for which the identity mapping $I : X \to X$ belongs to $KKM(X, X)$.

We say that (M, d) is an $\mathcal{NR}$-metric space, if there exists a closed convex subset (E, ρ) of a completely metrizable Hausdorff topological vector space (V, ρ) in which

$$\rho(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \leq \max(\rho(x_1, y_1), \rho(x_2, y_2)), \quad (*)$$

$$\text{for each } x_1, x_2, y_1, y_2 \in E, \ \alpha + \beta = 1, \alpha, \beta \geq 0,$$

such that (M, d) can be isometrically embeded into (E, ρ) and there exists a nonexpansive retraction $r : E \to M$.

Every hyperconvex space is an $\mathcal{NR}$-metric space [5, 32]. Note that when (V, ρ) is a Banach space, this space is an 1-CAR set in the sense of Agrawal et al.[1], and the condition $(*)$ holds.

Let (M_1, d_1) and (M_2, d_2) be two $\mathcal{NR}$-metric spaces. It is easy to show that $(M_1 \times M_2, d)$, where $d = (d_1^2 + d_2^2)^{1/2}$, is an $\mathcal{NR}$-metric space. Hence despite the fact that finite products of hyperconvex spaces in general are not hyperconvex (cf. Theorem 4.1 of Borkowski et al. [10]), they are $\mathcal{NR}$-metric space.

The following lemma shows that in every $\mathcal{NR}$-metric space (M, d) and for any subadmissible subset X of M, the identity mapping belongs to $KKM(X, X)$.

Lemma 2.8. *Let (M, d) be an $\mathcal{NR}$-metric space , then $r(convA) \subseteq co(A)$, for any $A \in \langle M \rangle$.*

Remark. The Fan-KKM Theorem implies that the identity mapping in normed spaces is an elements of $KKM(X, X)$ for any convex set X. The Khamsi-KKM Theorem [32] shows that when M is hyperconvex, then $I \in KKM(X, X)$ for $X \in \mathcal{A}(M)$. This result is also true for a metric topological vector spaces E such that all balls are convex. In fact by the Fan-KKM Theorem, the identity mapping belongs to $KKM(X, X)$ for each convex subset X of E. Hence the identity mapping also belongs to $KKM(X, X)$ with respect to metric of E for any admissible subset X of E. Horvath in [28-29] has established that in C-spaces, LC-spaces and LC-metric spaces M, $I \in KKM(M, M)$. Park [41-43] has shown that when $(M, D; \Gamma)$ is a G-convex space, then

$I \in$ KKM(M, M). A similar result has been obtained By Ben-El-Mechaiekh et al. in [9] for L-spaces.

The above lemma also shows that every $\mathcal{NR}$-metric space is a generalized convex space in the sense of Park [41-43], so by a result of Fakhar and Zafarani [24] and Theorem 2.2, we have the following theorem which improves Theorems 2.1 and 2.2 of Wu et al. [47].

Theorem 2.9.[2]. *Let (M, d) be an $\mathcal{NR}$-metric space and X be a nonempty subadmissible subset of M. Suppose that F is u.s.c., compact with closed acyclic values, then $F \in KKM(X, X)$.*

Now, we obtain the following theorem for the existence of an approximate fixed point for a wide class of uniform topological spaces. This result improves Corollary 4.3 of Ben-El-Mechaiekh et al. [9].

Theorem 2.10. [3]. *Let $(M, M; \Gamma)$ be a Γ-convex space supply with uniform space with basis $\mathcal{U}$. Assume that for each $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$, $V \subseteq U$ such that for each $x \in M$ and each $A \in \langle V[x] \rangle$, $\Gamma(A) \subseteq U[x]$. Suppose that $S : M \to M$ is a surjective function and $F \in S$-$KKM(M, M, M)$ such that $clF(M)$ is a totally bounded, then F has an approximate fixed point.*

Remark. If X is a convex subset of a topological space and V is a symmetric convex open neighborhood of 0. Then for V, we can define a Φ-mapping T as $T(x) = G(x) = \{y \in X : x - y \in V\}$. Hence $Gr(T) \subseteq V$, therefore Theorem 2.10 implies that any $F \in$ KKM(X, X) such that $clF(X)$ is totally bounded has an approximate fixed point with respect to V. Hence we obtain Theorem 2 of Park [42].

As a consequence of the above theorem we deduce the following approximate fixed point result for the metric spaces.

Theorem 2.11.[2, 3]. *Let (M, d) be a metric space, X be a nonempty subadmissible subset of M and $S : X \to X$ be a surjective function. Suppose that $F \in S$-$KKM(X, X, X)$ is such that $clF(X)$ is totally bounded, then F has an approximate fixed point.*

Remark. Similarly to Corollary 2.5, in Theorems 2.10 and 2.12, when F is closed and compact, and the space M is Hausdorff space, we can obtain a fixed point for the multivalued mapping F.

Recently Khamsi [33] obtained an abstract formulation to Sadovski's fixed point theorem for continuous functions, using convexity structures. Here we will obtain an analogous result for multivalued mappings which are u.s.c. and have the KKM property.

Motivated by the concept of c-measure of noncompactness introduced by Hahn [27] for topological vector spaces, we define this notion in a similar way for a topological space M with respect to a family $\mathcal{C}$ of abstract convexity. Let γ be a cone in a vector space with partial ordering $\leq$ and $\mathcal{M}$ a collection of nonempty subsets of a topological space M with the property that for any $A \in \mathcal{M}$, the sets $co_{\mathcal{C}}(A)$, $\overline{A}$, $A \cup \{x\}$, $(x \in M)$, and every subset of A belong to $\mathcal{M}$. Let c be a real number with $c \geq 1$. A function $\Psi : \mathcal{M} \to \gamma$ is called a c-measure of noncompactness with respect to $\mathcal{C}$, provided that the following conditions hold for any $Z \in \mathcal{M}$:

(1) $\Psi(coZ) \leq c\Psi(Z)$;
(2) if $x \in X$, then $\Psi(Z \cup \{x\}) = \Psi(Z)$;
(3) if $Z_1 \subset Z$, then $\Psi(Z_1) \leq \Psi(Z)$;
(4) $\Psi(\overline{Z}) = \Psi(Z)$.

If $F : M \to \mathcal{M}$, then F is called a Ψ-pseudocondensing mapping if, whenever $\Psi(Z) \leq c\Psi(F(Z))$ for $Z \in \mathcal{M}$, then Z is relatively compact.
In particular, if $c = 1$, then F is called Ψ-condensing.

Let M be a topological space and $\mathcal{C}$ be a family of closed subsets of M such that $\emptyset$ and M belong to $\mathcal{C}$. We will say that
(1) $\mathcal{C}$ has the intersection property(IP) if and only if $\cap A_i \in \mathcal{C}$ provided $A_i \in \mathcal{C}$.
(2) $\mathcal{C}$ has the chain intersection property(CIP) if and only if $\cap A_i \in \mathcal{C}$ provided (A_i) is a decreasing chain of elements of $\mathcal{C}$.
Suppose that $\mathcal{C}$ has (IP) and $A \subseteq M$, by $\mathcal{C}(A)$ we mean $\{B \in \mathcal{C} : A \subseteq B\}$ and $\mathcal{C}$-hull of A as in Example 2.1(a) will be denoted by $co_{\mathcal{C}}(A)$. If $\mathcal{C}$ has (CIP), then the subfamily $\mathcal{C}(A)$ satisfies the assumptions of Zorn's lemma. Therefore $\mathcal{C}(A)$ has minimal elements. We will still use the notation $co_{\mathcal{C}}(A)$ to designate such minimal elements.

Examples. Let (M, d) be a bounded hyperconvex metric space . Set

$$\mathcal{H} = \{H \subset M; H \neq \emptyset \text{ and is hyperconvex}\}.$$

By a result of Baillon [6], $\mathcal{H}$ satisfies CIP (but fails to satisfy IP, i.e. the intersection of two hyperconvex is not necessarily hyperconvex). Khamsi [33] has proved that $\alpha : 2^M \to [0, \infty)$, the Kuratowski measure of noncompactness defined by

$$\alpha(A) = \inf\{\varepsilon > 0; A \subset \bigcup_{i=1}^{i=n} A_i \ , A_i \subset M, \ diam(A_i) \leq \varepsilon\}$$

is a measure of noncompactness with respect to the family $\mathcal{H}$.

It is trivial that the family $\mathcal{A}(M)$ of admissible sets satisfies (IP).

Henceforth let $\mathcal{C}$ stand for a family of closed subsets of M with the (IP) or (CIP) such that $\emptyset$ and M belong to $\mathcal{C}$.

We will say that $\mathcal{C}$ satisfies the property (K) (for Kakutani) if and only if for each $C \in \mathcal{C}$ nonempty compact and any $F : C \to 2^C$ which is u.s.c., nonempty closed values with KKM property with respect to $\mathcal{C}$ has a fixed point. In Theorem 2.11 and its Remark, we have shown that if M is metric, then the family $\mathcal{A}(M)$ of admissible subsets of M satisfies (K).

Theorem 2.12.[3]. *Let M be a Hausdorff topological space and the family $\mathcal{C}$ has the property (K). Then for any nonempty $C \in \mathcal{C} \cap \mathcal{M}$, any u.s.c. $F : C \to 2^C$ which is Ψ-pseudocondensing mapping, nonempty closed values and $F \in KKM(C, C)$ with respect to $\mathcal{C}$ has a fixed point.*

As a corollary, we get the following result which improve the results of Kirk and Shin [36].

Corollary 2.13. *Let H be a bounded hyperconvex metric space and $F : H \to 2^{\mathcal{H}}$ a closed α-condensing with KKM property with respect to $\mathcal{H}$. Then F has a fixed point.*

Here we obtain a generalized Fan's matching theorem for a set with Γ-convex structure. In fact we obtain an open version of Fan's matching Theorem which improves Theorem 2.7 of Yuan [48] and is similar to Theorem 4.4 of Chang et al. [15]. Let us recall that a subset A of a topological space Y is called compactly open, if its intersection with any compact subsets of Y is open in its relative topology.

Theorem 2.14.[2]. *Let $(M, D; \Gamma)$ be Γ-convex space, X a nonempty set and Y a topological space. Suppose that $S : X \to D$, $T : M \to 2^Y$, $T \in S$-$KKM(X, M, Y)$ and $F : X \to 2^Y$ is compactly open valued such that $clT(\Gamma(S(X))$ is compact and is contained in $F(X)$. Then there exists $\{x_1, ..., x_j\} \subset X$ such that:*

$$T(\Gamma(S\{x_1, ..., x_j\}) \bigcap (\cap_{k=0}^{j} F(x_k)) \neq \emptyset.$$

As an application of the above Theorem, we have the following form of the Fan-Browder type fixed point Theorem, see Kirk et al. [31, Theorem 3.1] and [49].

Corollary 2.15. *Let (M, M, Γ) be a compact Γ-convex space such that the identity mapping $I \in KKM(M, M)$. Suppose that $R : M \to 2^M$ is a multivalued mapping with Γ-convex values such that $M = \cup\{IntR^{-}(y) : y \in M\}$. Then R has a fixed point.*

As an application of the metric KKM principle, we give the following version of Fan's best approximation in $\mathcal{NR}$- metric spaces which is similar to Theorem 2.9 of Kirk et al.[17].

Theorem 2.16.[2]. *Let $X \in \mathcal{A}(M)$ be a compact subset of an $\mathcal{NR}$-metric space (M,d). Suppose that $F : X \to 2^M$ is continuous with nonempty subadmissible values, then there exists an $x_0 \in X$, such that*

$$d(x_0, F(x_0)) = \inf_{x \in X} d(x, F(x_0)).$$

In particular, if $F(x_0)$ is compact and $x_0 \notin F(x_0)$, x_0 must be a boundary point of X.

As a consequence of the above Theorem we have the following fixed point theorem.

Theorem 2.17.[2]. *Let $X \in \mathcal{A}(M)$ be a compact subset of an $\mathcal{NR}$ metric space (M,d). Suppose that $F : X \to \mathcal{A}(M)$ is continuous. Then F has a fixed point if one of the following conditions holds for all $x \in BdX$ such that $x \notin F(x)$:*

(1) *There exists $y \in X$ such that $d(y, F(x)) < d(x, F(x))$.*
(2) *There exists $\alpha \in (0,1)$ such that $X \cap B(F(x), \alpha d(x, F(x))) \neq \emptyset$.*
(3) *$F(x) \cap X \neq \emptyset$.*

Remark. As a corollary of the theorem we obtain that each continuous map $T : X \to 2^X$ with admissible values, where X is a compact admissible subset of an $\mathcal{NR}$-metric space (M,d), has a fixed point.

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