WEAK HOLOMORPHY AND OTHER WEAK PROPERTIES

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Dedicated to the memory of Professor Ioana Cioranescu

ABSTRACT.[1] Let $\mathcal{A}(X)$ be a closed subspace of the space of all scalar functions on a Hausdorff space X which are bounded on all compact sets, endowed with the compact-open topology. Our main result – with a simple, short proof – is that, for a mapping f from X into a locally convex space E which has the property that the image $f(K)$ of each compact set $K \subset X$ is contained in an absolutely convex weakly compact set, $e' \circ f \in \mathcal{A}(X)$ for each e' in a separating set $S \subset E'$ implies $e' \circ f \in \mathcal{A}(X)$ for each $e' \in E'$. This is related to results of Grosse-Erdmann [7], [8] and Arendt, Nikolski [2] for vector valued holomorphic functions.

Weak conditions for holomorphy of a vector valued function have recently found renewed interest. In his Habilitationsschrift [7], K.-G. Grosse-Erdmann showed that it suffices to test weak holomorphy of a locally bounded function with values in a locally complete locally convex space on the elements of a separating subset in the dual of the range space; his proof was completely elementary, but rather lengthy. The result was utilized quite prominently (at two different points) in our article [5].

More recently, W. Arendt and N. Nikolski [2] gave a short proof of Grosse-Erdmann's result for functions with values in Fréchet spaces, using the theorem of Krein-Šmulian; also see the appendix of [1]. Finally, K.-G. Grosse-Erdmann [8] presented a new approach, based on the principal idea of [7], but making use of functional analytic tools to shorten the proof.

In the present short note we show with a very simple proof that, for a vector function with the property that the range of each compact set is contained in an absolutely convex weakly compact set, it suffices to test holomorphy on the elements of a separating subset of the dual of the (locally complete locally convex) range space. That is, we have a somewhat stronger hypothesis on the mapping to start with and do not recover Grosse-Erdmann's result unless the range space is semireflexive. Moreover, contrary to Grosse-Erdmann we also make use (as Arendt and Nikolski did) of the Dunford-Grothendieck theorem on the equivalence of holomorphy and weak holomorphy. But our present theorem really reads as follows:

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Let $\mathcal{A}(X)$ be a closed subspace of the space of all those scalar functions on a Hausdorff space X which are bounded on each compact set, endowed with the compact-open topology. If E is a locally convex space and $f : X \to E$ is a function such that for each compact subset K of X there exists an absolutely convex $\sigma(E, E')$-compact subset $C = C(K)$ of E with $f(K) \subset C$ and for which $e' \circ f \in \mathcal{A}(X)$ holds for all e' in a separating subset S of E', then $e' \circ f \in \mathcal{A}(X)$ holds for each $e' \in E'$.

Thus, our theorem applies not only to holomorphic functions, but also to harmonic functions, to solution spaces of hypoelliptic linear partial differential operators etc. The short proof relies on the method of linearizing non-linear vector valued functions which is related to Schwartz's ε-product and to the use of preduals in infinite dimensional holomorphy. In our applications of Grosse-Erdmann's theorem in [5] it was already known that the vector valued function was better than just locally bounded, and we believe that in many applications it is clear that the function satisfies the stronger hypothesis of our theorem.

The theorem and its proof

In the sequel, X will always be a Hausdorff topological space and E a locally convex space (which is by definition Hausdorff). $\mathcal{FB}(X, E)$ resp. $\mathcal{FP}(X, E)$ denotes the linear space of all the mappings $f : X \to E$ such that $f(K)$ is bounded resp. precompact in E for each compact subset K of X. $\mathcal{FR}(X, E)$ is the linear subspace of $\mathcal{FB}(X, E)$ of all mappings with the property that for each compact $K \subset X$ there exists an absolutely convex $\sigma(E, E')$-compact subset $C = C(K)$ of E such that $f(K)$ is contained in C. Note that if E is quasicomplete, $\mathcal{FR}(X, E)$ is precisely the space of all functions f from X to E for which $f(K)$ is relatively $\sigma(E, E')$-compact for each compact $K \subset X$; in this case, any function f from X to E with the property that $e' \circ f$ is continuous for each $e' \in E'$ belongs to $\mathcal{FR}(X, E)$. If E is the scalar field (of the real or complex numbers), we write $\mathcal{FB}(X)$ instead of $\mathcal{FB}(X, E)$. This space will always be endowed with the topology of uniform convergence on the compact subsets of X.

Lemma. *(a) Fix an arbitrary $f \in \mathcal{FB}(X, E)$. Then the linear mapping $I(f)$, defined by*

$$[I(f)](e') \;:=\; e' \circ f \quad \forall\, e' \in E',$$

is continuous from the strong dual E'_b into $\mathcal{FB}(X)$. Moreover, the transposed mapping ${}^t[I(f)]$ of $I(f)$ is continuous from $\mathcal{FB}(X)'_b$ into the strong bidual $E'' = (E'_b)'_b$ of E and satisfies ${}^t[I(f)](\delta_x) = f(x) \in E$ for each $x \in X$, where $\delta_x \in \mathcal{FB}(X)'$ is the evaluation at the point x.

(b) If $f \in \mathcal{FR}(X, E)$ resp. $\mathcal{FP}(X, E)$, then $I(f)$ is also continuous from $E'_\tau = E'$, endowed with the Mackey topology $\tau(E', E)$ (of uniform convergence on the absolutely convex $\sigma(E, E')$-compact subsets of E), resp. $E'_{pc} = E'$, endowed with the topology of uniform convergence on the precompact subsets of E, into $\mathcal{FB}(X)$.

Proof of (a). It is clear that $e' \circ f \in \mathcal{FB}(X)$ for each $e' \in E'$ and that $I(f)$ is a linear map from E' into $\mathcal{FB}(X)$. To show that $I(f)$ is continuous with respect to the topologies mentioned above, fix $\varepsilon > 0$ and a compact set $K \subset X$, and let

$$U := \{g \in \mathcal{FB}(X);\; \sup_{x \in K} |g(x)| \leq \varepsilon\}$$

denote the corresponding 0-neighborhood in $\mathcal{FB}(X)$ for the compact-open topology. By hypothesis on f, $f(K)$ is bounded in E so that $V := \varepsilon[f(K)]^\circ$ is a 0-neighborhood in E'_b for which obviously $[I(f)](V) \subset U$ holds.

To see the formula, fix $x \in X$, $f \in FB(X, E)$, and let $e' \in E'$ be arbitrary. Then we have

$$e'\left({}^t[I(f)](\delta_x)\right) = [\delta_x \circ I(f)](e') = e'(f(x)),$$

and the formula follows since e' was arbitrary.

A simple modification of the first part of the proof of (a) covers the two cases in (b). □

We mention in passing that much more than the second case in part (b) of the lemma is known. If $\mathcal{FP}(X,E)$ is also equipped with the topology of uniform convergence on the compact subsets of X and if E is quasicomplete, then the map $I : f \to I(f)$ yields a topological isomorphism

$$\mathcal{FP}(X,E) = E\varepsilon\mathcal{FB}(X) = \mathcal{L}_e(E'_{pc}, \mathcal{FB}(X)),$$

cf. [3], Corollary 15. Here ε indicates Schwartz's ε-product; the subindex e denotes the topology of uniform convergence on the equicontinuous subsets of E'. If E is even complete, there is topological equality with the complete ε-tensor product $E\check{\otimes}_\varepsilon\mathcal{FB}(X)$.

The setting of our main theorem is as follows. Let $\mathcal{A}(X)$ be a closed linear subspace of $\mathcal{FB}(X)$. The corresponding space $\mathcal{A}(X,E)$ of E-valued functions consists of all mappings $f \in \mathcal{FR}(X,E)$ which *belong to* $\mathcal{A}(X)$ *weakly*; that is, which satisfy $e'\circ f \in \mathcal{A}(X)$ for each $e' \in E'$. In this situation $\mathcal{A}(X,E)$ is automatically a closed subspace of $\mathcal{FR}(X,E)$ under the topology of uniform convergence on the compact subsets of X. Clearly, if $\mathcal{A}(X)$ is a subspace of the space $C(X)$ of all continuous functions on X, then any $f \in \mathcal{A}(X,E)$ must be continuous from X into $(E, \sigma(E,E'))$.

A subset S of E' is called *separating* if it separates the points of E or, equivalently, if

$$e \in E, \quad e'(e) = 0 \quad \forall e' \in S \quad \Rightarrow \quad e = 0.$$

Theorem. *Let $f \in \mathcal{FR}(X,E)$ be arbitrary. If $e'\circ f \in \mathcal{A}(X)$ holds for all e' in a separating subset S of E', then f must already be an element of $\mathcal{A}(X,E)$.*

PROOF. We have to show that $e'\circ f \in \mathcal{A}(X)$ holds for each $e' \in E'$. By the lemma, the linear mapping $I(f) : e' \to e'\circ f$ is continuous from E'_τ into $\mathcal{FB}(X)$, and by hypothesis we have $[I(f)](S) \subset \mathcal{A}(X)$. With a linear space $\mathcal{A}(X)$ we can always pass from S to its linear span, and hence we may assume without loss of generality that S is a linear subspace. Now by the Hahn-Banach theorem S separating implies S $\sigma(E',E)$-dense in E'. But a simple consequence of the Hahn-Banach theorem shows that then S is also dense in E', equipped with any topology which respects the duality of E and E', and thus in particular in E'_τ. Hence from $[I(f)](S) \subset \mathcal{A}(X)$ it clearly follows that $[I(f)](E') \subset \mathcal{A}(X)$ as $\mathcal{A}(X)$ is closed in $\mathcal{FR}(X)$. □

The preceding proof does not work for functions f which are only bounded on compact sets unless E is semireflexive, in which case one can indeed use the strong dual E'_b instead of E'_τ.

Note that by the theorem, in particular,

$$\mathcal{A}(X, E'_b) = \{f : X \to E';\ x \to [f(x)](e) \text{ belongs to } \mathcal{A}(X) \ \ \forall e \in E\}$$

since E is a separating subset of its bidual E''. (Compare with the definition of $A(X, E'_b)$ on page 14 of [5].)

We recall that a completely regular Hausdorff space X is called a *$k_{\mathbb{R}}$-space* if a scalar function g on X is continuous whenever the restriction of g to each compact subset of X is continuous. If X is a $k_{\mathbb{R}}$-space and f is a mapping of X into some completely regular Hausdorff space (e.g., into a locally convex space), then f must be continuous whenever the restriction of f to any compact subset of X is continuous.

Remark. *If $\mathcal{A}(X) \subset C(X)$, X is a $k_{\mathbb{R}}$-space and $f \in \mathcal{FP}(X,E)$ satisfies $e'\circ f \in \mathcal{A}(X)$ for all e' in a separating subset S of E', then f must already be continuous from X into E.*

To see this, one can use the theorem and [3], Remark 18. Here is a direct argument in which the theorem is not needed: By the hypotheses, f is clearly continuous from X into $(E, \sigma(E, S))$. Note that $\sigma(E, S)$ is a Hausdorff topology because S is separating. Now the restriction of f to each compact set $K \subset X$ is continuous into E since the topology of E and $\sigma(E, S)$ coincide on the precompact set $f(K) \subset E$. In view of the $k_{\mathbb{R}}$-assumption on X, this implies f continuous on X.

Concluding notes

Usually some completeness assumption on the range space E is necessary in the applications of our theorem. E.g., holomorphy and weak holomorphy (in an arbitrary number of variables, which means including infinite dimensional holomorphy, where 'locally bounded' has to be replaced by 'amply bounded' in general) are only equivalent when E is a *locally complete* space; i.e., when for each closed absolutely convex bounded subset B of E the space $E_B = \text{span}(B)$, endowed with the Minkowski functional of B as norm, is complete, cf. Grosse-Erdmann [8], Corollary 1. We also refer to [8] for a discussion of the history of results of the type of his theorem and for applications (e.g., to holomorphic functions with values in sequence spaces, function spaces and spaces of operators).

Moreover, in most examples $\mathcal{A}(X) \subset C(X)$ holds, and then one would usually define $\mathcal{A}(X, E)$ in a different way than we have done here, viz. as the intersection of our $\mathcal{A}(X, E)$ with the space $C(X, E)$ of all continuous mappings from X to E. (Our remark singles out a case where this is not necessary.) This usual definition occurs for instance in [3], [4] and [5]. We refer to [4] for examples of spaces $\mathcal{A}(X)$ to which our theorem applies. The examples include harmonic and polyharmonic functions as well as spaces of zero solutions of hypoelliptic differential operators with C^∞-coefficients.

It is clear that we have not dealt in this note with the question if $f \in \mathcal{FB}(X, E)$ must be continuous if $e' \circ f$ is an element of $\mathcal{A}(X) \subset C(X)$ for each $e' \in E'$ (however, cf. the remark). If the space of vector valued functions can be represented as an ε-product, this is a simple consequence of the formula $C(X, E) = E\varepsilon C(X)$ (for a $k_{\mathbb{R}}$-space X and quasicomplete E) and of the fact that the ε-product respects subspaces. In case $\mathcal{A}(X)$ is a complete nuclear space and E is complete, the continuity (and usually much more than that) follows from Grothendieck's famous "weak-strong principle" ([9], Chapitre II, page 80). We refer to Gramsch [6] for a discussion of related questions. In fact, [6] contains very similar ideas and methods as the present paper, but it appears that our theorem is not implied directly by the results of [6]. – Also note that Grosse-Erdmann [8], Proof of Theorem 1, Part (B) pointed out that it suffices to prove his theorem for Banach range spaces E because then the general case (that E is locally complete) follows very easily.

References

[1] W. Arendt, C.J.K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics **96**, Birkhäuser (2001).

[2] W. Arendt, N. Nikolski, Vector-valued holomorphic functions revisited, *Math. Z.* **234** (2000), 777–805.

[3] K.-D. Bierstedt, Tensor products of weighted spaces, *Bonner Math. Schriften* **81** (1975), 26–58.

[4] K.-D. Bierstedt, B. Gramsch, R. Meise, Lokalkonvexe Garben und gewichtete induktive Limites $\mathcal{F}$-morpher Funktionen, *Bonner Math. Schriften* **81**, 59–72.

[5] K.D. Bierstedt, S. Holtmanns, An operator representation for weighted spaces of vector valued holomorphic functions, *Result. Math.* **36** (1999), 9–20.

[6] B. Gramsch, An extension method of the duality theory of locally convex spaces with applications to extension kernels and the operational calculus, pp. 131–147 in *Functional Analysis: Surveys and Recent Results, Proc. Paderborn 1976*, North-Holland Math. Studies **27** (1977).

[7] K.-G. Grosse-Erdmann, *The Borel-Okada Theorem Revisited*, Habilitationsschrift, Hagen (1992).

[8] K.-G. Grosse-Erdmann, A weak criterion for vector-valued holomorphy, to appear in *Math. Proc. Cambridge Phlos. Soc.*

[9] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, *Memoirs Amer. Math. Soc.* **16** (1955, reprint 1966).

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