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Ultraholomorphic extension maps for spaces of ultradifferentiable jets

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Dedicated to the memory of Pascal Laubin

Abstract

The key results (4.3, 6.2, 7.2 and 7.6) provide ultraholomorphic approximation continuous linear maps for spaces of ultradifferentiable functions on an open subset of $\mathbb{R}^n$.

They lead to results about the existence of continuous linear extension maps from the spaces of the ultradifferentiable Whitney jets of Beurling or Roumieu type on a closed subset F of $\mathbb{R}^n$. Their values belong to spaces of functions defined on $\mathbb{R}^n \cup D$: they are ultradifferentiable on $\mathbb{R}^n$ and ultraholomorphic on D, an open subset of $\mathbb{C}^n$ such that $D \cap \mathbb{R}^n = \mathbb{R}^n \setminus F$. We consider the cases when the ultradifferentiable jets and functions are defined by means of a weight or of a sequence of positive numbers.

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1 Introduction

This paper is announced in the final remark of [8]: it contains the generalization of the results therein to the ultradifferentiable setting. We consider

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the Beurling and Roumieu types, defined by means of a weight ω or of a suitable sequence M of positive numbers. We first treat in detail the case of a sequence M for both the Beurling and Roumieu types. We next indicate in paragraph 7 how to treat the case of a weight.

The best way to describe what is going on consists in saying what happens in the C^∞-setting and then to mention how this generalizes in the "ultra" setting.

For a closed subset F of $\mathbb{R}^n$, we designate as usual by $\mathcal{E}(F)$ the Fréchet space of the Whitney jets on F (cf. [11]). Moreover if Ω is an open subset of $\mathbb{R}^n$, $BC^\infty(\Omega)$ is the Fréchet space of the C^∞-functions on Ω which are bounded on Ω as well as all their derivatives, endowed with the system of norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ defined by

$$\|f\|_m = \sup_{|\alpha| \leq m} 2^{(m+1)|\alpha|} \|D^\alpha f\|_\Omega .$$

Step 1.

In [7], the key result (cf. Theorem 4.1) states that *if Ω is a proper open subset of $\mathbb{R}^n$, then there is a continuous linear map T from $BC^\infty(\Omega)$ into $BC^\infty(\Omega)$ such that, for every $f \in BC^\infty(\Omega)$,*
a) *Tf has a holomorphic extension on*

$$\Omega^* = \{u + iv\colon u \in \Omega, v \in \mathbb{R}^n, |v| < d(u, \partial\Omega)\};$$

b) *for every $s \in \mathbb{N}$ and $\varepsilon > 0$, there is a compact subset K of Ω such that*

$$\sup_{|\alpha| \leq s} \|D^\alpha f - D^\alpha(Tf)\|_{\Omega \setminus K} \leq \varepsilon.$$

This fact is the basic material to prove the following result (cf. Theorem 1.1) about the existence of an analytic extension map: *if K is a non void compact subset of $\mathbb{R}^n$, then*
a) *every Whitney jet on K has a $BC^\infty(\mathbb{R}^n)$-extension which is real-analytic on $\mathbb{R}^n \setminus K$;*
b) *there is a continuous linear extension map from $\mathcal{E}(K)$ into $C^\infty(\mathbb{R}^n)$ if and only if there is such a map with values real-analytic outside K.*

Step 2.

In [5], L. Frerick and D. Vogt have got the generalization of this last result to the setting of the closed subsets: *if there is a continuous linear extension map from $\mathcal{E}(F)$ into $C^\infty(\mathbb{R}^n)$, then there also is such a map with* *values having a holomorphic extension on Ω^* if and only if, for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $\mathbb{R}^n \setminus F$ having non empty intersection with B is compact.* Their proof makes a deep use of the key result mentioned here above.

In [4], L. Frerick makes a deep analysis of this situation.

Step 3.

In order to refine these results, one has first to improve the key result of [7]. This has been done in [8] as follows.

If U is a proper open subset of $\mathbb{C}^n$, $\mathcal{H}_\infty(U)$ is the Fréchet space of the holomorphic functions on U which are bounded on U as well as all their derivatives, endowed with the system of norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ defined by

$$\|f\|_m = \sup_{|\alpha| \leq m} \|D^\alpha f\|_U.$$

The enhanced key result states that: *for every proper open subset Ω of $\mathbb{R}^n$, one can construct an open subset D of $\mathbb{C}^n$ such that $D \cap \mathbb{R}^n = \Omega$ and obtain a continuous linear map T from $\mathrm{BC}^\infty(\Omega)$ into $\mathcal{H}_\infty(D)$ such that for every $f \in \mathrm{BC}^\infty(\Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(Tf)(u + iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u + iv \in D$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

Step 4.

The generalization of Step 2 we have in mind consists in giving a better description of the analytic extension map with respect to the fact that its values have a holomorphic extension on some open subset of $\mathbb{C}^n$. Let us state such a result.

If Ω is a proper open subset of $\mathbb{R}^n$, let $\mathcal{H}_\infty \mathrm{C}^\infty(\Omega)$ designate the following Fréchet space. Its elements are the functions f defined on $\mathbb{R}^n \cup D$ such that
(1) $f|_{\mathbb{R}^n} \in \mathrm{C}^\infty(\mathbb{R}^n)$;
(2) $f|_D \in \mathcal{H}_\infty(D)$;
(3) $\lim_{z \to x} D^\alpha(f|_D)(z) = D^\alpha(f|_{\mathbb{R}^n})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^n}\Omega$.
It is endowed with the countable system of semi-norms $\{|||\cdot|||_m : m \in \mathbb{N}\}$ defined by

$$|||f|||_m = \sup_{|\alpha| \leq m} \|D^\alpha(f|_{\mathbb{R}^n})\|_{b_m \cup D}$$

where $b_m := \{x \in \mathbb{R}^n : |x| \leq m\}$.

This leads to the following result (cf. Theorem 5.1 of [8]): *let F be a proper closed subset of $\mathbb{R}^n$. If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, then the existence of a continuous linear extension map from $\mathcal{E}(F)$ into $\mathrm{C}^\infty(\mathbb{R}^n)$ implies the existence of such a map from $\mathcal{E}(F)$ into $\mathcal{H}_\infty \mathrm{C}^\infty(\mathbb{R}^n \setminus F)$.*

In [8], one also finds a result solving the case of a closed subset.

In the "ultra" situation, the setting is as follows:
(1) the generalization of Step 1 is available in [9] and [10];
(2) the argument of L. Frerick and D. Vogt applies, as we shall see;
(3) the equivalent to the enhanced key result of Step 3 and the ultraholomorphic extension maps generalizing Step 4 are the matter of this article.

2 Basic notations

For a C^∞-function f on an open subset Ω of $\mathbb{R}^n$, we set

$$\mathrm{D}^\alpha f(x) = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}}(x), \quad \forall \alpha \in \mathbb{N}_0^n, x \in \Omega,$$

and, for a holomorphic function f on an open subset U of $\mathbb{C}^n$,

$$\mathrm{D}^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial_{z_1}^{\alpha_1} \dots \partial_{z_n}^{\alpha_n}}(z), \quad \forall \alpha \in \mathbb{N}_0^n, z \in U.$$

Throughout the paragraphs 2 to 6, $\boldsymbol{M} = (M_r)_{r \in \mathbb{N}_0}$ is a sequence of positive numbers which is
(M1) *normalized*, i.e. $M_0 = 1$ and $M_r \geq 1$ for every $r \in \mathbb{N}$;
(M2) *logarithmically convex*, i.e. $M_r^2 \leq M_{r-1} M_{r+1}$ for every $r \in \mathbb{N}$;
(M3) *non quasi-analytic*, i.e. such that $\sum_{r=1}^\infty M_{r-1}/M_r < \infty$.

Moreover Ω is a proper open subset of $\mathbb{R}^n$ and Ω^* designates the open subset $\{u + iv : u \in \Omega, v \in \mathbb{R}^n, |v| < \mathrm{d}(u, \partial\Omega)\}$ of $\mathbb{C}^n$.

Then as in [6] for instance, one can introduce the now classical following spaces:
(a) for every non void compact subset K of $\mathbb{R}^n$, the spaces $\mathcal{E}_{(\boldsymbol{M})}(K) = \mathcal{E}^{(M_r)}(K)$ [resp. $\mathcal{E}_{\{\boldsymbol{M}\}}(K) = \mathcal{E}^{\{M_r\}}(K)$] of the ultradifferentiable Whitney jets of class $\boldsymbol{M}$ and of Beurling [resp. Roumieu] type on K;
(b) the spaces $\mathcal{E}_{(\boldsymbol{M})}(\mathbb{R}^n) = \mathcal{E}^{(M_r)}(\mathbb{R}^n)$ [resp. $\mathcal{E}_{\{\boldsymbol{M}\}}(\mathbb{R}^n) = \mathcal{E}^{\{M_r\}}(\mathbb{R}^n)$] of the ultradifferentiable functions of class $\boldsymbol{M}$ and of Beurling [resp. Roumieu] type on $\mathbb{R}^n$.

If F is a proper closed subset of $\mathbb{R}^n$, one can also introduce the spaces $\mathcal{E}_{(M)}(F)$ [resp. $\mathcal{E}_{\{M\}}(F)$] as the projective limit of the spaces $\mathcal{E}_{(M)}(F \cap b_m)$ [resp. $\mathcal{E}_{\{M\}}(F \cap b_m)$] where $b_m = \{x \in \mathbb{R}^n \colon |x| \leq m\}$ for every $m \in \mathbb{N}$.

We will also need the not so classical Fréchet space $\mathrm{C}(\boldsymbol{M}, \Omega) = \mathrm{C}(M_r, \Omega)$ introduced in Paragraph 2 of [9]. The elements of this space are the C^∞-functions f on Ω such that, for every $h > 0$, there is $k > 0$ verifying

$$|\mathrm{D}^\alpha f(x)| \leq k h^{|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n, x \in \Omega,$$

and $\mathrm{C}(\boldsymbol{M}, \Omega)$ is endowed with the fundamental system $\{\|\cdot\|_m : m \in \mathbb{N}\}$ of the norms defined by

$$\|f\|_m = \sup_{\alpha \in \mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|} \, \|\mathrm{D}^\alpha f\|_\Omega}{M_{|\alpha|}}.$$

3 Construction of the open subset D_Ω of $\mathbb{C}^n$

Given a proper open subset Ω of $\mathbb{R}^n$, the construction of the open subset D_Ω of $\mathbb{C}^n$ comes from a refinement of the construction of the sequence $(\lambda_r)_{r \in \mathbb{N}}$ made in the Paragraph 1 of [9].

For the sake of clarity and completeness, we give the construction explicitely. The reader may skip it at first reading and come back to it as needed. The point is that we want to be able to use the inequalities of [9] involving the numbers λ_r as well as to obtain supplementary results about the space $\mathrm{C}(\boldsymbol{M}, \Omega)$.

We first fix a compact cover $\{K_r \colon r \in \mathbb{N}\}$ of Ω subject to the following requirements: $(K_1)^\circ \neq \emptyset$, $\mathrm{d}(K_1, \partial\Omega) < 1$ and for every $r \in \mathbb{N}$, $(K_r)^{\circ,-} = K_r \subset (K_{r+1})^\circ$ as well as

$$\eta_r := \mathrm{d}(K_r, \mathbb{R}^n \setminus K_{r+1}) > \tfrac{1}{2}\mathrm{d}(K_r, \partial\Omega).$$

Of course the sequence $(\eta_r)_{r \in \mathbb{N}}$ strictly decreases to 0 and $\eta_1 < 1$.

Now by use of [6] (cf. p. 56), it is a direct matter to get a sequence $(a_r)_{r \in \mathbb{N}}$ (called $(u_r)_{r \in \mathbb{N}}$ in [9]) of $\mathrm{C}(\boldsymbol{M}, \mathbb{R}^n)$ such that, for every $r \in \mathbb{N}$,
(1) $a_r \equiv 1$ on a neighbourhood of $K_{r+2} \setminus (K_{r+1})^\circ$,
(2) $\mathrm{supp}(a_r) \subset (K_{r+3})^\circ \setminus K_r$.

Then for every r, $m \in \mathbb{N}$, we choose $d_{r,m} > 1$ such that

$$(r+1)d_{r,m}^2 < d_{r+1,m},$$
$$d_{r,m} < d_{r,m+1},$$
$$\|\mathrm{D}^\alpha a_r\|_{\mathbb{R}^n} \leq d_{r,m} 2^{-(m+r+1)|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

We set $p_r = d_{r,r}$ for every $r \in \mathbb{N}$ and fix a strictly increasing sequence $(\zeta_r)_{r \in \mathbb{N}}$ of positive numbers such that $p_{r+1} < 2^{\zeta_r}$ for every $r \in \mathbb{N}$, as well as a strictly decreasing sequence $(\varepsilon_r)_{r \in \mathbb{N}}$ of positive numbers such that $\varepsilon_r < 2^{-r\zeta_{r+1}}$ for every $r \in \mathbb{N}$. Finally we remark that

$$\Phi(\rho) := \pi^{-n/2} \int_{|y| \leq \rho} \mathrm{e}^{-|y|^2}\, dy \uparrow 1$$

as $\rho > 0$ increases to $+\infty$.

So if we introduce the numbers

$$\delta_r := \varepsilon_r (3np_r^2 p_{r+1} r^{\zeta_{r+1}+1} 2^{r+2+\zeta_{r+1}} M_{\zeta_{r+1}+1}^2)^{-1}, \quad \forall r \in \mathbb{N},$$

we can fix a strictly increasing sequence $(\lambda_r)_{r \in \mathbb{N}}$ of positive numbers by the following procedure.

We choose $\lambda_1 > 1$ verifying the conditions hereunder if they apply to λ_1 only and then the numbers $\lambda_2, \lambda_3, \ldots$ successively, submitted to the following requirements:

(1) $1 - \Phi(\lambda_r \delta_r) < \delta_r$;
(2) $\pi^{-n/2} \lambda_r^n \mathrm{e}^{-\lambda_r^2 r^{-2}} p_r^2 \mu(K_{r+3}) < 2^{-r}$, where μ is the Lebesgue measure;
(3) $\lambda_r^{-1} 2^{n+2} \pi^{-n/2} p_r^2 (1 + \mu(K_{r+3})) \leq 2^{-r}$;
(4) $\lambda_{r+1}^{-1} < \mathrm{d}(K_r, \mathbb{R}^n \setminus \Omega)$;
(5) $\mathrm{e}^{-\frac{1}{2}\lambda_r} \leq \lambda_r^{-(n+1)}$;
(6) $\lambda_r(\eta_p^2 - \lambda_{p+1}^{-2}) \geq \frac{1}{2}$ for every $p \in \{1, \ldots, r-1\}$;
(7) $\lambda_{r+1}^{-n} \leq \lambda_r^{-(n+1)}$;
(8) $\mathrm{e}^{\lambda_r^2 \lambda_{r+1}^{-2}} - 1 \leq \lambda_r^{-(n+1)}$;
(9) for every $p \in \mathbb{N}$, we set $R_p = \sup\{|u| : u \in K_p\}$ and, if $\lambda_1, \ldots, \lambda_p$ are fixed, we first choose $\Theta_p > 0$ such that $\left|\mathrm{e}^{i\theta} - 1\right| \leq \lambda_p^{-(n+1)}$ for every $\theta \in [-\Theta_p, \Theta_p]$ and next impose $4\lambda_p^2 \lambda_r^{-1} R_{r+2} \leq \Theta_p$ for every $r > p$.

Let us remark that the requirements (1) and (2) are exactly the conditions imposed in [9] for the definition of the sequence $(\lambda_r)_{r \in \mathbb{N}}$. So all the inequalities established in [9] are available.

Definition. Now we have at our disposal all we need to introduce the open subset D_Ω of $\mathbb{C}^n$ as the interior of

$$\bigcup_{r=0}^{\infty}\{u+iv\colon u\in K_{r+1}\setminus K_r, v\in\mathbb{R}^n, |v|<\lambda_{r+2}^{-1}\}$$

where $K_0 := \emptyset$.

The requirement (4) has been introduced in order to have $D_\Omega \subset \Omega^*$.

4 Key result about $\mathrm{C}(\boldsymbol{M},\Omega)$

Let Ω be a proper open subset of $\mathbb{R}^n$.

As in [9], given $f\in\mathrm{C}(\boldsymbol{M},\Omega)$, we define the sequence $(G_r(\cdot,f))_{r\in\mathbb{N}_0}$ of functions on $\mathbb{C}^n$ by the following recursion: we set $G_0(w,f)=0$ and

$$G_r(w,f)=\pi^{-n/2}\lambda_r^n\int_{\mathbb{R}^n}a_r(y)(f(y)-\sum_{k=0}^{r-1}G_k(y,f))\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(w_j-y_j)^2}\,dy$$

for every $r\in\mathbb{N}$ and $w\in\mathbb{C}^n$.

As the functions $a_r\in\mathrm{C}^\infty(\mathbb{R}^n)$ have compact support contained in Ω, this makes sense and the functions $G_r(\cdot,f)$ are holomorphic on $\mathbb{C}^n$. Moreover we have

$$\begin{aligned}&\mathrm{D}^\alpha G_r(w,f)\\&=\pi^{-n/2}\lambda_r^n\int_{\mathbb{R}^n}\mathrm{D}^\alpha\left(a_r(y)(f(y)-\sum_{k=0}^{r-1}G_k(y,f))\right)\cdot\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(w_j-y_j)^2}\,dy\end{aligned}$$

for every $\alpha\in\mathbb{N}_0^n$ and $r\in\mathbb{N}$.

We first estimate $|\mathrm{D}^\alpha G_r(u+iv,f)-\mathrm{D}^\alpha G_r(u,f)|$ for every $u+iv\in D_\Omega$, $\alpha\in\mathbb{N}_0^n$ and $r\in\mathbb{N}$. By use of the inequality (3) of the Proposition 1 of [9], we certainly have

$$|\mathrm{D}^\alpha G_r(u+iv,f)-\mathrm{D}^\alpha G_r(u,f)|\le\pi^{-n/2}\lambda_r^n\cdot d_{r,m}^2 2^{-m|\alpha|}M_{|\alpha|}\,\|f\|_m\cdot I_r$$

for every $m\in\mathbb{N}$, with

$$I_r:=\sup_{u+iv\in D_\Omega}\int_{K_{r+3}\setminus K_r}\left|\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(u_j+iv_j-y_j)^2}-\mathrm{e}^{-\lambda_r^2\sum_{j=1}^n(u_j-y_j)^2}\right|\,dy.$$

These numbers I_r have been estimated in [8] as follows.

Lemma 4.1 *We have* $I_1 \leq (1+\mathrm{e})\mu(K_4)$ *and*

$$I_r \leq 2^{n+2}(1+\mu(K_{r+3}))\lambda_r^{-(n+1)}, \quad \forall r \in \{2,3,\ldots\}. \blacksquare$$

Proposition 4.2 a) *For every* $m \in \mathbb{N}$, *there is* $C_m > 0$ *such that*

$$|D^\alpha G_r(u+iv,f) - D^\alpha G_r(u,f)| \leq C_m 2^{-m|\alpha|} M_{|\alpha|} \|f\|_m$$

for every $f \in \mathrm{C}(\boldsymbol{M},\Omega)$, $u+iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ *and* $r \in \{1,\ldots,m\}$.

b) *For every* $f \in \mathrm{C}(\boldsymbol{M},\Omega)$, m, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ *and* $u+iv \in D_\Omega$, *one has*

$$|D^\alpha G_{m+r}(u+iv,f) - D^\alpha G_{m+r}(u,f)| \leq 2^{-(m+r)} 2^{-m|\alpha|} M_{|\alpha|} \|f\|_m.$$

Proof. a) is a direct consequence of the Lemma 4.1.

b) It suffices to apply the Lemma 4.1 and to use of the requirement (3) of the definition of the numbers λ_r. $\blacksquare$

Definition. Given a proper open subset U of $\mathbb{C}^n$, let us designate by $\mathcal{H}_\infty(\boldsymbol{M},U)$ the vector space of the holomorphic functions g on U such that, for every $h > 0$, there is $k > 0$ verifying

$$\|\mathrm{D}^\alpha g\|_U \leq k h^{|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n.$$

So for every $m \in \mathbb{N}$,

$$|\cdot|_m : \mathcal{H}_\infty(\boldsymbol{M},U) \to \mathbb{R}; \quad g \mapsto \sup_{\alpha \in \mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|} \|\mathrm{D}^\alpha g\|_U}{M_{|\alpha|}}$$

is a norm on $\mathcal{H}_\infty(\boldsymbol{M},U)$ such that $|\cdot|_m \leq |\cdot|_{m+1}$ and we endow $\mathcal{H}_\infty(\boldsymbol{M},U)$ with the Fréchet space structure coming from $\{|\cdot|_m : m \in \mathbb{N}\}$.

Now everything is in order to obtain the key result about the space $\mathrm{C}(\boldsymbol{M},\Omega)$ in view of the extension theorems.

Theorem 4.3 *For every proper open subset* Ω *of* $\mathbb{R}^n$, *there is a continuous linear map* T_Ω *from* $\mathrm{C}(\boldsymbol{M},\Omega)$ *into* $\mathcal{H}_\infty(\boldsymbol{M},D_\Omega)$ *such that for every* $f \in \mathrm{C}(\boldsymbol{M},\Omega)$, $\varepsilon > 0$ *and* $s \in \mathbb{N}$, *there is a compact subset* K *of* Ω *such that*

$$|D^\alpha(T_\Omega f)(u+iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u+iv \in D_\Omega$ *and* $\alpha \in \mathbb{N}_0^n$ *verifying* $u \in \Omega \setminus K$ *and* $|\alpha| \leq s$.

Proof. We just have to set $(T_\Omega f)(u+iv) = \sum_{m=0}^{\infty} G_r(u+iv, f)$ for every $f \in \mathrm{C}(\boldsymbol{M}, \Omega)$ and $u+iv \in D_\Omega$.

For every $f \in \mathrm{C}(\boldsymbol{M}, \Omega)$, the Proposition 8 of [9] says that $T_\Omega f$ is a holomorphic function on Ω^* hence on D_Ω. It is also clear that the construction of $T_\Omega f$ is linearly depending on f. To conclude that T_Ω is a continuous linear map from $\mathrm{C}(\boldsymbol{M}, \Omega)$ into $\mathcal{H}_\infty(\boldsymbol{M}, D_\Omega)$, we then have just to note that, for every $f \in \mathrm{C}(\boldsymbol{M}, \Omega)$, $m \in \mathbb{N}$, $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$, we successively have

$$\begin{aligned}
|\mathrm{D}^\alpha T_\Omega f(u+iv)| &\leq |\mathrm{D}^\alpha T_\Omega f(u)| + \sum_{r=1}^{m} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \\
&\qquad + \sum_{r=m+1}^{\infty} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \\
&\leq (c_m + mC_m + 2^{-m}) 2^{-(m-1)|\alpha|} M_{|\alpha|} \, \|f\|_m
\end{aligned}$$

by use of ([9], Proposition 7) and of the Proposition 4.2 to get the second inequality.

Let us now prove the second part of the statement: let $f \in \mathrm{C}(\boldsymbol{M}, \Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$ be fixed.

The Theorem 1 of [9] provides $d_0 \in \mathbb{N}$ such that

$$|\mathrm{D}^\alpha T_\Omega f(u) - \mathrm{D}^\alpha f(u)| \leq \frac{\varepsilon}{3}$$

for every $u \in \Omega \setminus K_{d_0}$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$.

The part b) of the Proposition 4.2 leads to

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq 2^{-r} 2^{-|\alpha|} M_{|\alpha|} \, \|f\|_1$$

for every $u+iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ and integer $r \geq 2$. So we can fix an integer $m \geq 2$ such that

$$\sum_{r=m+1}^{\infty} |\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq \frac{\varepsilon}{3}$$

for every $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$.

Now we turn our attention to the evaluation of

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)|$$

for every $r \in \{1,\dots,m\}$, $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$ and $u \in \Omega \setminus K_d$ with $d \in \mathbb{N}$ such that $d \geq d_0$. We already know that it is

$$\leq \pi^{-n/2}\lambda_m^n \cdot p_m^2 2^{-|\alpha|} M_{|\alpha|} \, \|f\|_1 \cdot I_{r,u+iv}$$

with

$$I_{r,u+iv} \leq \int_{K_{r+3}\setminus K_r} \left| \mathrm{e}^{\lambda_r^2|v|^2} \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j-y_j)} - 1 \right| dy.$$

For $u+iv \in D_\Omega$ verifying $u \in K_{d+1} \setminus K_d$ with $d \geq \sup\{m+2, d_0\}$, we have

$$\begin{aligned}\left| \mathrm{e}^{\lambda_r^2|v|^2} \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j-y_j)} - 1 \right| \\ \leq \mathrm{e}^{\lambda_r^2\lambda_{d+2}^{-2}} \left| \mathrm{e}^{-2i\lambda_r^2 \sum_{j=1}^n v_j(u_j-y_j)} - 1 \right| + (\mathrm{e}^{\lambda_r^2\lambda_{d+2}^{-2}} - 1)\end{aligned}$$

with $\exp(\lambda_r^2\lambda_{d+2}^{-2}) \leq \mathrm{e}$ and $\exp(\lambda_r^2\lambda_{d+2}^{-2}) - 1 \to 0$ if $d \to \infty$. Moreover we have

$$\left| 2\lambda_r^2 \sum_{j=1}^n v_j(u_j - y_j) \right| \leq 2\lambda_r^2\lambda_{d+2}^{-1}(|u| + |y|) \leq 4\lambda_m^2\lambda_{d+2}^{-1}R_{d+1}.$$

So we can choose $d_1 \geq \sup\{m+2, d_0\}$ such that

$$|\mathrm{D}^\alpha G_r(u+iv, f) - \mathrm{D}^\alpha G_r(u, f)| \leq \frac{\varepsilon}{3m}$$

for every $r \in \{1,\dots,m\}$, $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| \leq s$ and $u+iv \in D$ such that $u \in \Omega \setminus K_{d_1}$.

Hence the conclusion.∎

5 Beurling type extensions

Case 1 : *F is compact or $\mathbb{R}^n \setminus F$ is bounded.*

Definitions. Given a proper open subset U of $\mathbb{C}^n$, we designate by $\mathcal{F}(U)$ the vector space of the functions f defined on $\mathbb{R}^n \cup U$ such that
(1) $f|_{\mathbb{R}^n} \in \mathrm{C}^\infty(\mathbb{R}^n)$,
(2) $f|_U \in \mathcal{H}(U)$,
(3) $\lim_{z\to x} \mathrm{D}^\alpha(f|_U)(z) = \mathrm{D}^\alpha(f|_{\mathbb{R}^n})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^n}(\mathbb{R}^n \cap U)$.

The space $\mathcal{F}(\boldsymbol{M}, U)$ is the vector space of the elements f of $\mathcal{F}(U)$ such that, for every $m \in \mathbb{N}$ and $h > 0$, there is $k > 0$ such that

$$\|\mathrm{D}^\alpha f\|_{b_m \cup U} \leq k h^{|\alpha|} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^n,$$

endowed with the fundamental system of semi-norms $\{\|\cdot\|_m : m \in \mathbb{N}\}$ defined by

$$\|f\|_m := \sup_{\alpha \in \mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|} \|\mathrm{D}^\alpha f\|_{b_m \setminus U}}{M_{|\alpha|}} + \sup_{\alpha \in \mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|} \|\mathrm{D}^\alpha f\|_U}{M_{|\alpha|}};$$

it is a Fréchet space (we recall that $b_m := \{x \in \mathbb{R}^n \colon |x| \leq m\}$).

Theorem 5.1 *Let F be a proper closed subset of $\mathbb{R}^n$.*

If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, then the existence of a continuous linear extension map E from $\mathcal{E}_{(M)}(F)$ into $\mathcal{E}_{(M)}(\mathbb{R}^n)$ implies the existence of such a map E_F from $\mathcal{E}_{(M)}(F)$ into $\mathcal{F}(\boldsymbol{M}, D_{\mathbb{R}^n \setminus F})$.

Proof. If F is compact, we choose a function $\psi \in \mathcal{E}_{(M)}(\mathbb{R}^n)$ identically 1 on a neighbourhood of F with compact support and check that

$$E_1 \colon \mathcal{E}_{(M)}(F) \to \mathcal{E}_{(M)}(\mathbb{R}^n); \quad \varphi \mapsto \psi \cdot E\varphi$$

has a meaning and defines a continuous linear extension map. So in both cases (i.e. F is compact or $\mathbb{R}^n \setminus F$ is bounded), we may very well suppose that $(E\cdot)|_{\mathbb{R}^n \setminus F}$ is a continuous linear map from $\mathcal{E}_{(M)}(F)$ into $\mathrm{C}(\boldsymbol{M}, \mathbb{R}^n \setminus F)$.

Now to every jet $\varphi \in \mathcal{E}_{(M)}(F)$, we associate the function $E_F\varphi$ defined on $\mathbb{R}^n \cup D_{\mathbb{R}^n \setminus F}$ as follows

$$\begin{cases} (E_F\varphi)(x) &= (E\varphi)(x), & \forall x \in F, \\ (E_F\varphi)(z) &= T_{\mathbb{R}^n \setminus F}((E\varphi)|_{\mathbb{R}^n \setminus F})(z), & \forall z \in D_{\mathbb{R}^n \setminus F}. \end{cases}$$

By use of the key theorem 4.3, it is a direct matter to check that E_F is a linear extension map from $\mathcal{E}_{(M)}(F)$ into $\mathcal{F}(\boldsymbol{M}, D_{\mathbb{R}^n \setminus F})$. Its continuity is straightforward by use of the continuity of the different maps its definition involves. ∎

Remark. If F is a non void compact subset of $\mathbb{R}^n$, we have $b_m \setminus D_{\mathbb{R}^n \setminus F} = F$ for m large enough in the definition of the norm $\|\cdot\|_m$ of $\mathcal{F}(\boldsymbol{M}, D_{\mathbb{R}^n \setminus F})$.

Case 2 : *F is closed.*

Definitions. Let us abbreviate "connected component" by "c.c.". Given a proper closed subset F of $\mathbb{R}^n$, let us set

$$\Omega_1 := \cup\{\omega\colon \omega = \text{c.c. of } \mathbb{R}^n \setminus F, \omega \cap b_1 \neq \emptyset\},$$

introduce by recursion the sets

$$\Omega_j := \cup\{\omega\colon \omega = \text{c.c. of } \mathbb{R}^n \setminus F, \omega \cap b_j \neq \emptyset, \omega \cap (\cup_{k=1}^{j-1}\Omega_k) = \emptyset\}$$

for $j = 2, 3, \ldots$ and write $J := \{j \in \mathbb{N}\colon \Omega_j \neq \emptyset\}$. For every $j \in J$, the construction of Paragraph 3 applied to Ω_j provides an open subset D_{Ω_j}.

We then set $D_F := \cup_{j\in J} D_{\Omega_j}$ and introduce the following Fréchet space $\mathcal{G}(\boldsymbol{M}, D_F)$. Its elements are the functions $f \in \mathcal{F}(D_F)$ which restriction to D_{Ω_j} belongs to $\mathcal{H}_\infty(\boldsymbol{M}, D_{\Omega_j})$ for every $j \in J$, endowed with the countable system of semi-norms $\{|||\cdot|||_m : m \in \mathbb{N}\}$ defined by

$$|||f|||_m = \sup_{\alpha\in\mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|}}{M_{|\alpha|}} \|\mathrm{D}^\alpha(f|_{\mathbb{R}^n})\|_{b_m\cap F} + \sup_{j\leq m}\sup_{\alpha\in\mathbb{N}_0^n} \frac{2^{(m+1)|\alpha|}}{M_{|\alpha|}} \|\mathrm{D}^\alpha f\|_{D_{\Omega_j}}.$$

Theorem 5.2 *Let F be a proper closed subset of $\mathbb{R}^n$.*
If there is a continuous linear extension map $E\colon \mathcal{E}_{(M)}(F) \to \mathcal{E}_{(M)}(\mathbb{R}^n)$, the following assertions are equivalent:
(1) there also is such map E_1 from $\mathcal{E}_{(M)}(F)$ into $\mathcal{G}(\boldsymbol{M}, D_F)$;
(2) for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $\mathbb{R}^n\setminus F$ having non empty intersection with B is compact.

Proof. (1) $\Rightarrow$ (2). The argument of Frerick and Vogt developed in [5] applies.

(2) $\Rightarrow$ (1). If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, the condition (2) is automatically satisfied and the previous theorem provides the result.

If F is not compact and if $\mathbb{R}^n \setminus F$ is not bounded, we proceed as follows.

If $n \geq 2$, as F is not compact, the condition (2) implies that all the connected components of $\mathbb{R}^n \setminus F$ are bounded and, as $\mathbb{R}^n \setminus F$ is not bounded, J is infinite. If $n = 1$, as F is not compact, the condition (2) implies that one and only one connected component ω of Ω may be unbounded: it is of the type $]-\infty, a[$ or $]b, +\infty[$. We then choose a function $\psi \in \mathrm{C}(\boldsymbol{M}, \mathbb{R})$ identically 1 on a neighbourhood of $[a, +\infty[$ or $]-\infty, b]$ and 0 on $]-\infty, a-1]$ or $[b+1, +\infty[$ respectively and check that

$$E_2\colon \mathcal{E}_{(M)}(F) \to \mathcal{E}_{(M)}(\mathbb{R}) \quad \varphi \mapsto \psi.E\varphi$$

is a continuous linear extension map such that $(E_2\cdot)|_{\Omega_j}$ is a continuous linear map from $\mathcal{E}_{(M)}(F)$ into $\mathrm{C}(\boldsymbol{M}, \Omega_j)$ for every $j \in J$.

So up to a substitution, we may very well suppose that, for every $j \in J$, $(E\cdot)|_{\Omega_j}$ is a continuous linear map from $\mathcal{E}_{(M)}(F)$ into $\mathrm{C}(\boldsymbol{M}, \Omega_j)$.

Now we apply the Theorem 4.3 for every $j \in J$ and get continuous linear extension maps T_{Ω_j} from $\mathrm{C}(\boldsymbol{M}, \Omega_j)$ into $\mathcal{H}_\infty(\boldsymbol{M}, D_{\Omega_j})$.

To every jet $\varphi \in \mathcal{E}_{(M)}(F)$, we then associate the function $E_1\varphi$ defined on $\mathbb{R}^n \cup D_F$ by

$$\begin{cases} (E_1\varphi)(x) &= (E\varphi)(x), & \forall x \in F, \\ (E_1\varphi)(z) &= T_{\Omega_j}((E\varphi)|_{\Omega_j})(z), & \forall z \in D_{\Omega_j}, j \in J. \end{cases}$$

It is straightforward to check that E_1 so defined is a continuous linear extension map from $\mathcal{E}_{(M)}(F)$ into $\mathcal{G}(\boldsymbol{M}, D_F)$. ∎

6 Roumieu type extensions

Definitions. Let Ω be a proper open subset of $\mathbb{R}^n$.

We recall the definition of the Hausdorff LB-spaces $\mathrm{C}\{\boldsymbol{M}, \Omega\} = \mathrm{C}\{M_r, \Omega\}$ introduced in the Paragraph 4 of [9].

For every $m \in \mathbb{N}$, $\mathrm{C}_m\{\boldsymbol{M}, \Omega\}$ is the Banach space consisting of the C^∞-functions f on Ω such that

$$|||f|||_m := \sup_{\alpha \in \mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_\Omega}{m^{|\alpha|} M_{|\alpha|}} < \infty$$

endowed with the norm $|||\cdot|||_m$; $\mathrm{C}\{\boldsymbol{M}, \Omega\}$ is the inductive limit of these spaces $\mathrm{C}_m\{\boldsymbol{M}, \Omega\}$.

It is a direct matter to introduce by the same procedure the following Hausdorff LB–space: $\mathcal{H}_\infty\{\boldsymbol{M}, U\}$.

Given a proper open subset U of $\mathbb{C}^n$ and positive integers m and k, $\mathcal{F}_k\{\boldsymbol{M}, b_m \cup U\}$ is the vector space of the functions f defined on $b_m \cup U$ such that

(a) $f|_{b_m} \in \mathrm{C}^\infty(b_m)$;

(b) $f|_U \in \mathcal{H}(U)$;

(b) $\lim_{z \to x} \mathrm{D}^\alpha(f|_U)(z) = \mathrm{D}^\alpha(f|_{\mathbb{R}^n})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and $x \in \partial_{\mathbb{R}^n}(\mathbb{R}^n \cap U)$;

such that

$$\|f\|_k := \sup_{\alpha \in \mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_{b_m \setminus U}}{k^{|\alpha|} M_{|\alpha|}} + \sup_{\alpha \in \mathbb{N}_0^n} \frac{\|\mathrm{D}^\alpha f\|_U}{k^{|\alpha|} M_{|\alpha|}} < \infty,$$

endowed of course with the norm $\|\cdot\|_k$.

We then define $\mathcal{F}\{\boldsymbol{M}, b_m \cup U\}$ as the inductive limit of the Banach spaces $\mathcal{F}_k\{\boldsymbol{M}, b_m \cup U\}$ and finally $\mathcal{F}\{\boldsymbol{M}, U\}$ as the projective limit of the LB-spaces $\mathcal{F}\{\boldsymbol{M}, b_m \cup U\}$.

Proposition 6.1 a) *For every $m \in \mathbb{N}$, there is $C_m > 0$ such that*

$$|D^\alpha G_r(u+iv, f) - D^\alpha G_r(u, f)| \leq C_m(m+1)^{|\alpha|} M_{|\alpha|} \,|||f|||_m$$

for every $f \in \mathrm{C}_m\{\boldsymbol{M}, \Omega\}$, $u+iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ and $r \in \{1, \ldots, m\}$.

b) *For every $m \in \mathbb{N}$, $f \in \mathrm{C}_m\{\boldsymbol{M}, \Omega\}$, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and $u+iv \in D_\Omega$, one has*

$$|D^\alpha G_{m+r}(u+iv, f) - D^\alpha G_{m+r}(u, f)| \leq 2^{-(m+r)}(m+1)^{|\alpha|} M_{|\alpha|} \,|||f|||_m \,.$$

Proof. It is a direct consequence of the Lemma 4.1, of the inequality (15) of the Proposition 9 of [9] and of the requirement (3) of the definition of the numbers λ_r.∎

Now comes the appropriate key result.

Theorem 6.2 *There is a continuous linear map T_Ω from $\mathrm{C}\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_\infty\{\boldsymbol{M}, D_\Omega\}$ such that for every $f \in \mathrm{C}\{\boldsymbol{M}, \Omega\}$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(T_\Omega f)(u+iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

In fact, for every $m \in \mathbb{N}$, T_Ω is a continuous linear map from $\mathrm{C}_m\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_{4m,\infty}\{\boldsymbol{M}, D_\Omega\}$.

Proof. The proof is quite similar to the one of the Theorem 4.3; we just indicate the appropriate modifications.

For every $f \in \mathrm{C}\{\boldsymbol{M}, \Omega\}$, the Proposition 16 of [9] asserts that $G(\cdot, f)$ is a holomorphic function on Ω^* hence on D_Ω. The continuity of the map T_Ω from $\mathrm{C}\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_\infty\{\boldsymbol{M}, \Omega\}$ (and more precisely from $\mathrm{C}_m\{\boldsymbol{M}, \Omega\}$ into $\mathcal{H}_{4m,\infty}\{\boldsymbol{M}, \Omega\}$ for every $m \in \mathbb{N}$) is provided by the Proposition 15 of [9] and the Proposition 6.1 since they lead to

$$|\mathrm{D}^\alpha(T_\Omega f)(u+iv)| \leq (b_m + mC_m + 2^{-m})(4m)^{|\alpha|} M_{|\alpha|} \,|||f|||_m$$

for every $m \in \mathbb{N}$, $f \in \mathrm{C}_m\{\boldsymbol{M}, \Omega\}$, $u + iv \in D$ and $\alpha \in \mathbb{N}_0^n$.

To establish the second part of the statement, one has just to substitute the Theorem 4 of [9] to the use of the Theorem 1 of [9], the part b) of the Proposition 6.1 to the use of part b) of the Proposition 4.2 as well as the inequality (15) of the Proposition 9 of [9] to the use of the inequality (3) of the Proposition 1 of [9].∎

Now we are ready to get the extension results.

Theorem 6.3 *Let F be a proper closed subset of $\mathbb{R}^n$.*

If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, then the existence of a continuous linear extension map E from $\mathcal{E}_{\{M\}}(F)$ into $\mathcal{E}_{\{M\}}(\mathbb{R}^n)$ implies the existence of such a map from $\mathcal{E}_{\{M\}}(F)$ into $\mathcal{F}\{\boldsymbol{M}, D_{\mathbb{R}^n\setminus F}\}$.

Proof. a) If $\mathbb{R}^n \setminus F$ is bounded, we proceed as follows.

We first choose $m_0 \in \mathbb{N}$ such that the closure of $\mathbb{R}^n \setminus F$ is contained in the interior of b_{m_0}. For every positive integers $m \geq m_0$ and k, we then define the map $T_{F,m,k}$ on $\mathcal{E}_{k,\{M\}}(b_m)$ by

$$\begin{cases} (T_{F,m,k}\varphi)(x) = \varphi(x), & \forall x \in F \cap b_m, \\ (T_{F,m,k}\varphi)(z) = T_{\mathbb{R}^n\setminus F}(\varphi|_{\mathbb{R}^n\setminus F})(z), & \forall z \in D_{\mathbb{R}^n\setminus F} \end{cases}$$

(as $b_m \setminus F = \mathbb{R}^n \setminus F$ for every integer $m \geq m_0$, this makes sense); it clearly is a continuous linear map from $\mathcal{E}_{k,\{M\}}(b_m)$ into $\mathcal{F}_{4k}\{\boldsymbol{M}, b_m \cup D_{\mathbb{R}^n\setminus F}\}$. Therefore, for every integer $m \geq m_0$, the map $T_{F,m}$ it canonically defines from $\mathcal{E}_{\{M\}}(b_m)$ into $\mathcal{F}\{\boldsymbol{M}, b_m \cup D_{\mathbb{R}^n\setminus F}\}$ also is continuous and linear, as well as finally their canonical restriction T_F from $\mathcal{E}_{\{M\}}(\mathbb{R}^n)$ into $\mathcal{F}\{\boldsymbol{M}, D_{\mathbb{R}^n\setminus F}\}$. Hence the conclusion by consideration of the map $T_F E$.

b) If F is compact, we first choose a function $\psi \in \mathcal{E}_{(M)}(\mathbb{R}^n)$ identically 1 on a neighbourhood of F with compact support H and a positive integer m_0 such that $H \subset (b_{m_0})^\circ$. For every positive integers $m \geq m_0$ and k, it is a direct matter to check that the map

$$M_{\psi,m,k}\colon \mathcal{E}_{k,\{M\}}(b_m) \to \mathrm{C}_{2k}\{\boldsymbol{M}, \mathbb{R}^n\}; \quad \varphi \mapsto \psi\varphi$$

is continuous and linear. So for every integer $m \geq m_0$, the map $M_{\psi,m}$ they canonically define from $\mathcal{E}_{\{M\}}(b_m)$ into $\mathrm{C}\{\boldsymbol{M}, \mathbb{R}^n\}$ also is continuous and linear, as well as finally their canonical restriction M_ψ from $\mathcal{E}_{\{M\}}(\mathbb{R}^n)$ into

$C\{\boldsymbol{M},\mathbb{R}^n\}$. In this way so far we have obtained a new continuous linear extension map: the map $M_\psi E$ from $\mathcal{E}_{\{M\}}(F)$ into $C\{\boldsymbol{M},\mathbb{R}^n\}$.

Now for every positive integers $m \geq m_0$ and k, the map

$$T_{F,m,k}\colon C_k\{\boldsymbol{M},\mathbb{R}^n\} \to \mathcal{F}_{4k}\{\boldsymbol{M}, b_m \cup D_{\mathbb{R}^n\setminus F}\}$$

defined by

$$\varphi \mapsto \begin{cases} \varphi & \text{on } F, \\ T_{\mathbb{R}^n\setminus F}(\varphi|_{\mathbb{R}^n\setminus F}) & \text{on } D_{\mathbb{R}^n\setminus F} \end{cases}$$

has a meaning and clearly is continuous and linear, as well as the canonical maps

$$T_{F,m}\colon C\{\boldsymbol{M},\mathbb{R}^n\} \to \mathcal{F}\{\boldsymbol{M}, b_m \cup D_{\mathbb{R}^n\setminus F}\}$$

and

$$T_F\colon C\{\boldsymbol{M},\mathbb{R}^n\} \to \mathcal{F}\{\boldsymbol{M}, D_{\mathbb{R}^n\setminus F}\}$$

they successively define.

Hence the conclusion by consideration of the map $T_F M_\psi E$.∎

In order to get a result for the general case, i.e. when F is a proper closed subset of $\mathbb{R}^n$, we need of course to introduce a space: $\mathcal{G}\{\boldsymbol{M}, D_F\}$. For this purpose, we use the same notations Ω_j, J and D_F as in the definition of the space $\mathcal{G}(\boldsymbol{M}, D_F)$. Next we denote by $D_{j(m)}$ the union of the unbounded connected components of $\mathbb{R}^n \setminus F$ with the open subsets D_{Ω_j} contained in b_m. Then for every positive integers m and k, we define the Banach space $\mathcal{G}_k\{\boldsymbol{M}, b_m \cup D_{j(m)}\}$ as the vector space of the functions f defined on $b_m \cup D_{j(m)}$ such that

(a) $f|_{b_m} \in C^\infty(b_m)$;

(b) $f|_{D_{j(m)}} \in \mathcal{H}_\infty\{\boldsymbol{M}, D_{j(m)}\}$;

(c) $\lim_{z\to x} D^\alpha(f|_{D_{j(m)}})(z) = D^\alpha(f|_{b_m})(x)$ for every $\alpha \in \mathbb{N}_0^n$ and element x of $\partial_{\mathbb{R}^n}(b_m \setminus D_{j(m)})$;

(d) $\displaystyle \|f\|_k := \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|D^\alpha f\|_{b_m\cup D_{j(m)}}}{k^{|\alpha|} M_{|\alpha|}} < \infty$

endowed of course with the norm $\|\cdot\|_k$.

Finally we introduce the LB-space $\mathcal{G}\{\boldsymbol{M}, b_m \cup D_{j(m)}\}$ as the inductive limit of these Banach spaces and $\mathcal{G}\{\boldsymbol{M}, D_F\}$ as the projective limit of these LB-spaces.

Now everything is set up to state the result.

Theorem 6.4 *Let F be a proper closed subset of $\mathbb{R}^n$.*

If there is a continuous linear extension map $E\colon \mathcal{E}_{\{M\}}(F) \to \mathcal{E}_{\{M\}}(\mathbb{R}^n)$, the following assertions are equivalent:
(1) there also is such a map E_F from $\mathcal{E}_{\{M\}}(F)$ into $\mathcal{G}\{\boldsymbol{M}, D_F\}$;
(2) for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $R^n \backslash F$ having non empty intersection with B is compact.

Proof. (1) $\Rightarrow$ (2). The consideration of the map $(E_F\cdot)|_{\mathbb{R}^n}$ tells us that the values of E are real-analytic on $\mathbb{R}^n \setminus F$ and the argument of Frerick and Vogt applies.

(2) $\Rightarrow$ (1). The proof of the Theorem 5.2 can be adapted to this situation. The map E_2 coincides with $M_\psi E$ where M_ψ is defined by

$$M_\psi\colon \mathcal{E}_{\{M\}}(\mathbb{R}) \to \mathcal{E}_{\{M\}}(\mathbb{R}); \quad f \mapsto \psi f$$

and is clearly a continuous linear map.

So again for every $j \in J$, we may very well suppose that $(E\cdot)|_{\Omega_j}$ is a continuous linear map from $\mathcal{E}_{\{M\}}(F)$ into $\mathrm{C}\{\boldsymbol{M}, \Omega_j\}$. Therefore we can introduce the map E_1 and it is a direct matter to check that it suits our purpose.∎

7 The weighted spaces case

In this paragraph we indicate how to treat the case when the ultradifferentiable jets and functions are defined by means of a weight. The idea is to apply the same method, using the results of [10] instead of those of [9].

Definitions. For the definition of a weight, we use the modification introduced by Braun, Meise and Taylor in [3] to Beurling's method of [1]. So a *weight* is a function $\boldsymbol{\omega}$ from $[0, +\infty[$ into itself which is continuous, increases and verifies the following conditions:
(ω1) there is $l \geq 1$ such that $\boldsymbol{\omega}(2t) \leq l(1 + \boldsymbol{\omega}(t))$ for every $t \geq 0$;
(ω2) $\int_1^\infty \frac{\boldsymbol{\omega}(t)}{1+t^2}\, dt < \infty$;
(ω3) $\lim_{t\to+\infty} \frac{\ln(1+t)}{\boldsymbol{\omega}(t)} = 0$;
(ω4) the function $\varphi(\cdot) = \boldsymbol{\omega}(e^{\cdot})$ is convex on $[0, +\infty[$.
By the Proposition 1.2(b) of [2], there is then a weight $\boldsymbol{\sigma} \leq \boldsymbol{\omega}$ such that $\boldsymbol{\sigma}(1) = 0$ and $\boldsymbol{\sigma}(t) = \boldsymbol{\omega}(t)$ for large t. As in what follows, the values of $\boldsymbol{\omega}(t)$

are used only for large t, we are going to suppose moreover that we have $\boldsymbol{\omega}(1) = 0$ hence $\varphi(0) = 0$.

The *Young's conjugate* φ^* of φ is the function defined on $[0, +\infty[$ by $\varphi^*(y) = \sup_{x \geq 0} (xy - \varphi(x))$. It is a convex and increasing function which verifies $\varphi^*(0) = 0$; moreover $\varphi^*(y)/y$ increases and $\lim_{y \to \infty} \varphi^*(y)/y = \infty$.

The condition $(\omega 1)$ of the definition of a weight provides the existence of a positive integer j_0 such that $\varphi(x+1) \leq j_0(\varphi(x)+1)$ for every $x \geq 0$. Finally by the Lemma 1.4 of [3], there is $y_0 > j_0$ such that $\varphi^*(y) - y \geq j_0 \varphi^*(y/j_0) - j_0$ for every $y \geq y_0$.

As we will use [10] to play the fundamental role that [9] had in the $\boldsymbol{M}$-case, we refer to [10] for the definition of the spaces $\mathcal{E}_{(\boldsymbol{\omega})}(K)$ and $\mathcal{E}_{\{\boldsymbol{\omega}\}}(K)$ for K a non empty compact subset of $\mathbb{R}^n$, $\mathcal{E}_{(\boldsymbol{\omega})}(\mathbb{R}^n)$ and $\mathcal{E}_{\{\boldsymbol{\omega}\}}(\mathbb{R}^n)$, as well as $\mathrm{C}(\boldsymbol{\omega}, \Omega)$ and $\mathrm{C}\{\boldsymbol{\omega}, \Omega\}$ for Ω a proper open subset of $\mathbb{R}^n$. Moreover if F is a proper closed subset of $\mathbb{R}^n$, we define the spaces $\mathcal{E}_{(\boldsymbol{\omega})}(F)$ and $\mathcal{E}_{\{\boldsymbol{\omega}\}}(F)$ as the projective limits of the spaces $\mathcal{E}_{(\boldsymbol{\omega})}(F \cap b_m)$ and $\mathcal{E}_{\{\boldsymbol{\omega}\}}(F \cap b_m)$ respectively.

Now we follow the pattern of the $\boldsymbol{M}$-case.

So the first thing to do is to obtain the equivalent of the construction of the open subset D_Ω of $\mathbb{C}^n$ while preserving the inequalities established in [10]. This is direct: one has just to proceed as in Paragraph 3, replacing the material coming from [9] by the corresponding one of [10].

In what follows the notation D_Ω will refer to this new open subset of $\mathbb{C}^n$.

The next goal is to get the key result about $\mathrm{C}(\boldsymbol{\omega}, \Omega)$. The use of the inequality 5 of [10], of the Lemma 4.1 and of the requirement (3) of the definition of the numbers λ_r lead directly to the following fact.

Proposition 7.1 a) *For every integer $m \geq j_0$, there is $C_m > 0$ such that*

$$|D^\alpha G_r(u + iv, f) - D^\alpha G_r(u, f)| \leq C_m \mathrm{e}^{p(m)\varphi^*(\frac{|\alpha|}{p(m)})} \|f\|_m$$

for every $f \in \mathrm{C}(\boldsymbol{\omega}, \Omega)$, $u + iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ and $r \in \{1, \ldots, m\}$, where $p(m)$ denotes the integral part of m/d.

b) *For every integer $m \geq j_0$, there is a constant $q_m > 0$ such that, for every $f \in \mathrm{C}(\boldsymbol{\omega}, \Omega)$, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and $u + iv \in D_\Omega$, one has*

$$|D^\alpha G_{m+r}(u + iv, f) - D^\alpha G_{m+r}(u, f)| \leq 2^{-(m+r)} q_m \mathrm{e}^{p(m)\varphi^*(\frac{|\alpha|}{p(m)})} \|f\|_m . \blacksquare$$

Definition. Given a proper open subset U of $\mathbb{C}^n$, we designate by $\mathcal{H}_\infty(\boldsymbol{\omega}, U)$ the vector space of the holomorphic functions g on U such that, for every $m \in \mathbb{N}$,

$$|g|_m := \sup_{\alpha\in\mathbb{N}_0^n} \|\mathrm{D}^\alpha g\|_U \,\mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})} < \infty,$$

endowed with the Fréchet space structure coming from $\{|\cdot|_m : m \in \mathbb{N}\}$.

Now everything is set up to state the key result.

Theorem 7.2 *For every proper open subset Ω of $\mathbb{R}^n$, there is a continuous linear map T_Ω from $\mathrm{C}(\boldsymbol{\omega}, \Omega)$ into $\mathcal{H}_\infty(\boldsymbol{\omega}, D_\Omega)$ such that for every $f \in \mathrm{C}(\boldsymbol{\omega}, \Omega)$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(T_\Omega f)(u+iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u+iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

Proof. By the Proposition 6 of [10], $T_\Omega f$ is a holomorphic function on Ω^* hence on D_Ω. Of course T_Ω is a linear map; its continuity is established as in the proof of the Theorem 4.3 by use of the Proposition 5 of [10] instead of the Proposition 7 of [9].

To obtain the second part, one has just to proceed as in the proof of the Theorem 4.3 substituting the Theorem 1 of [10] to the Theorem 1 of [9], the part b) of the Proposition 7.1 to that of the Proposition 4.2 as well as the inequality 5 of [10] to the inequality 3 of [9].∎

Definition. Given a proper open subset U of $\mathbb{C}^n$, $\mathcal{F}(\boldsymbol{\omega}, U)$ is the vector space of the elements f of $\mathcal{F}(U)$ such that for every $m \in \mathbb{N}$,

$$\|f\|_m := \sup_{\alpha\in\mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_{b_m \cup U} \,\mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})} < \infty,$$

endowed with the Fréchet space structure coming from the system of seminorms $\{\|\cdot\|_m : m \in \mathbb{N}\}$.

An easy adaptation of the proof of the Theorem 5.1 leads then to the following result.

Theorem 7.3 *Let F be a proper closed subset of $\mathbb{R}^n$.*

If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, then the existence of a continuous linear extension map from $\mathcal{E}_{(\omega)}(F)$ into $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ implies the existence of such a map from $\mathcal{E}_{(\omega)}(F)$ into $\mathcal{F}(\boldsymbol{\omega}, D_{\mathbb{R}^n\setminus F})$.∎

Definition. Let us use the notations Ω_j, J and D_F as in the definition of the space $\mathcal{G}(\boldsymbol{M}, D_F)$. We then define the Fréchet space $\mathcal{G}(\boldsymbol{\omega}, D_F)$ as the vector space of the elements $f \in \mathcal{F}(D_F)$ which restriction to D_{Ω_j} belongs to $\mathcal{H}_\infty(\boldsymbol{\omega}, D_{\Omega_j})$ for every $j \in J$, endowed with the countable system of semi-norms $\{|||\cdot|||_m : m \in \mathbb{N}\}$ defined by

$$|||f|||_m := \sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_{b_m \cap F} \, \mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})} + \sup_{j \le m} \sup_{\alpha \in \mathbb{N}_0^n} \|\mathrm{D}^\alpha f\|_{D_{\Omega_j}} \, \mathrm{e}^{-m\varphi^*(\frac{|\alpha|}{m})}.$$

This time a straightforward modification of the proof of the Theorem 5.2 leads to the folowing result.

Theorem 7.4 *Let F be a proper closed subset of $\mathbb{R}^n$.*
If there is a continuous linear extension map from $\mathcal{E}_{(\omega)}(F)$ into $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$, the following assertions are equivalent:
(1) *there also is such map from $\mathcal{E}_{(\omega)}(F)$ into $\mathcal{G}(\boldsymbol{\omega}, D_F)$;*
(2) *for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $\mathbb{R}^n \setminus F$ having non empty intersection with B is compact.*■

Now we turn our attention to the Roumieu type. Of course this time the Paragraph 4 of [10] becomes the basic ingredient for the proofs. Guessing that at this moment the method is well established, we limit ourselves to the essential.

First of all, given a proper open subset U of $\mathbb{C}^n$, one introduces the spaces $\mathcal{H}_\infty\{\boldsymbol{\omega}, U\}$, $\mathcal{F}_k\{\boldsymbol{\omega}, b_m \cup U\}$, $\mathcal{F}\{\boldsymbol{\omega}, b_m \cup U\}$ and $\mathcal{F}\{\boldsymbol{\omega}, U\}$. With the notations of the Paragraph 6 in mind, one also introduces the spaces $\mathcal{G}_k\{\boldsymbol{\omega}, b_m \cup D_{j(m)}\}$, $\mathcal{G}\{\boldsymbol{\omega}, b_m \cup D_{j(m)}\}$ and $\mathcal{G}\{\boldsymbol{\omega}, D_F\}$.

Now everything is at hand to state the results.

Proposition 7.5 a) *For every $m \in \mathbb{N}$, there is $C_m > 0$ such that*

$$|D^\alpha G_r(u+iv, f) - D^\alpha G_r(u, f)| \le C_m \mathrm{e}^{\frac{1}{j_0 m}\varphi^*(j_0 m|\alpha|)} \, |||f|||_m$$

for every $f \in \mathrm{C}_m\{\boldsymbol{\omega}, \Omega\}$, $u + iv \in D_\Omega$, $\alpha \in \mathbb{N}_0^n$ and $r \in \{1, \ldots, m\}$.

b) *For every $m \in \mathbb{N}$, there is a constant $l_m > 0$ such that, for every $f \in \mathrm{C}_m\{\boldsymbol{\omega}, \Omega\}$, $r \in \mathbb{N}$, $\alpha \in \mathbb{N}_0^n$ and $u + iv \in D_\Omega$, one has*

$$|D^\alpha G_{m+r}(u+iv, f) - D^\alpha G_{m+r}(u, f)| \le 2^{-(m+r)} l_m \mathrm{e}^{\frac{1}{j_0 m}\varphi^*(j_0 m|\alpha|)} \, |||f|||_m. ■$$

Theorem 7.6 *For every proper open subset Ω of $\mathbb{R}^n$, there is a continuous linear map T_Ω from $\mathrm{C}\{\boldsymbol{\omega}, \Omega\}$ into $\mathcal{H}_\infty\{\boldsymbol{\omega}, D_\Omega\}$ such that for every $f \in \mathrm{C}\{\boldsymbol{\omega}, \Omega\}$, $\varepsilon > 0$ and $s \in \mathbb{N}$, there is a compact subset K of Ω such that*

$$|D^\alpha(T_\Omega f)(u+iv) - D^\alpha f(u)| \leq \varepsilon$$

for every $u + iv \in D_\Omega$ and $\alpha \in \mathbb{N}_0^n$ verifying $u \in \Omega \setminus K$ and $|\alpha| \leq s$.

In fact, for every $m \in \mathbb{N}$, T_Ω is a continuous linear map from $\mathrm{C}_m\{\boldsymbol{\omega}, \Omega\}$ into $\mathcal{H}_{j_0^2 m,\infty}\{\boldsymbol{\omega}, D_\Omega\}$. ∎

Theorem 7.7 *Let F be a proper closed subset of $\mathbb{R}^n$ for which there is a continuous linear extension map from $\mathcal{E}_{\{\omega\}}(F)$ into $\mathcal{E}_{\{\omega\}}(\mathbb{R}^n)$.*

a) *If F is compact or if $\mathbb{R}^n \setminus F$ is bounded, then there also is such a map from $\mathcal{E}_{\{\omega\}}(F)$ into $\mathcal{F}\{\boldsymbol{\omega}, D_{\mathbb{R}^n\setminus F}\}$.*

b) *The following assertions are equivalent:*
(1) *there also is such map from $\mathcal{E}_{\{\omega\}}(F)$ into $\mathcal{G}\{\boldsymbol{\omega}, D_F\}$;*
(2) *for every bounded subset B of $\mathbb{R}^n$, the boundary of the union of the connected components of $\mathbb{R}^n \setminus F$ having non empty intersection with B is compact.* ∎

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