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ON THE COHOMOLOGY OF THE LIE SUPERALGEBRA OF CONTACT VECTOR FIELDS ON $S^{1/1}$

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Abstract

We investigate the first cohomology space attached to the embedding of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{1|1}$ in the Lie Superalgebra of superpseudodifferential operators. Following the paper [11], we show that this space is four-dimensional with only even cocycles and we calculate explicitly four 1-cocycles representing non-trivial generating cohomology classes.

1 Introduction

The classifications of multi-parameter deformations of homomorphisms of Lie algebras and in particular representations have been studied in many recent papers [1, 2, 10, 11]. The first cohomology space classify the infinitesimal deformations, while the obstructions are living in the second cohomology space. The study of multi-parameter deformations of the standard embedding of the Lie algebra $\mathrm{Vect}(S^1)$ of vector fields on the circle S^1 inside the Lie algebra $\Psi\mathcal{DO}$ of pseudodifferential operators on S^1 was carried out in [11]. In this paper we address ourselves to the computation of the first cohomology space of the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on the supercircle $S^{1|1}$ with coefficients in the Lie superalgebra $\mathcal{S}\Psi\mathcal{DO}$ of superpseudodifferential operators on $S^{1|1}$. It is a first step towards that classification for the natural embedding of $\mathcal{K}(1)$ inside $\mathcal{S}\Psi\mathcal{DO}$. Namely, we first compute the first cohomology space of the $\mathcal{K}(1)$-module of tensor densities $\mathfrak{F}_\lambda = \{F\alpha^\lambda, F \in C^\infty(S^{1|1})\}$, where $\alpha = dx + \theta d\theta$ is the contact 1-form and the action of $\mathcal{K}(1)$ is given by Lie derivatives. The first cohomology space of $\mathcal{K}(1)$ with coefficients in the Poisson superalgebra $\mathcal{SP}$ of superpseudodifferential symbols of $\mathcal{S}\Psi\mathcal{DO}$ will be a corollary of the later one, since, $\mathcal{SP}$ has a decomposition to a direct sum of modules of tensor densities. After that we compute the first cohomlogy space in the $\mathcal{K}(1)$-module $\mathcal{S}\Psi\mathcal{DO}$, using the same method as in [11]. The main result of this paper can be stated as follows (Theorem (6.1)): *The first cohomology space $H^1(\mathcal{K}(1),\ \mathcal{S}\Psi\mathcal{DO})$ is four-dimensional and it is generated by the 1-cocycles (6.1): $\Theta_0, \Theta_1, \Theta_2$ and Θ_3.* In our approach to the proof of Theorem (6.1), we follow the lines by [11]. That is we apply successive differentials of the spectral sequences corresponding to the complex $C^*(\mathcal{K}(1),\ \mathcal{SP})$.

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2 Superpseudodifferential operators on $S^{1|1}$

2.1 Lie superalgebra structure

We first recall the definition of the algebra of superpseudodifferential operators on the supercircle $S^{1|1}$(cf, [4, 9]).

The supercircle $S^{1|1}$ is the superextention of the circle S^1 with local coordinates (x, θ), where $x \in S^1$ and $\theta^2 = 0$. A C^∞ function on $S^{1|1}$ has the form $F = f(x) + 2g(x)\theta$ with $f, g \in C^\infty(S^1)$. The vector field $\eta = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$ on $S^{1|1}$ sends F to $\eta(F) = 2g(x) + f'(x)\theta$ so that $\eta^2 = \frac{1}{2}[\eta, \eta] = \frac{\partial}{\partial x}$. The usual Leibniz rule $\frac{\partial}{\partial x} \circ f = f'(x) + f(x)\frac{\partial}{\partial x}$ on $C^\infty(S^1)$, is replaced on $C^\infty(S^{1|1})$ by:

$$\eta \circ F = \eta(F) + \sigma(F)\eta \qquad (2.1)$$

where the involution σ is the grading automorphism on $C^\infty(S^{1|1})$, equal to 1 on the even part and to -1 on the odd part (in other words, η is a superderivation).

The formula (2.1), generalises by induction on m to the graded Leibniz formula

$$\eta^m \circ F = \sum_{k=0}^{\infty} \binom{m}{k}_s \eta^k(\sigma^{m-k}(F))\eta^{m-k} \qquad (2.2)$$

for all integers $m \geq 0$. where the supersymmetric binomial coefficients $\binom{m}{k}_s$ are defined by:

$$\binom{m}{k}_s = \begin{cases} \binom{[\frac{m}{2}]}{[\frac{k}{2}]} & \text{if } k \text{ is even} \quad \text{or } m \text{ is odd} \\ 0, & \text{otherwise} \end{cases}$$

with $[x]$ is usual denoting the integral part of a real number x, and for $l \in \mathbb{Z}_{\geq 0}$, the binomial coefficient $\binom{x}{l} = x(x-1)\cdots(x-l+1)$. Let us introduce the superalgebra of superpseudodifferential operators $\mathcal{S}\Psi\mathcal{DO}$ on $S^{1|1}$ by:

$$\mathcal{S}\Psi\mathcal{DO} = \{\sum_{k \in \mathbb{Z}_{\geq 0}} F_k \eta^{\omega - k}, \ w \in \mathbb{Z}, \ F_k \in C^\infty(S^{1|1})\},$$

where the composition of superpseudodifferential operators is generated by the graded Leibniz formula (2.2):

$$F\eta^m \circ G\eta^n = \sum_{k=0}^{\infty} \binom{m}{k}_s F\eta^k(\sigma^{m-k}(G))\eta^{m+n-k}, \ m, n \in \mathbb{Z} \text{ and } F, \ G \in C^\infty(S^{1|1}). \qquad (2.3)$$

As usual, the composition of operators induces a Lie superalgebra structure on $\mathcal{S}\Psi\mathcal{DO}$ with the super-commutator defined on homogeneous elements by:

$$[A, B] = A \circ B - (-1)^{p(A)p(B)} BA,$$

where we let p denote the function of parity.

2.2 Symbols of superpseudodifferential operators on $S^{1|1}$

In this subsection, we will define the Poisson bracket of superpseudodifferential symbols. We first list some definitions and notations from [11]. Let $\mathcal{P}(S^1)$ be the ring of symbols of pseudodifferential operators on S^1

$$A(x, \xi) = \sum_{-\infty}^{n} a_i(x)\xi^i,$$

where $a_i(t) \in C^\infty(S^1)$, and the variable ξ corresponds to $\frac{\partial}{\partial x}$. The space $\mathcal{P}(S^1)$ is a Poisson Lie algebra with the bracket given by

$$\{A(x,\xi), B(x,\xi)\} = \frac{\partial}{\partial \xi}A(x,\xi)\frac{\partial}{\partial x}B(x,\xi) - \frac{\partial}{\partial x}A(x,\xi)\frac{\partial}{\partial \xi}B(x,\xi),$$

where the multiplication is naturally defined.

Analogously, we introduce in the super-case, the super-commutative ring

$$\mathcal{SP} = C^\infty(S^{1|1}) \otimes (\mathbb{C}[\xi, \xi^{-1}]] \oplus \mathbb{C}[\xi, \xi^{-1}]]\zeta)$$

of symbols of superpseudodifferential operators on $S^{1|1}$

$$S(x,\xi,\zeta) = \sum_{-\infty}^{n} F_k(x)\xi^k + \Big(\sum_{-\infty}^{n} G_k(x)\xi^k\Big)\zeta,$$

where $F_k, G_k \in C^\infty(S^{1|1})$, $\zeta = \overline{\theta} + \theta\xi$ corresponds to η and $\overline{\theta}$ corresponds to $\frac{\partial}{\partial\theta}$; with $\overline{\theta}^2 = \zeta^2 = 0$ and $\zeta \cdot F\xi^m = \sigma(F)\xi^m\zeta$, $F \in C^\infty(S^{1|1})$. Then, the multiplication of symbols is obvious.

We define the Poisson bracket on $\mathcal{SP}$ by

$$\{S,T\} = \frac{\partial}{\partial\xi}(S)\frac{\partial}{\partial x}(T) - \frac{\partial}{\partial x}(S)\frac{\partial}{\partial\xi}(T) - (-1)^{p(S)}\Big(\frac{\partial}{\partial\theta}(S)\frac{\partial}{\partial\overline{\theta}}(T) + \frac{\partial}{\partial\overline{\theta}}(S)\frac{\partial}{\partial\theta}(T)\Big), \quad (2.4)$$

where S, $T \in \mathcal{SP}$ (cf, [7])

3 The space of tensor densities on $S^{1|1}$

Let us first recall the Vect(S^1)-module of tensor densities on S^1. Consider the one parameter action of $Vect(S^1)$ on $C^\infty(S^1)$ given by

$$L^\lambda_{X(x)\partial}(f(x)) = X(x)f'(x) + \lambda X'(x)f(x), \quad (3.1)$$

where $f \in C^\infty(S^1)$ and $\lambda \in \mathbb{R}$. Denote $\mathcal{F}_\lambda$ the $Vect(S^1)$-module structure on $C^\infty(S^1)$ given by (3.1). Note that the adjoint $Vect(S^1)$-module is isomorphic to $\mathcal{F}_{-1}$. Geometrically, $\mathcal{F}_\lambda$ is the space of tensor densities of degree λ on S^1, i.e. the set of all expressions: $f(x)(dx)^\lambda$, where $f \in C^\infty(S^1)$.

We have analogous definition of tensor densities in the super-case (see [9]). Let $\alpha = dx + \theta d\theta$ be the contact 1-form on $S^{1|1}$ and let $\mathcal{K}(1)$ be the Lie superalgebra of vector fields on $S^{1|1}$ preserving the 1-form α. The Lie Superalgebra $\mathcal{K}(1)$ is also known as the algebra of Neveu-Schwarz without central charge or the Lie superalgebra of contact vector fields on $S^{1|1}$.

Every vector field in $\mathcal{K}(1)$ has the form

$$v_F = \frac{1}{2}(F + \sigma(F))\eta^2 + \eta(F)\eta, \; F \in C^\infty(S^{1|1}). \quad (3.2)$$

We introduce a one parameter action of $\mathcal{K}(1)$ on $C^\infty(S^{1|1})$ by the rule:

$$\mathfrak{L}^{\lambda}_{v_F}(G) = F\cdot\eta^2(G) + \frac{(-1)^{p(F)(p(G)+1)}}{2}\eta(F)\cdot\eta(G) + \lambda\eta^2(F)\cdot G, \tag{3.3}$$

where $F,\ G\in C^\infty(S^{1|1})$. We denote this $\mathcal{K}(1)$-module by $\mathfrak{F}_\lambda$.

Geometrically, The space $\mathfrak{F}_\lambda$ is no other then the space of all tensor densities on $S^{1|1}$ of degree λ:

$$\phi = f(x,\theta)\alpha^\lambda,\ f(x,\theta)\in C^\infty(S^{1|1}), \tag{3.4}$$

where the action (3.3) of $\mathcal{K}(1)$ is the Lie derivative action on tensor densities.

Remarks 3.1. *1) The action (3.3) of $\mathcal{K}(1)$ on $\mathfrak{F}_\lambda$ is given explicitly by*

$$\mathfrak{L}^{\lambda}_{v_F}(G) = \mathcal{L}^{\lambda}_{a\partial}(g_0) + 2bg_1 + 2(\mathcal{L}^{\lambda+\frac{1}{2}}_{a\partial}(g_1) + J_1(b,g_0))\theta \tag{3.5}$$

where $F = a+2\theta b$, $G = g_0 + 2\theta g_1$ and the operator J_1 is defined on $\mathcal{F}_\lambda\otimes\mathcal{F}_\mu$ by

$$J_1(f,g) = -\lambda f g' + \mu g f'.$$

As a $\mathrm{Vect}(S^1)$-module (i.e. $b=0$) the space of tensor densities $\mathfrak{F}_\lambda$ is isomorphic to $\mathcal{F}_\lambda\oplus\mathcal{F}_{\lambda+\frac{1}{2}}$, which is the $\mathbb{Z}_2$-grading of $\mathfrak{F}_\lambda$. In particular, the Lie superalgebra $\mathcal{K}(1)$ is isomorphic to $\mathcal{F}_{-1}\oplus\mathcal{F}_{-\frac{1}{2}}$ as $\mathrm{Vect}(S^1)$-module.

2) The adjoint $\mathcal{K}(1)$-module is isomorphic to the module $\mathfrak{F}_{-1}$. This isomorphism induces a super Poisson bracket on $C^\infty(S^{1|1})$ given by:

$$\{F,G\} = \mathfrak{L}^{-1}_{v_F}(G) = FG' - F'G + \frac{(-1)^{p(F)(p(G)+1)}}{2}\eta(F)\cdot\eta(G). \tag{3.6}$$

4 The structure of $\mathcal{SP}$ as a $\mathcal{K}(1)$-module

The natural embedding of $\mathcal{K}(1)$ inside $\mathcal{S\Psi DO}$ given by the expression (3.2) induces a $\mathcal{K}(1)$-module structure on $\mathcal{S\Psi DO}$. Analogously, we have a $\mathcal{K}(1)$-module structure on $\mathcal{SP}$ given by the natural embedding of $\mathcal{K}(1)$:

$$\pi : v_F \longmapsto \frac{1}{2}(F+\sigma(F))\xi + \eta(F)\zeta. \tag{4.1}$$

The Poisson super-algebra $\mathcal{SP}$ is $\mathbb{Z}$-graded, where we give x,θ the degree zero and ξ,ζ the degree one.

Then we have

$$\mathcal{SP} = \widetilde{\bigoplus}_{n\in\mathbb{Z}}\mathcal{SP}_n \tag{4.2}$$

where, $\widetilde{\bigoplus}_{n\in\mathbb{Z}} = (\bigoplus_{n<0})\bigoplus\prod_{n\geq 0}$ and $\mathcal{SP}_n = \{F\xi^{-n} + G\xi^{-n-1}\zeta,\ F,\ G\in C^\infty(S^{1|1})\}$ is the homogeneous subspace of degree $-n$.

Each element of $\mathcal{S\Psi DO}$ can be written as

$$A = \sum_{k\in\mathbb{Z}}(F_k + G_k\eta^{-1})\eta^{2k},$$

where $F_k,\ G_k \in C^\infty(S^{1|1})$. We define the order of A by

$$ord(A) = sup\{k; F_k \neq 0 \text{ or } G_k \neq 0\}.$$

This definition of order equips $\mathcal{S}\Psi\mathcal{DO}$ with a decreasing filtration as follows: let us set

$$\mathbf{F}_n = \{A \in \mathcal{S}\Psi\mathcal{DO},\ ord(A) \leq -n\}$$

where $n \in \mathbb{Z}$. So one has

$$\ldots \subset \mathbf{F}_{n+1} \subset \mathbf{F}_n \subset \ldots . \tag{4.3}$$

This filtration is compatible with the multiplication and the super Poisson bracket, that is, for $A \in \mathbf{F}_n$ and $B \in \mathbf{F}_m$, one has $A \circ B \in \mathbf{F}_{n+m}$ and $\{A, B\} \in \mathbf{F}_{n+m-1}$, where we identify $\mathcal{SP}$ with $\mathcal{S}\Psi\mathcal{DO}$. This filtration makes $\mathcal{S}\Psi\mathcal{DO}$ as an associative filtered superalgebra. Moreover, this filtration is compatible with the natural action of $\mathcal{K}(1)$ on $\mathcal{S}\Psi\mathcal{DO}$. Indeed, if $v_F \in \mathcal{K}(1)$ and $A \in \mathbf{F}_n$, then

$$v_F.A = [v_F, A] \in \mathbf{F}_n.$$

The induced $\mathcal{K}(1)$-module on the quotient $\mathbf{F}_n/\mathbf{F}_{n+1}$ is isomorphic to the $\mathcal{K}(1)$-module $\mathcal{SP}_n$. Therefore, the $\mathcal{K}(1)$-module on the associated graded space of the filtration (4.3), is isomorphic to the graded $\mathcal{K}(1)$-module $\mathcal{SP}$, that is

$$\mathcal{SP} \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} \mathbf{F}_n/\mathbf{F}_{n+1}.$$

Proposition 4.1. *As a $\mathcal{K}(1)$-module we have*

$$\mathcal{SP} \simeq \widetilde{\bigoplus}_{n\in\mathbb{Z}} (\mathfrak{F}_n \oplus \mathfrak{F}_{n+\frac{1}{2}}).$$

Proof. The $\mathcal{K}(1)$-module $\mathcal{SP}_n$ of the grading (4.2) has the direct sum decomposition of the two $\mathcal{K}(1)$-modules, ${\mathcal{SP}_n}^1$ and ${\mathcal{SP}_n}^2$, defined by

$${\mathcal{SP}_n}^1 = \{(F + \sigma(F))\xi^{-n} + \eta(F)\xi^{-n-1}\zeta,\ F \in C^\infty(S^{1|1})\}, \tag{4.4}$$

and

$${\mathcal{SP}_n}^2 = \{F\xi^{-n-1}\zeta - 2\theta F\xi^{-n},\ F \in C^\infty(S^{1|1})\}. \tag{4.5}$$

The action of $\mathcal{K}(1)$ on ${\mathcal{SP}_n}^1$ is induced by the embedding (4.1) as follows:

$$\begin{aligned}
& v_F \cdot (\frac{1}{2}(G + \sigma(G))\xi^{-n} + \eta(G)\xi^{-n-1}\zeta) \\
=\ & \{\pi(v_F),\ (G + \sigma(G))\xi^{-n} + \eta(G)\xi^{-n-1}\zeta\} \\
=\ & (\mathfrak{L}^n_{v_F}(G) + \sigma(\mathfrak{L}^n_{v_F}(G)))\xi^{-n} + \eta(\mathfrak{L}^n_{v_F}(G))\xi^{-n-1}\zeta.
\end{aligned}$$

The natural map $\varphi_1 : \mathfrak{F}_n \longrightarrow {\mathcal{SP}_n}^1$ defined by

$$\varphi_1(F) = (F + \sigma(F))\xi^{-n} + \eta(F)\xi^{-n-1}\zeta,$$

provides us with an isomorphism of $\mathcal{K}(1)$-modules.

The action of $\mathcal{K}(1)$ on $\mathcal{SP}_n{}^2$ is given by

$$\begin{aligned} v_F \cdot (G\xi^{-n-1}\zeta - 2\theta G\xi^{-n}) &= \{\pi(v_F), G\xi^{-n-1}\zeta - 2\theta G\xi^{-n}\} \\ &= \mathfrak{L}_{v_F}^{n+\frac{1}{2}}(G)\xi^{-n-1}\zeta - 2\theta\mathfrak{L}_{v_F}^{n+\frac{1}{2}}(G)\xi^{-n}. \end{aligned}$$

The natural map $\varphi_2 : \mathfrak{F}_{n+\frac{1}{2}} \longrightarrow \mathcal{SP}_n{}^2$ defined by:

$$\varphi_2(F) = F\xi^{-n-1}\zeta - 2\theta F\xi^{-n},$$

provides us with an isomorphism of $\mathcal{K}(1)$-modules.

□

5 The first cohomology space $H^1(\mathcal{K}(1), \mathcal{SP})$

In this section, we will compute the first cohomology space of $\mathcal{K}(1)$ with coefficients in $\mathcal{SP}$. To do this, we first recall some fundamental concepts from cohomology theory ([6]).

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra acting on a super vector space $V = V_0 \oplus V_1$. The space $\mathrm{Hom}(\mathfrak{g}, V)$ is $\mathbb{Z}_2$-graded via

$$\mathrm{Hom}(\mathfrak{g}, V)_b = \oplus_{a\in\mathbb{Z}_2}\mathrm{Hom}(\mathfrak{g}_a, V_{a+b});\ b \in \mathbb{Z}_2. \tag{5.1}$$

Let $Z^1(\mathfrak{g}, V) = \{c \in \mathrm{Hom}(\mathfrak{g}, V)\colon c([g,h]) = g\cdot c(h) - (-1)^{p(g)p(h)}h\cdot c(g),\ \forall g, h \in \mathfrak{g}\}$ be the space of 1-cocycles for the Chevalley-Eilenberg differential. According to the $\mathbb{Z}_2$-grading (5.1), each $c \in Z^1(\mathfrak{g}, V)$, is broken to $(c', c'') \in \mathrm{Hom}(\mathfrak{g}_0, V) \oplus \mathrm{Hom}(\mathfrak{g}_1, V)$ subject to the following three equations:

$$\begin{array}{llcl} (E_1) & c'([g_1, g_2]) - g_1.c'(g_2) + g_2.c'(g_1) & = & 0,\ g_1, g_2 \in \mathfrak{g}_0 \\ (E_2) & c''([g, h] - g.c''(h) + h.c'(g) & = & 0,\ g \in \mathfrak{g}_0, h \in \mathfrak{g}_1 \\ (E_3) & c'([h_1, h_2] - h_1c''(h_2) - h_2c''(h_1) & = & 0,\ h_1, h_2 \in \mathfrak{g}_1. \end{array} \tag{5.2}$$

In the sequel let us consider the Lie super algebra $\mathcal{K}(1)$ acting on $\mathfrak{F}_\lambda$. The first cohomology space $H^1(\mathcal{K}(1), \mathfrak{F}_\lambda)$ inherits the $\mathbb{Z}_2$-grading from (5.1) and it decomposes to a odd part and a even part as follows:

$$H^1(\mathcal{K}(1), \mathfrak{F}_\lambda) = H^1(\mathcal{K}(1), \mathfrak{F}_\lambda)_0 \oplus H^1(\mathcal{K}(1), \mathfrak{F}_\lambda)_1.$$

We calculate each part independently. The following proposition is the main result of this section:

Proposition 5.1. *1) The first cohomology space $H^1(\mathcal{K}(1), \mathfrak{F}_\lambda)_0$ has the following structure:*

$$H^1(\mathcal{K}(1), \mathfrak{F}_\lambda)_0 = \begin{cases} \mathbb{R}^2 & \text{if } \lambda = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The space $H^1(\mathcal{K}(1), \mathfrak{F}_0)_0$ is generated by the cohomology classes of the 1-cocycles

$$c_0(v_F) = \frac{1}{4}(F + \sigma(F)) + \frac{1}{2}F \text{ and } c_1(v_F) = \eta^2(F) \tag{5.3}$$

2) The cohomology space

$$H^1(\mathcal{K}(1),\mathfrak{F}_\lambda)_1=\begin{cases}\mathbb{R} \text{ if } \lambda=\frac{1}{2},\frac{3}{2}\\ 0 \text{ otherwise.}\end{cases}$$

It is spanned by the non-trivial cohomology class corresponding to the 1*-cocycle*

$$c_2(v_F)=\eta^3(F)\quad \text{if } \lambda=\frac{1}{2}, \tag{5.4}$$

and

$$c_3(v_F)=\eta^5(F)\quad \text{if } \lambda=\frac{3}{2}. \tag{5.5}$$

To prove this proposition, we will need the following two results:

Proposition 5.2. *[6] The space of first cohomology of* $\mathrm{Vect}(S^1)$ *with coefficients in the space of tensor densities* $\mathfrak{F}_\lambda$ *has the following sructure:*

$$H^1(\mathrm{Vect}(S^1);\mathcal{F}_\lambda)=\begin{cases}\mathbb{R}^2 &, \quad \text{if } \lambda=0\\ \mathbb{R} &, \quad \text{if } \lambda=1 \text{ or } 2\\ 0 &, \quad \text{otherwise}\end{cases} \tag{5.6}$$

It is spanned by the classes of the following non-trivial 1*-cocycles:*

$$\begin{aligned}\beta_0\Big(f(x)\frac{d}{dx}\Big) &= f(x) \text{ and } \beta_1\Big(f(x)\frac{d}{dx}\Big)=f'(x), \text{ if } \lambda=0,\\ \beta_2\Big(f(x)\frac{d}{dx}\Big) &= f''(x), \text{ if } \lambda=1 \text{ and}\\ \beta_3\Big(f(x)\frac{d}{dx}\Big) &= f'''(x), \text{ if } \lambda=2.\end{aligned} \tag{5.7}$$

Moreover, we have the following lemma

Lemma 5.3. *Let* $C_0=(C_{00},C_{11})$ *be a even* 1*-cocycle from* $\mathcal{K}(1)$ *to* $\mathfrak{F}_\lambda$, *where* $C_{00}:\mathrm{Vect}(S^1)\longrightarrow\mathcal{F}_\lambda$ *and* $C_{11}:\mathcal{F}_{\frac{1}{2}}\longrightarrow\mathcal{F}_{\lambda+\frac{1}{2}}$ *are given by the grading (5.1). Then, if* C_{00} *is a coboundary over* $\mathrm{Vect}(S^1)$ *then,* C_0 *is a coboundary over* $\mathcal{K}(1)$.

Proof. Recall that a 1-coboundary of $\mathrm{Vect}(S^1)$ with coefficients in $\mathcal{F}_\lambda$ has the form $c(a(x)\frac{\partial}{\partial x})=\mathcal{L}^\lambda_{a(x)\frac{\partial}{\partial x}}(f)$ for some $f\in\mathcal{F}_\lambda$. Now let $C_{00}(v_F)=\mathcal{L}^\lambda_{a(x)\frac{\partial}{\partial x}}(f)$ for some $f\in\mathcal{F}_\lambda$ be a coboundary where $F=a(x)+2\theta b(x)$. If we apply the equations (E_2) and (E_3) from (5.2) to C_0, we will obtain $C_{11}(F)=2\theta J_1(b(x),f)$ and then, $C_0(v_F)=\mathfrak{L}^\lambda_{v_F}(f)$ is a coboundary of $\mathcal{K}(1)$. □

Remark 5.4. *We have the same Lemma for odd* 1*-cocycle* $C_1=(C_{01},C_{10})$, *where* $C_{01}:\mathrm{Vect}(S^1)\longrightarrow\mathcal{F}_{\lambda+\frac{1}{2}}$ *and* $C_{10}:\mathcal{F}_{\frac{1}{2}}\longrightarrow\mathcal{F}_\lambda$.

Proof of Proposition 5.1. Since the space of 1-cocycles from $\mathcal{K}(1)$ to $\mathfrak{F}_\lambda$ is $\mathbb{Z}_2$-graded, we first assume that C is a even non-trivial 1-cocycle. According to the $\mathbb{Z}_2$-graduation (5.1) of even cocycles , $C = C' + C''$ where the linear maps $C' : \mathrm{Vect}(S^1) \longrightarrow \mathcal{F}_\lambda$ and $C'' : \mathcal{F}_{\frac{1}{2}} \longrightarrow \mathcal{F}_{\lambda+\frac{1}{2}}$ are the homogenous parts. The equation (E_1) from (5.2), implies that C' is a 1-cocycle of $\mathrm{Vect}(S^1)$ and the lemma (5.3) implies that C' is non-trivial. By proposition (5.2), C' is cohomologous to one of the cocycles (5.7). To compute C'', we apply the equations (E_2) and (E_3) from (5.2) to the cocycle C. We have solutions only if $C' = \beta_0$ or $C' = \beta_1$ and we obtain that C is one of the cocycles c_0 or c_1.

Next, if C is odd, the same arguments show that (E_2) and (E_3) are compatible if and only if $C' = \beta_2$ or $C' = \beta_3$, and then we obtain c_2 and c_3. □

The first cohomology space of $\mathcal{K}(1)$ with coefficients in the space of symbols $\mathcal{SP}$ inherits the grading (4.2) of $\mathcal{SP}$, so it suffices to compute it in each degree. Combining propositions (4.1) and (5.1), we obtain the main result of this section, that can be stated as follows:

Theorem 5.5. *The first cohomology space of $\mathcal{K}(1)$ with coefficients in the space of symbols $\mathcal{SP}$ is four-dimensional with only even 1-cocycles. It is spanned by the classes of the following non-trivial 1-cocycles*

$$\begin{aligned} C_0(v_F) &= \frac{1}{4}(F+\sigma(F)) + \frac{1}{2}F, \\ C_1(v_F) &= \eta^2(F), \\ C_2(v_F) &= \mathrm{ad}^3_\zeta(\pi(v_F))\xi^{-2}\overline{\zeta} \text{ and} \\ C_3(v_F) &= \mathrm{ad}^5_\zeta(\pi(v_F))\xi^{-3}\overline{\zeta}. \end{aligned} \tag{5.8}$$

where $\mathrm{ad}_\zeta(\pi(v_F)) = \{\zeta, \pi(v_F)\}$ *with π is the map (4.1) and* $\overline{\zeta} = \overline{\theta} - \theta\xi$ $(\overline{\zeta}^2 = 0)$.

Proof. According to propositions (4.1) and (5.1), the cohomology space of $\mathcal{K}(1)$ with coefficients in $\mathcal{SP}_n$ has the following structure

$$H^1(\mathcal{K}(1), \mathcal{SP}_n) = \begin{cases} \mathbb{R}^3 & , \quad \text{if } n = 0 \\ \mathbb{R} & , \quad \text{if } n = 1 \\ 0 & , \quad \text{otherwise .} \end{cases} \tag{5.9}$$

In the case $n = 0$, the cohomology space $H^1(\mathcal{K}(1), \mathcal{SP}_0)$ is generated by the non-trivial cohomology classes of the cocycles $\tilde{C}_0, \tilde{C}_1$ and C_2 corresponding to the cocycles c_0, c_1 and c_2 of proposition (5.1) via the isomorphism in proposition (4.1). They are given by

$$\begin{aligned} \tilde{C}_0(v_F) &= \frac{1}{2}\Big(F + \sigma(F) + \eta(F)\xi^{-1}\zeta - \frac{1}{4}\eta(F - \sigma(F))\xi^{-1}\zeta\Big), \\ \tilde{C}_1(v_F) &= \mathrm{ad}^2_\zeta(\pi(v_F))\xi^{-1} \text{ and} \\ C_2(v_F) &= \mathrm{ad}^3_\zeta(\pi(v_F))\xi^{-2}\overline{\zeta}. \end{aligned} \tag{5.10}$$

In the case $n = 1$, the cohomology space $H^1(\mathcal{K}(1), \mathcal{SP}_1)$ is generated by the non-trivial cohomology class of the cocycle C_3 corresponding to the 1-cocycle c_3 and it is given by

$$C_3(v_F) = \mathrm{ad}_\zeta^5(\pi(v_F))\xi^{-3}\overline{\zeta}. \tag{5.11}$$

As a 1-cocycle of $\mathcal{SP}$, $\tilde{C}_0$ is cohomologous to C_0. Indeed, $C_0 - \tilde{C}_0 = \mathrm{ad}_{\frac{1}{2}\theta\xi^{-1}\zeta}\Big(\pi(v_F)\Big)$ and $\tilde{C}_1 = C_1 + \frac{1}{2}C_2$. This completes the proof of the theorem.

□

6 The first cohomology space $H^1(\mathcal{K}(1),\ \mathcal{S\Psi DO})$

In this section, we will compute the cohomology space of $\mathcal{K}(1)$ with coefficients in the filtered module $\mathcal{S\Psi DO}$. A straightforward but long computations, using spectral sequences associated to Grad($\mathcal{S\Psi DO}$) [8] and Theorem (5.5) leads to the following theorem:

Theorem 6.1. *The first cohomology space $H_0^1(\mathcal{K}(1), \mathcal{S\Psi DO})$ of $\mathcal{K}(1)$ with coefficients in the space $\mathcal{S\Psi DO}$ is four-dimensional with only even 1-cocycles. It is spanned by the classes of the following non-trivial 1-cocycles*

$$\begin{aligned}
\Theta_0(v_F) &= \frac{1}{4}(F+\sigma(F)) + \frac{1}{2}F,\\
\Theta_1(v_F) &= \eta^2(F),\\
\Theta_2(v_F) &= \sum_{n=1}^{\infty}(-1)^n\frac{n-2}{n}\sigma(\bar{\eta}^{2n+1}(F))\bar{\eta}^{-2n+1} + \sum_{n=1}^{\infty}(-1)^n\frac{n-3}{n+1}\bar{\eta}^{2n+2}(F)\bar{\eta}^{-2n},\\
\Theta_3(v_F) &= \sum_{n=2}^{\infty}(-1)^n\frac{n-1}{n}\sigma(\bar{\eta}^{2n+1}(F))\bar{\eta}^{-2n+1} + \sum_{n=2}^{\infty}(-1)^n\frac{n-1}{n+1}\bar{\eta}^{2n+2}(F)\bar{\eta}^{-2n},
\end{aligned} \tag{6.1}$$

where $\bar{\eta} = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial x}$.

Proof. Since the cohomology space $H^1(\mathcal{K}(1),\ \mathcal{S\Psi DO})$ is obviously upper-bounded by $H^1(\mathcal{K}(1),\ \mathcal{SP})$, we have to find explicit expressions for the non trivial cocycles generating the former cohomology space. To construct these cocycles, we follow the lines in [10] based on the computations of successive differentials of the spectral sequences corresponding to the complex $C^*(\mathcal{K}(1),\ \mathcal{SP})$. So, we consider a cocycle with values in $\mathcal{SP}$, but we compute its boundary as it was with values in $\mathcal{S\Psi DO}$ and keep a symbolic part of the result. This gives a new cocycle of degree equal to the degree of the previous one plus one. We iterate this procedure, we establish a recurrent formula between successive terms. The cocycles c_0 and c_1 survive in the same form, we will denote them Θ_0 and Θ_1 when seen as cocycles with values in $\mathcal{S\Psi DO}$. The previous procedure applied to c_2 and c_3 leads to the cocycles Θ_2 and Θ_3.

□

Remark 6.2. *The parts of Θ_2 and Θ_3 which are maps from* $\mathrm{Vect}(S^1)$ *to* $\Psi\mathcal{D}(S^1)$ *in the grading (5.1) are a multiple by a coefficient of the 1-cocycles θ_2 and θ_3 in [10].*

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