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A GENERALIZATION OF TIETZE'S THEOREM ON LOCAL CONVEXITY FOR OPEN SETS

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Abstract

Let S be a nonempty subset of a real topological linear space L and s a point in cl$S$. A point s of *weak local C-convexity* of S is defined as follows: if there exists a neighbourhood N of s such that $s \in$clC_s, where C_s is a component of $S \cap N$, then $[x, y] \subseteq S$ for each pair of points $x, y \in C_s$, otherwise $[x, y] \subseteq S$ for each pair of points x, y in any component of $S \cap N$. It is proved that an open connected subset S of L whose boundary consists exclusively of C-wlc points of S is convex. This is a version of the Sacksteder-Straus-Valentine generalization of Tietze's local characterization of convexity for open sets.

Key words: weak local C-convexity point, Tietze-type theorem.
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Let S be a nonempty subset of a real topological linear space L. A point s in cl$S$ is said to be a *point of weak local convexity* of S if and only if there is some neighbourhood N of s such that for each pair of points x, y in $S \cap N$, $[x, y] \subseteq S$ [2, Def.4.2]. A point s of *weak local C-convexity* of S is defined as follows: if there exists a neighbourhood N of s such that $s \in$clC_s, where C_s is a component of $S \cap N$, then $[x, y] \subseteq S$ for each pair of points $x, y \in C_s$, otherwise $[x, y] \subseteq S$ for each pair of points x, y in any component of $S \cap N$ (cf. [2, Def.4.5]). Furthermore [1],[2, Def.4.3], a point s in cl$S$ is said to be a *point of strong local convexity (C-convexity)* if and only if $S \cap N$ (each component of $S \cap N$) is convex for some neighbourhood N of s in L. For the sake of brevity, we call points of weak and strong local convexity (C-convexity) of S, respectively, wlc and slc (C-wlc and C-slc) points of S. (xyz) will represent the two-dimensional flat determined by three noncollinear points x, y, z.

Tietze's famous characterization of convexity states that a closed connected subset S of L consisting exclusively of wlc points is convex [2, Th.4.4]. In [1], a generalization was proved that a connected compact subset S of a complete locally convex real topological linear space consisting exclusively of C-slc points is convex. In [3, Cor.2.3], the author proved essentially that an open connected subset S of L whose boundary consists exclu-

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sively of wlc points is convex. The purpose of this note is to prove a generalization of this result kept in the spirit of [1]. The straightforward argument differs from that in [3] and thanks to the assumption of the openness of S is simpler.

Theorem. *If S is an open connected subset of a real topological linear space L with the boundary consisting exclusively of C-wlc points of S, then S is convex.*

Proof. Since S is open and connected, and L is locally starshaped [2, Th.1.4], an easy argument reveals that S must be polygonally connected. For this, we fix a point x_0 in S and consider the subset $\mathcal{W}$ of S consisting of points which can be joined to x_0 via polygonal paths in S. The local starshapedness of S implies immediately that $\mathcal{W}$ is simultaneously open and closed in S, so that it must coincide with connected S. Select arbitrarily distinct points x, y of S. Let $[x,y] \nsubseteq S$. Hence, there exists in S a simple polygonal path $\mathcal{P} = [x_0, x_1] \cup \dots \cup [x_n, x_{n+1}] (n \geq 1, x_0 = x, x_{n+1} = y)$ with the minimal number $n+1$ of nondegenerate line segments. Consider the subpath $[x_0, x_1] \cup [x_1, x_2]$. Of course, the points x_0, x_1, x_2 are noncollinear. By [2, Th.1.8], we can identify $(x_0x_1x_2)$ in the topology induced from L with the Euclidean plane R^2. Without loss of generality, assume that x_1 is the origin of L. Since $[x_1, x_2]$ is compact and $S \cap (x_0x_1x_2)$ is relatively open in $(x_0x_1x_2)$, there exists a relatively open circle B in $(x_0x_1x_2)$ centered at x_1 such that $[x_1, x_2] + B \subseteq S$. If $x_0 \in B$, then $[x_0, x_2] \subseteq S$ and $\mathcal{P}$ can be replaced by a path consisting of n line segments , a contradiction. Denote thus $(t, x_1] = B \cap [x_0, x_1]$. Then $\mathrm{conv}((t, x_1] \cup [x_1, x_2]) \subseteq S$. Since S is open, there is a point $x_0' \in S$ such that $x_0 \in (x_0', x_1)$. Suppose, to reach a contradiction, that $\mathrm{conv}((x_0', x_1] \cup [x_1, x_2]) \nsubseteq S$. Then there exists the largest subsegment $(w, x_1]$ of $[x_0', x_1]$ such that $\mathrm{conv}((w, x_1] \cup [x_1, x_2]) \subseteq S$. Suppose that $[x_2, w] \nsubseteq S$. Since $x_2 + B \subseteq S$, there exists a largest subsegment $[x_2, u)$ of $[x_2, w]$ contained in S. $u \in \mathrm{bdry}S$, so that, by initial assumption, u is a C-wlc point of S. Consequently, there exists a relatively open circle D in $(x_0x_1x_2)$ centered at u such that for the component C_u of $D \cap S$ for which $u \in \mathrm{cl}C_u$, if $p, q \in C_u$, then $[p, q] \subseteq S$. Pick out a point $a \in D \cap (x_2, u)$. $a \in S$ and S is open, so that there is a relatively open circle D_a in $(x_0x_1x_2)$ centered at a and contained in $D \cap S$. D_a and $D\cap \mathrm{conv}((w, x_1] \cup [x_1, x_2])$ lie in the same component of $D \cap S$ having u in its closure, so that by assumption $u \in \mathrm{conv}(D_a \cup (D \cap \mathrm{conv}((w, x_1] \cup [x_1, x_2]))) \subseteq S$ and the segment $[x_2, u)$ can be extended beyond u in S, a contradiction. Hence, $[x_2, w] \subseteq S$. But S is open and $[x_2, w]$ is compact, so that there exists a point $w' \in (x_0', w)$ such that $\mathrm{conv}((w', x_1] \cup [x_1, x_2]) \subseteq S$, contradictory to the choice of w. Hence, $[x_0, x_2] \subseteq \mathrm{conv}((x_0', x_1] \cup [x_1, x_2]) \subseteq S$. Thus we can replace the path $\mathcal{P}$ by the path $[x_0, x_2] \cup \dots \cup [x_n, x_{n+1}]$ consisting of at most n line segments, contradictory to the choice of $\mathcal{P}$. Hence, $[x, y] \subseteq S$ and S is convex, by the arbitrary choice of x, y.

The proof is complete. □

It is still an open question if the assumptions of the theorems in [1] (cf. [2, Ths.4.5 and 4.6]) can be weakened in any way.

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References

[1] Sacksteder,R., Straus,E.G. and Valentine,F.A.: A generalization of a theorem of Tietze and Nakajima on local convexity. *J. London Math. Soc.* **36** (1961) 52 - 56.
[2] Valentine,F.A.: *Convex Sets*. McGraw-Hill, New York, 1964.
[3] Cel,J.: An optimal Krasnosel'skii-type theorem for an open starshaped set. *Geom. Dedicata* **66** (1997) 293 - 301.

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