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ON THE JAYNES-CUMMINGS HAMILTONIAN SUPERSYMMETRIC CHARACTERISTICS

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Abstract

We study some degeneracies of the Jaynes-Cummings Hamiltonian eigenstates. More precisely, we underscore operators connecting the degenerated eigenstates or explaining their non-degeneracy. These operators actually are supercharges and the supersymmetry underlying the Jaynes-Cummings model is thus exhibited. We also consider two extensions of the Hamiltonian to show the unicity of their supercharges.

Key-words: Jaynes-Cummings Hamiltonian, degeneracy, supersymmetry.

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1 Introduction

The Jaynes-Cummings Hamiltonian (H_{JC}) [1] is associated with a model describing the interaction between a spin-$\frac{1}{2}$ particle and a one-mode magnetic field having an oscillating component along one axis and a constant component along another axis [2]. This model, extensively used in quantum optics [3], is one of the simplest examples of quantum systems combining bosons and fermions, a typical feature of supersymmetry [4].

Numerous studies of this model have already been realized. For example, we know that, under some hypotheses, it may be seen as an extension of the supersymmetric harmonic oscillator system [2]. Moreover, the diagonalisation of H_{JC} [2] allows to construct the creation and annihilation operators of H_{JC}, and then, to underscore

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the coherent states in the stationary or evolution contexts [2]. Another approach consists in the understanding of H_{JC} as an element of the u(1/1) superalgebra [5]. The coherent states of this superalgebra can then be obtained [5]. Two extensions of H_{JC} can also be considered [6]. We will come back on that point later.

The supersymmetric characteristics of H_{JC} have only been viewed through the two above-mentioned extensions [6]. We will show here that H_{JC} has also supersymmetric characteristics by its own.

The main purpose of this paper is to prove that the H_{JC}-eigenstates are degenerated only for one value of the energy and then to explain this degeneracy.

The contents of this paper are the following. Section 2 is devoted to the energy spectrum and the eigenstates of H_{JC}. Section 3 is divided into two parts : in the first one, we prove the existence of only one supercharge when there is no degeneracy for the H_{JC}-eigenstates (3.1); then, we prove the existence of two operators connecting the degenerated eigenstates associated with a particular value of the energy (3.2). Finally, we present, in section 4, a few remarks about two extensions of H_{JC}.

Our units are taken with the constant $\hbar$ equal to unity.

2 Energy spectrum and eigenstates of H_{JC}

The Jaynes-Cummings model can be described by the Hamiltonian [1]

$$H_{JC} = \omega(a^\dagger a + \frac{1}{2}) + \frac{\omega_0}{2}\sigma_3 + g(a^\dagger \sigma_- + a\sigma_+), \tag{2.1}$$

where $a^\dagger$ and a are respectively the creation and annihilation operators of the bosonic harmonic oscillator and where $\sigma_\pm = \sigma_1 \pm \sigma_2$, σ_3 refer to the Pauli matrices.

In order to find the energy spectrum and the eigenstates of H_{JC}, we have to solve the equation

$$H_{JC} \mid \psi >= E \mid \psi > \tag{2.2}$$

in the basis of the vectors

$$\begin{pmatrix} 0 \\ \mid n > \end{pmatrix} =\mid n, - > \text{ and } \begin{pmatrix} \mid n > \\ 0 \end{pmatrix} =\mid n, + > . \tag{2.3}$$

If we note Δ the difference between the two angular frequencies ω and ω_0, we obtain results which can be summarized as follows :

a)for all the values of g, we have to distinguish two cases

(i) either $E = \frac{\Delta}{2}$ and the corresponding eigenstate is

$$\mid E_0 >=\mid 0, - > . \tag{2.4}$$

(ii) or $E = \omega k \pm \frac{\Delta}{2} r(k)$,$k \in \mathbb{N}_\circ$, and the corresponding eigenstates are

$$\mid E_k^\pm >= \frac{1}{R(k)}(g\sqrt{k} \mid k-1, + > + \frac{\Delta}{2}(r(k)+1) \mid k, - >). \tag{2.5}$$

$$| E_k^- >= \frac{1}{R(k)}(\frac{\Delta}{2}(r(k)+1) \mid k-1, + > -g\sqrt{k} \mid k, - >), \qquad (2.6)$$

where

$$r(k) = (1 + \frac{4g^2 k}{\Delta^2})^{\frac{1}{2}} \qquad (2.7)$$

and

$$R(k) = (\frac{\Delta^2}{2} r(k)(1 + r(k)))^{\frac{1}{2}}. \qquad (2.8)$$

b)if there exists $k \in \mathbb{N}_\circ$ such as $\frac{\Delta}{2} = \omega k + \frac{\Delta}{2} r(k)$, Δ has to be negative and g has to take the values

$$g = \pm\sqrt{\omega(\omega k - \Delta)} \qquad (2.9)$$

Then the corresponding eigenstates are

$$| E_k^{\mp} >= \sqrt{\frac{\omega k}{2\omega k - \Delta}}(\mid k-1, + > \mp\sqrt{\frac{\omega k - \Delta}{\omega k}} \mid k, - >). \qquad (2.10)$$

c)if there exists $k \in \mathbb{N}_\circ$ such as $\frac{\Delta}{2} = \omega k - \frac{\Delta}{2} r(k)$, Δ has to be positive and, also here, g has to take the values

$$g = \pm\sqrt{\omega(\omega k - \Delta)}$$

Then the corresponding eigenstates are also (2.10).

In the particular case where $\Delta = 0$ and $g = 0$, the results are the same as those of the supersymmetric harmonic oscillator [4].

3 Explanation of degeneracy

There is degeneracy of the H_{JC}-eigenstates when $E = \frac{\Delta}{2}$ only. This fact can be explained through supersymmetry or more precisely through the existence of supercharges.

Let us assume $\Delta = 0$. A similar way of thinking in the case $\Delta \neq 0$ would lead us to the same conclusions.

First of all, let us search for the supercharges of H_{JC}.

3.1 Supercharges of H_{JC}

Let us recall that supersymmetric quantum mechanics needs the positive nature of the energies. So we translate [6] H_{JC} by adding a positive constant c to it. Thus, in order to find the supercharges of H_{JC}, we have to solve the equation

$$Q^2 = H_{JC} + c \qquad (3.1)$$

whose solution is

$$Q = \sqrt{\omega} a \sigma_+ + \sqrt{\omega} a^\dagger \sigma_- + \frac{1}{2} \frac{g}{\sqrt{\omega}} I, \tag{3.2}$$

fixing the constant c, without loss of generality, as the value

$$c = \frac{g^2}{4\omega} \tag{3.3}$$

Moreover, the operators σ_+, σ_- and I generating the Clifford algebra [7] Cl_2 (characterized here by its fundamental irreductible representation), we deduce that $Q =$ (3.2) is the only supercharge of $H_{JC} + c$.

Furthermore, the effect of Q on the H_{JC}-eigenstates explain the non-degeneracy of these states in the general case, as the H_{JC}-eigenstates are also eigenstates with respect to Q.

In order to understand the eigenstates degeneracy in the case $E = c$, we have to find operators connecting these states. That is the purpose of the next section.

3.2 Existence of operators connecting the H_{JC} degenerated eigenstates

In the case $k = 1$ and $g = \omega$, the two operators connecting the degenerated eigenstates for $E = c$ are

$$P = \begin{pmatrix} f(N)a^\dagger & f(N) \\ -g(N)a^{\dagger 2} & -g(N)a^\dagger \end{pmatrix} \text{ and } P^\dagger = \begin{pmatrix} af(N) & -a^2 g(N) \\ f(N) & -ag(N) \end{pmatrix} \tag{3.4}$$

where f and g are real functions of N. These operators satisfy the typical relations of supersymmetric quantum mechanics [4] i.e.

$$P^2 = P^{\dagger 2} = 0 \quad , \quad \{P, P^\dagger\} = H_{JC} + c \tag{3.5}$$

characterizing the Lie superalgebra sqm(2), but only on the space generated by $\mid E_0 >$ and $\mid E_1^- > = \mid 0, + > - \mid 1, - >$, with the condition $f(0) = g(1)$. On the whole Fockspace, the relations of sqm(2) are not ascertained.

A similar way of thinking for other values of k leads us to the same conclusion. Because the two operators connecting degenerated eigenstates only act on the above-mentioned space of these eigenstates, the unicity of $Q =$ (3.2) is not in the balance again.

4 Two generalizations of H_{JC} in the case $\Delta = 0$

The first one consists in superposing on H_{JC} a second Hamiltonian H_2 defined by this expression [6]

$$H_2 = \omega(a^\dagger a + \frac{1}{2} + \frac{1}{2}\sigma_3) + ig(a^\dagger \sigma_- - a\sigma_+). \tag{4.1}$$

It is unitarily equivalent to H_{JC}. The resulting Hamiltonian is thus [6]

$$H = \begin{pmatrix} H_{JC} & 0 \\ 0 & H_2 \end{pmatrix}. \tag{4.2}$$

One supercharge of $H + c$ is given by this expression

$$Q = \sqrt{\omega} a \xi_+ + \sqrt{\omega} a^\dagger \xi_- + \frac{g}{2\sqrt{\omega}} \eta \tag{4.3}$$

where

$$\xi_+ = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \xi_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}; \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \tag{4.4}$$

The odd parity of these operators and all their anticommutation relations lead us to consider five more operators generating with ξ_+, ξ_- and η the Lie superalgebra osp(2/2). As there exists only one representation of this superalgebra [8] with 4 by 4 matrices, we can conclude that $Q = (4.3)$ is the only supercharge of $H + c$ connecting the degenerated eigenstates.

The second extension of H_{JC} consists in adding a positive constant Δ' to $H + c$, where $H = (4.2)$. Also here, there is only one supercharge for $H + c + \Delta'$ which is

$$Q^{\Delta'} = Q + \sqrt{\Delta'} R \tag{4.5}$$

where $Q = (4.3)$ and

$$R = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.6}$$

Indeed, another supercharge should have the form

$$Q_1^{\Delta'} = Q + \sqrt{\Delta'} R' \tag{4.7}$$

and should satisfy the relations

$$\{Q_1^{\Delta'}, Q^{\Delta'}\} = 0 \tag{4.8}$$

and

$$Q_1^{\Delta'^2} = H + c + \Delta'. \tag{4.9}$$

In other words, the operator R' should have the form

$$R' = \begin{pmatrix} d & 0 & il & 0 \\ 0 & d & 0 & l \\ -il & 0 & -d & 0 \\ 0 & l & 0 & -d \end{pmatrix} \tag{4.10}$$

with

$$l^2 + d^2 = 1. \tag{4.11}$$

Taking the expressions (4.6) and (4.10) for R and R', we have

$$\{R, R'\} = 2lI \tag{4.12}$$

and then

$$\{Q_1^{\Delta'}, Q^{\Delta'}\} = 2(H + c + 2lI) \neq 0. \tag{4.13}$$

That is in contradiction with (4.8) and allows us to conclude that $Q^{\Delta'} = (4.5)$ is the only supercharge of $H + c + \Delta'$ connecting the degenerated eigenstates.

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