WEAK HOLOMORPHY
AND OTHER WEAK PROPERTIES

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Dedicated to the memory of Professor Ioana Cioranescu

ABSTRACT. Let $A(X)$ be a closed subspace of the space of all scalar functions on a Hausdorff space $X$ which are bounded on all compact sets, endowed with the compact-open topology. Our main result — with a simple, short proof — is that, for a mapping $f$ from $X$ into a locally convex space $E$ which has the property that the image $f(K)$ of each compact set $K \subset X$ is contained in an absolutely convex weakly compact set, $e' \cdot f \in A(X)$ for each $e' \in E'$ implies $e' \cdot f \in A(X)$ for each $e' \in E'$. This is related to results of Grosse-Erdmann [7], [8] and Arendt, Nikolski [2] for vector valued holomorphic functions.

Weak conditions for holomorphy of a vector valued function have recently found renewed interest. In his Habilitationsschrift [7], K.-G. Grosse-Erdmann showed that it suffices to test weak holomorphy of a locally bounded function with values in a locally complete locally convex space on the elements of a separating subset in the dual of the range space; his proof was completely elementary, but rather lengthy. The result was utilized quite prominently (at two different points) in our article [5].

More recently, W. Arendt and N. Nikolski [2] gave a short proof of Grosse-Erdmann's result for functions with values in Fréchet spaces, using the theorem of Krein-Šmulian; also see the appendix of [1]. Finally, K.-G. Grosse-Erdmann [8] presented a new approach, based on the principal idea of [7], but making use of functional analytic tools to shorten the proof.

In the present short note we show with a very simple proof that, for a vector function with the property that the range of each compact set is contained in an absolutely convex weakly compact set, it suffices to test holomorphy on the elements of a separating subset of the dual of the (locally complete locally convex) range space. That is, we have a somewhat stronger hypothesis on the mapping to start with and do not recover Grosse-Erdmann's result unless the range space is semifinite. Moreover, contrary to Grosse-Erdmann we also make use (as Arendt and Nikolski did) of the Dunford-Grothendieck theorem on the equivalence of holomorphy and weak holomorphy. But our present theorem really reads as follows:

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