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# HYPOCONTINUOUS MULTILINEAR MAPPINGS IN TOPOLOGICAL MODULES

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**Abstract.** Basic results concerning equihypocontinuous sets of multilinear mappings between topological modules are obtained.

The study of the notion of a hypocontinuous bilinear mapping, a notion which is intermediate between that of a continuous bilinear mapping and that of a separately continuous bilinear mapping, is well-known in Functional Analysis [2]. Although less discussed than hypocontinuous bilinear mappings, hypocontinuous multilinear mappings may be found, for example, in [8].

In this paper we initiate the investigation of the concept of equihypocontinuity for sets of multilinear mappings between topological modules, by proving that basic facts of the classical theory remain valid in this quite general context. Our main result, whose proof depends on a version of the Banach-Steinhaus theorem for continuous linear mappings between topological modules, asserts that, under certain conditions, separate equicontinuity implies equihypocontinuity. Clarifying examples are also included in the paper.

Throughout this paper $R$ denotes a commutative topological ring with a non-zero identity element. $E$, $F$, $E_1, \dots, E_m$ denote unitary topological $R$-modules, where $m$ is an integer with $m \geq 2$, $\mathcal{L}_a(E_1, \dots, E_m; F)$ denotes the $R$-module of all $m$-linear mappings from $E_1 \times \cdots \times E_m$ into $F$ and $\mathcal{L}(E; F)$ denotes the $R$-module of all continuous linear

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mappings from $E$ into $F$.

**Definition 1.** A set $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is said to be *separately equicontinuous* if, for each $j \in \{1, \ldots, m\}$ and for each $x_i \in E_i$ $(i \in \{1, \ldots, m\}, i \neq j)$, the set of linear mappings

$$\{x_j \in E_j \mapsto A(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) \in F; A \in \mathfrak{X}\}$$

is equicontinuous.

In the case where $\mathfrak{X}$ consists of a single mapping $A$, we shall say that $A$ is *separately continuous*.

**Example 2.** Let $\mathcal{M}$ be a covering of $E$ by bounded sets, and consider $\mathcal{L}(E; F)$ endowed with the $R$-module topology of $\mathcal{M}$-convergence. Then the bilinear mapping

$$u\colon E \times \mathcal{L}(E; F) \to F,$$

given by $u(x, A) = A(x)$ for $x \in E$ and $A \in \mathcal{L}(E; F)$, is separately continuous. In particular, $u$ is separately continuous when $\mathcal{L}(E; F)$ is endowed with the $R$-module topology $\tau_s$ of pointwise convergence or with the $R$-module topology $\tau_b$ of bounded convergence.

Indeed, for each $A \in \mathcal{L}(E; F)$, the mapping $u(\cdot, A)\colon x \in E \mapsto u(\cdot, A)(x) = A(x) \in F$ is obviously continuous. On the other hand, let us see that, for each $x \in E$, the mapping $u(x, \cdot)\colon A \in \mathcal{L}(E; F) \mapsto u(x, A) \in F$ is continuous. In fact, let $W$ be an arbitrary neighborhood of 0 in $F$. By assumption, there exists a $B \in \mathcal{M}$ such that $x \in B$. Since $U = \{A \in \mathcal{L}(E; F); A(B) \subset W\}$ is a neighborhood of 0 for the topology of $\mathcal{M}$-convergence and the relation $A \in U$ implies $u(x, A) \in W$, then $u(x, \cdot)$ is continuous.

**Definition 3.** For each $i \in \{1, \ldots, m\}$ let $\mathcal{M}_i$ be a set of bounded subsets of $E_i$. For a fixed $j \in \{1, \ldots, m\}$ a set $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is said to be $(\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_{j+1}, \ldots, \mathcal{M}_m)$-*equihypocontinuous* if:
(a) each $A \in \mathfrak{X}$ is separately continuous;
(b) for each $B_i \in \mathcal{M}_i$ $(i \in \{1, \ldots, m\}, i \neq j)$ the set of linear mappings

$$\{x_j \in E_j \mapsto A(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) \in F; x_i \in B_i \text{ for } i \neq j, A \in \mathfrak{X}\}$$

is equicontinuous.

If $\mathfrak{X}$ consists of a single mapping $A$, we shall say that $A$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_{j+1}, \ldots, \mathcal{M}_m)$-*hypocontinuous*.

A set $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is said to be $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$-*equihypocontinuous* if $\mathfrak{X}$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_{j+1}, \ldots, \mathcal{M}_m)$- equihypocontinuous for all $j \in \{1, \ldots, m\}$.

If $\mathfrak{X}$ consists of a single mapping $A$, we shall say that $A$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$-*hypocontinuous.*

**Remark 4.** If $\mathcal{M}_j$ is a covering of $E_j$ for all $j \in \{1, \ldots, m\}$ and if $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ satisfies (b) for all $j \in \{1, \ldots, m\}$, then $\mathfrak{X}$ is separately equicontinuous.

**Example 5.** Let $(R, ||\cdot||)$ be a commutative normed ring with an identity element 1 such that $||1|| = 1$ and such that there exists an invertible element $\lambda \in R$ with $||\lambda|| < 1$. Let $E$ be an infrabarrelled topological $R$-module [5] and $F$ an arbitrary topological $R$-module. Let $\mathcal{M}_1$ be the set of all bounded subsets of $E$ and let $\mathcal{M}_2$ be the set of all $\tau_b$-bounded subsets of $\mathcal{L}(E; F)$, and consider $u$ as in Example 2. Then $u$ is $(\mathcal{M}_1, \mathcal{M}_2)$-hypocontinuous. Indeed, it is clear that $u$ is $\mathcal{M}_1$-hypocontinuous for arbitrary $E$. On the other hand, if $\mathfrak{X} \subset \mathcal{L}(E; F)$ is $\tau_b$-bounded, then the set

$$\{x \in E \mapsto u(x, A) \in F; A \in \mathfrak{X}\}$$

(which is precisely $\mathfrak{X}$) is equicontinuous by Theorem 14 of [5]. Therefore $u$ is $\mathcal{M}_2$-hypocontinuous.

**Example 6.** Let $(R, ||\cdot||)$ be as in Example 5.

(a) Let $E$ be a topological $R$-module which is not barrelled [7], [4]. By Theorem 9 of [4], there exist a topological $R$-module $F$ and a $\tau_s$-bounded subset of $\mathcal{L}(E; F)$ which is not equicontinuous. Let $\mathcal{M}_1$ be the set of all bounded subsets of $E$ and let $\mathcal{M}_2$ be the set of all $\tau_s$-bounded subsets of $\mathcal{L}(E; F)$, and consider $u$ as in Example 2. Then $u$ is $\mathcal{M}_1$-hypocontinuous but $u$ is not $\mathcal{M}_2$-hypocontinuous.

(b) Let $E$ be a topological $R$-module which is not infrabarrelled. By Theorem 14 of [5], there exist a topological $R$-module $F$ and a $\tau_b$-bounded subset of $\mathcal{L}(E; F)$ which is not equicontinuous. Let $\mathcal{M}_1$ be the set of all bounded subsets of $E$ and let $\mathcal{M}_2$ be the set of all $\tau_b$-bounded subsets of $\mathcal{L}(E; F)$, and consider $u$ as in Example 2. Then $u$ is $\mathcal{M}_1$-hypocontinuous but $u$ is not $\mathcal{M}_2$-hypocontinuous.

Before we proceed, let us recall the following

**Definition 7** [3]. $R$ satisfies condition (N) if, for all neighborhoods $L_1$ and $L_2$ of 0 in $R$, the set $L_1 L_2$ is a neighborhood of 0 in $R$.

**Proposition 8.** Assume that $R$ satisfies condition (N) and that the product of any neighborhood of 0 in $R$ by any neighborhood of 0 in $E_i$ is a neighborhood of 0 in $E_i$ for $i \in \{1, \ldots, m\}$. If $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is equicontinuous, then $\mathfrak{X}$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$-equihypocontinuous.

**Proof.** Let $j \in \{1, \ldots, m\}$ be fixed and let us prove that $\mathfrak{X}$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_{j+1}, \ldots, \mathcal{M}_m)$-equihypocontinuous. For this purpose, let $B_i \in \mathcal{M}_i$ be arbitrary for $i \neq j$, and let $W$ be an arbitrary neighborhood of 0 in $F$. By the equicontinuity of $\mathfrak{X}$ at $(0, \ldots, 0) \in E_1 \times \cdots \times E_m$, there exists a neighborhood $V_i$ of 0 in $E_i$ $(i = 1, \ldots, m)$ such that $A(x_1, \ldots, x_m) \in W$ for all $x_i \in V_i$ and for all $A \in \mathfrak{X}$. By the boundedness of $B_i$, there exists a neighborhood $L$ of 0 in $R$ such that $LB_i \subset V_i$ for $i \neq j$. Consequently,

$$
\begin{aligned}
& A(B_1 \times \cdots \times B_{j-1} \times (L^{m-1}V_j) \times B_{j+1} \times \cdots \times B_m) \\
& = A((LB_1) \times \cdots \times (LB_{j-1}) \times V_j \times (LB_{j+1}) \times \cdots \times (LB_m)) \\
& \subset A(V_1 \times \cdots \times V_{j-1} \times V_j \times V_{j+1} \times \cdots \times V_m) \subset W
\end{aligned}
$$

for all $A \in \mathfrak{X}$ (where $L^{m-1} = \{\lambda_1 \ldots \lambda_{m-1}; \lambda_1, \ldots, \lambda_{m-1} \in L\}$), $L^{m-1}V_j$ being a neighborhood of 0 in $E_j$ by hypothesis. Therefore $\mathfrak{X}$ is $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_{j-1} \times \mathcal{M}_{j+1} \times \cdots \times \mathcal{M}_m)$-equihypocontinuous. Finally, by the arbitrariness of $j$, $\mathfrak{X}$ is $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_m)$-equihypocontinuous, as asserted.

The converse of Proposition 8 is not true in general, as the following example shows.

**Example 9.** Let $(\mathbb{K}, |\cdot|)$ be a spherically complete field and $E$ a separated locally convex space over $\mathbb{K}$ [9]. Let $E'_b$ be the vector space $E'$ over $\mathbb{K}$ of all continuous linear forms on $E$ endowed with the locally convex topology $\tau_b$ of bounded convergence. We claim that $u\colon E \times E'_b \to \mathbb{K}$ is continuous if and only if $E$ is normable ($u$ being as in Example 2).

Indeed, if $u$ is continuous, there exists a neighborhood $U$ of 0 in $E$ and a $\mathbb{K}$-convex, bounded and closed subset $B$ of $E$ such that the relations $x \in U$, $\varphi \in B^p$ imply $|u(x, \varphi)| = |\varphi(x)| < 1$, where $B^p = \{\varphi \in E'; |\varphi(x)| < 1 \text{ for all } x \in B\}$. Thus $U \subset (B^p)^p = B$ (where $(B^p)^p = \{x \in E; |\varphi(x)| < 1 \text{ for all } \varphi \in B^p\}$), the last equality being a consequence of Proposition 2.b of [10]. Therefore $B$ is a neighborhood of 0 in $E$, and hence $E$ is normable ([6], 2.3).

Conversely, assume that there exists a norm $\|\cdot\|$ on $E$ which defines its topology.

Then $\tau_b$ is defined by the norm $||\varphi|| = \sup_{x \neq 0} \frac{|\varphi(x)|}{||x||}$, and the inequality

$$|u(x, \varphi)| = |\varphi(x)| \leq ||\varphi|| \, ||x|| \quad (x \in E, \, \varphi \in E')$$

guarantees the continuity of $u$.

Now assume that $E$ is metrizable but is not normable. Then $E$ is an infrabarrelled topological vector space over $\mathbb{K}$ by Proposition 3 of [1] and Corollary 16 of [5]. Moreover, by Example 5 and what we have just seen above, the mapping $u\colon E \times E'_b \to \mathbb{K}$ is $(\mathcal{M}_1, \mathcal{M}_2)$-hypocontinuous but is not continuous, where $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is the set of all bounded subsets of $E$ (resp. $E'_b$).

**Proposition 10**. Let $j \in \{1, \ldots, m\}$ and let $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ be $(\mathcal{M}_1, \ldots, \mathcal{M}_{j-1}, \mathcal{M}_{j+1}, \ldots, \mathcal{M}_m)$-equihypocontinuous. Then, for each $B_i \in \mathcal{M}_i$ $(i \neq j)$ and for each bounded subset $B$ of $E_j$, the set

$$\begin{aligned} &\mathfrak{X}(B_1 \times \cdots \times B_{j-1} \times B \times B_{j+1} \times \cdots \times B_m) \\ &= \{A(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m); x_i \in B_i \text{ for } i \neq j, x_j \in B, A \in \mathfrak{X}\} \end{aligned}$$

is bounded.

**Proof.** Let $W$ be an arbitrary neighborhood of 0 in $F$. By hypothesis, there exists a neighborhood $V_j$ of 0 in $E_j$ such that the relations $x_i \in B_i$ $(i \neq j)$, $x_j \in V_j$, $A \in \mathfrak{X}$ imply $A(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) \in W$. By the boundedness of $B$, there exists a neighborhood $L$ of 0 in $R$ such that $LB \subset V_j$; hence

$$L\mathfrak{X}(B_1 \times \cdots \times B_{j-1} \times B \times B_{j+1} \times \cdots \times B_m) \subset W.$$

Therefore $\mathfrak{X}(B_1 \times \cdots \times B_{j-1} \times B \times B_{j+1} \times \cdots \times B_m)$ is bounded, as was to be shown.

**Proposition 11**. Assume that $\mathcal{M}_i$ is a covering of $E_i$ for $i = 1, \ldots, m$ and that $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$-equihypocontinuous. Then, for each $j \in \{1, \ldots, m\}$ and for each $B_i \in \mathcal{M}_i$ $(i \neq j)$, the set

$$\begin{aligned} \{(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) &\in B_1 \times \cdots \times B_{j-1} \times E_j \times B_{j+1} \times \cdots \times B_m \\ &\mapsto A(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) \in F; A \in \mathfrak{X}\} \end{aligned}$$

is equicontinuous.

**Proof.** Let $(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_m) \in B_1 \times \cdots \times B_{j-1} \times E_j \times B_{j+1} \times \cdots \times B_m$ be arbitrary and let $W$ be an arbitrary neighborhood of 0 in $F$. Take a neighborhood $W_1$ of 0 in $F$ such that $\underbrace{W_1 + \cdots + W_1}_{m \text{ times}} \subset W$.

For all $(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_m) \in E_1 \times \cdots \times E_{j-1} \times E_j \times E_{j+1} \times \cdots \times E_m$ and for all $A \in \mathcal{L}_a(E_1, \ldots, E_m; F)$, we have

$$\begin{aligned} A(x_1, \ldots, x_m) - A(a_1, \ldots, a_m) &= A(x_1 - a_1, a_2, \ldots, a_m) + A(x_1, x_2 - a_2, a_3, \ldots, a_m) \\ &+ \cdots + A(x_1, \ldots, x_{j-2}, x_{j-1} - a_{j-1}, a_j, a_{j+1}, \ldots, a_m) \\ &+ \cdots + A(x_1, \ldots, x_{m-2}, x_{m-1} - a_{m-1}, a_m) + A(x_1, \ldots, x_{m-1}, x_m - a_m). \end{aligned}$$

Since $\mathcal{M}_j$ is a covering of $E_j$, there exists a $B_j \in \mathcal{M}_j$ such that $a_j \in B_j$. By hypothesis, for each $i \in \{1, \ldots, m\}$ there exists a neighborhood $V_i$ of 0 in $E_i$ such that the relations $x_1 \in B_1, \ldots, x_{i-1} \in B_{i-1}, x_i \in V_i, x_{i+1} \in B_{i+1}, \ldots, x_m \in B_m, A \in \mathfrak{X}$ imply $A(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) \in W_1$. Therefore the relations $x_1 \in B_1, \ldots, x_{j-1} \in B_{j-1}, x_j \in E_j, x_{j+1} \in B_{j+1}, \ldots, x_m \in B_m, x_1 - a_1 \in V_1, \ldots, x_{j-1} - a_{j-1} \in V_{j-1}, x_j - a_j \in V_j, x_{j+1} - a_{j+1} \in V_{j+1}, \ldots, x_m - a_m \in V_m, A \in \mathfrak{X}$ imply $A(x_1, \ldots, x_m) - A(a_1, \ldots, a_m) \in W$. By the arbitrariness of $(a_1, \ldots, a_{j-1}, a_j, a_{j+1}, \ldots, a_m)$, the proof is complete.

**Remark 12.** In the bilinear case the assumption that $A \in \mathcal{L}_a(E_1, E_2; F)$ is $\mathcal{M}_1$-hypocontinuous is sufficient to ensure the continuity of the mapping

$$(x_1, x_2) \in B_1 \times E_2 \mapsto A(x_1, x_2) \in F,$$

where $\mathcal{M}_1$ and $\mathcal{M}_2$ are arbitrary and $B_1 \in \mathcal{M}_1$.

Indeed, let $(a_1, a_2) \in B_1 \times E_2$ be arbitrary and let $W$ be an arbitrary neighborhood of 0 in $F$. Take a neighborhood $W_1$ of 0 in $F$ such that $W_1 + W_1 \subset W$. For all $(x_1, x_2) \in E_1 \times E_2$, we have

$$A(x_1, x_2) - A(a_1, a_2) = A(x_1 - a_1, a_2) + A(x_1, x_2 - a_2).$$

Since $A$ is separately continuous, there exists a neighborhood $V_1$ of 0 in $E_1$ such that $A(x_1 - a_1, a_2) \in W_1$ if $x_1 \in B_1$ and $x_1 - a_1 \in V_1$. By the $\mathcal{M}_1$-hypocontinuity of $A$, there exists a neighborhood $V_2$ of 0 in $E_2$ such that $A(x_1, x_2 - a_2) \in W_1$ if $x_1 \in B_1$, $x_2 \in E_2$ and

$x_2 - a_2 \in V_2$. Consequently, $A(x_1, x_2) - A(a_1, a_2) \in W_1 + W_1 \subset W$ if $(x_1, x_2) \in B_1 \times E_2$, $x_1 - a_1 \in V_1$ and $x_2 - a_2 \in V_2$, proving our claim.

Our next result establishes a relation between equihypocontinuity and uniform equicontinuity.

**Proposition 13.** Let $\mathfrak{X} \subset \mathcal{L}_a(E_1, \dots, E_m; F)$ be $(\mathcal{M}_1 \times \cdots \times \mathcal{M}_m)$-equihypocontinuous. Then, for each $B_i \in \mathcal{M}_i$, the set

$$\{(x_1, \dots, x_m) \in B_1 \times \cdots \times B_m \mapsto A(x_1, \dots, x_m) \in F; A \in \mathfrak{X}\}$$

is uniformly equicontinuous.

**Proof.** Let $W$ be an arbitrary neighborhood of 0 in $F$ and let $W_1$ be a neighborhood of 0 in $F$ such that $\underbrace{W_1 + \cdots + W_1}_{m \text{ times}} \subset W$. By hypothesis, for each $i \in \{1, \dots, m\}$ there exists a neighborhood $V_i$ of 0 in $E_i$ such that the relations $(x_1, \dots, x_m), (y_1, \dots, y_m) \in B_1 \times \cdots \times B_m$, $x_i - y_i \in V_i$, $A \in \mathfrak{X}$ imply $A(x_1, \dots, x_{i-1}, x_i - y_i, y_{i+1}, \dots, y_m) \in W_1$. Therefore the relations $(x_1, \dots, x_m), (y_1, \dots, y_m) \in B_1 \times \cdots \times B_m$, $x_i - y_i \in V_i$ $(i \in \{1, \dots, m\})$, $A \in \mathfrak{X}$ imply

$$\begin{aligned} A(x_1, \dots, x_m) - A(y_1, \dots, y_m) &= A(x_1 - y_1, y_2, \dots, y_m) + A(x_1, x_2 - y_2, y_3, \dots, y_m) \\ &+ \cdots + A(x_1, \dots, x_{i-1}, x_i - y_i, y_{i+1}, \dots, y_m) \\ &+ \cdots + A(x_1, \dots, x_{m-1}, x_m - y_m) \in \underbrace{W_1 + \cdots + W_1}_{m \text{ times}} \subset W. \end{aligned}$$

This completes the proof.

We close our paper by proving that, under certain conditions, separate equicontinuity implies equihypocontinuity.

**Theorem 14.** Assume that there exists a $j \in \{1, \dots, m\}$ such that $E_i$ is a barrelled topological $R$-module for all $i \in \{1, \dots, m\} - \{j\}$. If $\mathfrak{X} \subset \mathcal{L}_a(E_1, \dots, E_m; F)$ is separately equicontinuous, then $\mathfrak{X}$ is $(\mathcal{M}_1, \dots, \mathcal{M}_{i-1}, \mathcal{M}_{i+1}, \dots, \mathcal{M}_m)$-equihypocontinuous for all $i \in \{1, \dots, m\} - \{j\}$.

**Proof.** Without loss of generality we may assume that $i = 1$ and $j = m$. In fact, suppose that $i < j$ (argue in a similar way if $j < i \le m$). For each $A \in \mathfrak{X}$ define

$$\widetilde{A}\colon E_i \times \cdots \times \underbrace{E_1}_{i\text{-th position}} \times \cdots \times \underbrace{E_m}_{j\text{-th position}} \times \cdots \times E_j \to F$$

by $\widetilde{A}(x_i, \ldots, x_1, \ldots, x_m, \ldots, x_j) = A(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m)$; then $\widetilde{A} \in \mathcal{L}_a(E_i, \ldots, E_1, \ldots, E_m, \ldots, E_j; F)$. It is easily seen that the set $\widetilde{\mathfrak{X}} = \{\widetilde{A}; A \in \mathfrak{X}\}$ is separately equicontinuous and that $\mathfrak{X}$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_{i-1}, , \mathcal{M}_{i+1}, \ldots, \mathcal{M}_m)$-equihypocontinuous if and only if $\widetilde{\mathfrak{X}}$ is $(\mathcal{M}_2, \ldots, \mathcal{M}_1, \ldots, \mathcal{M}_m, \ldots, \mathcal{M}_j)$-equihypocontinuous.

In view of what we have just seen, we will assume that $i = 1$ and $j = m$, and we will prove that $\mathfrak{X}$ is $(\mathcal{M}_2, \ldots, \mathcal{M}_m)$-equihypocontinuous. For this purpose, let $B_\ell \in \mathcal{M}_\ell$ be arbitrary for $\ell \in \{2, \ldots, m\}$. We claim that the set

$$\mathfrak{X}_1 = \{x_1 \in E_1 \mapsto A(x_1, x_2, \ldots, x_m) \in F; x_\ell \in B_\ell \text{ for } \ell \geq 2, A \in \mathfrak{X}\}$$

of continuous linear mappings from $E_1$ into $F$ is equicontinuous. By Theorem 3.1 of [7], it is enough to show that $\mathfrak{X}_1$ is pointwise bounded ($E_1$ is barrelled). In order to do so, let us consider for each $\ell \in \{2, \ldots, m\}$, $x_1 \in E_1, \ldots, x_{\ell-1} \in E_{\ell-1}$, the set

$$\mathfrak{X}_{x_1, \ldots, x_{\ell-1}} = \{x_\ell \in E_\ell \mapsto A(x_1, \ldots, x_{\ell-1}, x_\ell, x_{\ell+1}, \ldots, x_m) \in F; \\ x_{\ell+1} \in B_{\ell+1}, \ldots, x_m \in B_m, A \in \mathfrak{X}\}$$

of continuous linear mappings from $E_\ell$ into $F$. For all $x_1 \in E_1, \ldots, x_{m-1} \in E_{m-1}$, $\mathfrak{X}_{x_1, \ldots, x_{m-1}}$ is equicontinuous because $\mathfrak{X}$ is separately equicontinuous; thus $\mathfrak{X}_{x_1, \ldots, x_{m-1}}(B_m)$ is a bounded subset of $F$ by Theorem 25.5 of [11]. Therefore, for all $x_1 \in E_1, \ldots, x_{m-2} \in E_{m-2}$, $\mathfrak{X}_{x_1, \ldots, x_{m-2}}$ is pointwise bounded; thus $\mathfrak{X}_{x_1, \ldots, x_{m-2}}$ is equicontinuous because $E_{m-1}$ is barrelled. By continuing in this way we finally conclude that $\mathfrak{X}_{x_1}(B_2)$ is bounded for all $x_1 \in E_1$. Therefore $\mathfrak{X}_1$ is pointwise bounded, which completes the proof of the theorem.

**Corollary 15.** If $E_i$ is a barrelled topological $R$-module for all $i \in \{1, \ldots, m\}$ and if $\mathfrak{X} \subset \mathcal{L}_a(E_1, \ldots, E_m; F)$ is separately equicontinuous, then $\mathfrak{X}$ is $(\mathcal{M}_1, \ldots, \mathcal{M}_m)$-equihypocontinuous.

**Proof.** Follows immediately from Theorem 14.

**Corollary 16.** If $E_2$ is a barrelled topological $R$-module and if $\mathfrak{X} \subset \mathcal{L}_a(E_1, E_2; F)$ is separately equicontinuous, then $\mathfrak{X}$ is $\mathcal{M}_1$-equihypocontinuous.

**Proof.** Follows immediately from Theorem 14.

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