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DISTRIBUTIVE SEMILATTICES AND BOOLEAN LATTICES

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RÉSUMÉ

Nous montrons que, dans la définition classique d'un lattis de Boole comme lattis distributif complémenté, le mot lattis peut être remplacé par demi-lattis. La notion de 0-distributivité, introduite dans [9], est appliquée ici aux $\wedge$-demi-lattis et caractérisée comme suit : un $\wedge$-demi-lattis borné est 0-distributif si et seulement si tout filtre maximal est premier. Le théorème de séparation dû à Stone est étendu aux demi-lattis : un $\wedge$-demi-lattis filtrant supérieurement est distributif si et seulement si tout couple formé d'un filtre et d'un idéal disjoints peut être séparé par un filtre premier. Enfin il est établi que le théorème de Nachbin (un lattis distributif borné est booléen si et seulement si filtres premiers et filtres maximaux coïncident) reste vrai si l'on substitue à l'adjectif distributif les mots « très faiblement complémenté », c'est-à-dire : l'idéal zéro est le noyau d'une seule congruence, l'identité.

1. INTRODUCTION

It is well known (and the result is due to Nachbin [7]) that a bounded distributive lattice is Boolean if and only if every prime filter is maximal. In a recent paper [1], D. Adams tried to weaken Nachbin's conditions and obtained the following result : a lattice L is Boolean if it satisfies the conditions

(a) L is bounded;

(b) L is weakly complemented;

(c) any ideal of L is maximal if and only if it is prime;

(d) any filter of L is maximal if and only if it is prime.

Certainly the hypothesis of distributivity is suppressed, but conditions (c) and (d) are both very strong since double-sided. Consequently it seems inadequate to claim that Nachbin's condition has been replaced by a « much weaker condition », owing to the supplementary fact that weak complementedness and distributivity are completely independent.

We shall show that the previous conditions are redundant : (a), (b) and (d) are sufficient and (b) can even be weakened. More precisely, if in Nachbin's statement we replace distributivity by 0-distributivity, a weakened form that we introduced in [9], the additional assumption of very weak complementedness is needed (theorem 4).

But in the beginning of our paper we are concerned with semilattices. We first show that a bounded $\wedge$-semilattice which is distributive and complemented is a Boolean lattice (theorem 1). Then the meaning of 0-distributivity in an up-directed

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$\wedge$-semilattice is clarified : 0-distributivity is equivalent to the fact that any maximal filter is prime (theorem 2). Thereafter we establish that the distributivity of an up-directed $\wedge$-semilattice can be characterized, as in the case of lattices, by the possibility of separating by a prime filter any filter F and any ideal I such that $F \cap I = \varnothing$.

2. PRELIMINARIES

Throughout this paper the word semilattice will mean $\wedge$*-semilattice*, i.e. a set with an associative, commutative and idempotent binary operation. A partially ordered set A such that for any $x, y \in A$, the set of upper bounds of $\{a, b\}$ is not void, is said to be *up-directed*.

A *filter* of a semilattice S is a non-empty subset F of S such that $x \wedge y \in F$ if and only if $a \in F$ and $b \in F$. The principal filter generated by an element a of S, i.e. the set $\{x : x \in S, x \geqslant a\}$, will be denoted by $[a)$. A filter F of S is *prime* if, whenever two filters F_1 and F_2 are such that $\varnothing \neq F_1 \cap F_2 \subseteq F$, then F_1 or F_2 belongs to F. A proper filter F is *maximal* if the only filter strictly containing F is S. When S has a least element 0, a proper filter F is maximal if and only if, for any $a \in S - F$, there exists an element $b \in F$ such that $a \wedge b = 0$.

An *ideal* I of a semilattice S is a non-empty subset of S such that

(I_1) $y \leqslant x$ and $x \in I$ imply $y \in I$;

(I_2) for any $x, y \in I$, there exists $z \in I$ such that $z \geqslant x$ and $z \geqslant y$.

The notions of prime and maximal ideals can be defined in the same way as for filters. In an up-directed semilattice S, a filter F is prime if and only if $S - F$ is an ideal of S.

We warn the reader that what we call filter (resp. ideal) is usually named ideal (resp. filter). Our convention makes it possible to apply the theory of $\wedge$-semilattices to the study of lattices without any change of terminology : a subset of a lattice $L = \langle L; \vee, \wedge \rangle$ is a filter (resp. an ideal) if and only if it is a filter (resp. an ideal) of $L = \langle L; \wedge \rangle$.

The concept of distributive semilattice is due to G. Grätzer and E. T. Schmidt [3]. It has been investigated by T. Katriňák [6], J. Varlet [10], G. Grätzer [4] and J. Rhodes [8]. A semilattice is *distributive* if $c \geqslant a \wedge b$ $(a, b, c \in S)$ implies the existence of $a_1, b_1 \in S$ such that $a_1 \geqslant a$, $b_1 \geqslant b$ and $a_1 \wedge b_1 = c$. A lattice is distributive if and only if it is distributive as a semilattice.

In [9] we introduced the notion of 0-distributive lattice, in order to generalize that of pseudo-complemented lattice. As a matter of fact the concept of 0-distributivity applies to semilattices bounded below. A semilattice S with least element 0 will be said *0-distributive* if, for any $a \in S$, the subset $I = \{x \in S : x \wedge a = 0\}$ is an ideal. In the same way we can define a 1-distributive semilattice.

A semilattice S with 0 is *weakly complemented* if it satisfies one of the three equivalent conditions :

(C_1) for any pair a, b $(a < b)$ of elements of S, there is an element c such that $a \wedge c = 0$ and $b \wedge c \neq 0$;

(C_2) for any pair a, b of distinct elements of S, there exists an element c disjoint from one of these elements but not from the other;

(C_3) for any pair a, b of distinct elements of S, there is a maximal filter containing one of them but not the other.

Finally we shall make use of another notion introduced by G. Grätzer and E. T. Schmidt in [2], p. 152 : a lattice with 0 is *very weakly complemented* if the zero ideal is the kernel of a unique congruence, the identity.

3. MAIN RESULTS

Our terminology will be coherent only if 0-distributivity and 1-distributivity are implied by distributivity in any bounded semilattice. Fortunately we have .:

Lemma. A distributive semilattice with 0 (resp. 1) *is* 0-*distributive* (resp. 1-*distributive*).

Proof. Let S be a distributive semilattice with 0. We have to show that for any $a \in \mathrm{S}$, $\mathrm{I} = \{x \colon x \wedge a = 0\}$ is an ideal. Let b, c be any two elements of I. From $b \geqslant a \wedge c = 0$, we deduce the existence of elements d, e such that $d \geqslant a$, $e \geqslant c$ and $d \wedge e = b$. The element e is an upper bound of $\{b, c\}$ and belongs to I since $e \wedge a = e \wedge (a \wedge d) = (e \wedge d) \wedge a = b \wedge a = 0$.

The second part is also easy to establish. In the distributive semilattice S with 1, $\mathrm{F} = \{x \colon x \vee a = 1\}$ is not empty and contains $y \geqslant x$ whenever it contains x. It remains to show that $x, y \in \mathrm{F}$ implies $x \wedge y \in \mathrm{F}$. Let k be any upper bound of a and $x \wedge y$. Since $k \geqslant x \wedge y$, there exist m, n such that $m \geqslant x$, $n \geqslant y$ and $m \wedge n = k$. From $m \geqslant a$ and $m \geqslant x$, we deduce $m = 1$. Similarly $n = 1$, hence $k = 1$.

The notion of *complement* makes sense in any partially ordered set P with least element 0 and greatest element 1. Precisely, b is a complement of a if the only upper bound of a and b is 1 and their only lower bound is 0. We shall express this fact by $a \vee b = 1$ and $a \wedge b = 0$ even when the operations $\wedge$ and $\vee$ are undefined for some pairs of P.

The classical definition of a Boolean lattice can be improved as follows.

Theorem 1. *A bounded distributive* $\wedge$*-semilattice is a Boolean lattice if and only if it is complemented.*

Proof. The « only if » part is obvious. To establish the « if » part, we first show that in a bounded distributive $\wedge$-semilattice S, when an element a has a complement a', this complement is unique and $a' = a^+$, the dual pseudo-complement of a. Let us suppose both elements b and c are complements of a in S. By the lemma, there is no loss of generality in assuming $b > c$. Since $c > a \wedge b = 0$, there exist x, y such that $x \geqslant a$, $y \geqslant b$ and $x \wedge y = c$. But then $x \geqslant a$ and $x \geqslant c$ imply $x = 1$, hence $y = c$, which contradicts $y \geqslant b$. Clearly the unique complement a' of a is the least element of S whose join with a is equal to 1; in other terms, $a' = a^+$.

The next observation is that when a and b are both complemented and $a \geqslant b$, then $a' \leqslant b'$: complementation in a distributive $\wedge$-semilattice is order-reversing. In fact, $b \vee x = 1 \Leftrightarrow x \geqslant b^+$. From $a \geqslant b$ and $b \vee b^+ = 1$, we deduce $a \vee b^+ = 1$ and $b^+ \geqslant a^+$.

We are now in a position to prove that any bounded, distributive and complemented $\wedge$-semilattice S is a (Boolean) lattice. Let a, b be any pair of elements of S; a', b' their respective complements. Let us show a and b have a least upper bound. If $a' \wedge b' = c'$, then $c'' = c$ is an upper bound of a and b. We will show that c is precisely the least upper bound of a and b. Let x be an upper bound of a and b satisfying $x < c$. Owing to the distributivity of S, we have for any upper bound d of c' and x: $d \geqslant c' = a' \wedge b' \Rightarrow \exists\, e, f \colon e \geqslant a'$, $f \geqslant b'$ and $e \wedge f = d$. But $e \geqslant a'$ and

$e\,(\geqslant d \geqslant x) \geqslant a$ imply $e = 1$. Similarly, $f = 1$ hence $d = 1$. The element x would be a second complement of c', a contradiction. In conclusion, any pair of S has a supremum and S is a lattice. Since a lattice is distributive if and only if it is distributive as a semilattice, the proof is over.

Theorem 2. *A bounded semilattice is* 0*-distributive if and only if any maximal filter is prime* (*).

Proof. First let us consider a semilattice S in which any maximal filter is prime. We have to prove that S is 0-distributive, that is, for any $a \in S$, the set $I = \{x \in S : a \wedge x = 0\}$ is an ideal. For any pair b, c of I, the set of all upper bounds of b and c is a filter F. The set $G = \{y \in S : y \geqslant a \wedge f,\ f \in F\}$ is also a filter. If G is distinct from S, it is contained in a maximal filter M. By hypothesis M is prime and, since $[b) \cap [c) = F \subseteq M$, either $[b) \subseteq M$ or $[c) \subseteq M$. But $b \in M$ (or $c \in M$) would imply $0 \in M$, a contradiction. Hence $G = S$, there exists $f \in F$ such that $a \wedge f = 0$ and consequently b, c have in I the upper bound f.

Let us now consider a 0-distributive $\wedge$-semilattice S. Let F be a maximal filter which is not prime : there exist two filters G and H such that $G \cap H \subseteq F$, but neither $G \subseteq F$ nor $H \subseteq F$. So we can find $x \in G - F$ and $y \in H - F$. Since F is maximal, there exist z and t in F such that $x \wedge z = y \wedge t = 0$. Since $x \wedge (z \wedge t) = y \wedge (z \wedge t) = 0$, an element $u \geqslant x, y$ can be found satisfying $u \wedge (z \wedge t) = 0$. Since u, z, t all belong to F, F contains 0, a contradiction.

Stone characterized distributive lattices by means of the following separation property : a lattice is distributive if and only if when a filter F and an ideal I are disjoint, there exists a prime filter containing F and disjoint from I. This result can be generalized to semilattices as follows.

Theorem 3. *An up-directed semilattice is distributive if and only if for any filter* F *and any ideal* I *such that* $F \cap I = \varnothing$, *there exists a prime filter containing* F *and disjoint from* I.

Proof. The necessity of the condition has been proved in [6], theorem 2. Now we establish the sufficiency. Let us consider any three elements a, b, c such that $c \geqslant a \wedge b$. We denote by F_1 (resp. F_2) the (non-empty) set of upper bounds of a and c (resp. b and c). F_1 and F_2 are filters, as well as $F = \{z \geqslant x \wedge y : x \in F_1, y \in F_2\}$. Let us suppose F does not contain c. By hypothesis there exists a prime filter P containing F but not c. Since $[a) \cap [c) = F_1 \subseteq P$ and P is prime, a has to belong to P. For a similar reason P contains b. Consequently, $c \geqslant a \wedge b$ is in P, a contradiction. In conclusion, F contains c, that is, there exist $d \geqslant a$ and $e \geqslant b$ such that $d \wedge e = c$, q.e.d.

Comment. Theorem 3 provides us with a sufficient condition for an up-directed semilattice to be distributive and it is surprising that the fact is mentioned neither in [6] nor in [4]. Let us consider the following separation properties of the semilattice S.

(P_1) when an ideal and a filter are disjoint, they can be separated by a prime filter;

(P_2) a filter and an element not belonging to it can be separated by a prime filter;

(*) The hypothesis of boundedness can be replaced by the slightly weaker one : the semilattice is bounded below and up-directed.

(P_2') an ideal and an element not belonging to it can be separated by a prime filter;

(P_3) any two distinct elements can be separated by a prime filter.

The corresponding properties with maximal filter instead of prime filter will be denoted by (M_i). Clearly (P_i) implies (P_j) and (M_i) implies (M_j) whenever $i \leqslant j$. We point out that (M_3) is nothing else than (C_3).

The proof of theorem 3 shows that (P_2) is a sufficient condition for an up-directed semilattice to be distributive. It is natural to ask whether (P_3) would suffice to ensure distributivity. We conjecture the answer is no. If the answer were affirmative, it would lead directly (by using definition (C_3) of weak complementedness and theorems 2 and 3) to the following proposition : an up-directed, 0-distributive and weakly complemented semilattice is distributive. This statement has been proved for lattices in [9]. Maybe the next step would be (by using theorem 1) : a bounded weakly complemented semilattice is Boolean if and only if prime and maximal filters coincide.

Furthermore example (D) below shows that weak complementedness does not imply (M_1) : it suffices to consider the filter [1) and its set-complement, which is an ideal. Does weak complementedness imply (M_2) ? Here also an affirmative answer would imply (by using theorem 2) the same conclusions as above. All this explains why we shall concentrate in our last theorem on lattices and not on semilattices. The best result we can provide is :

Theorem 4. *A bounded very weakly complemented lattice* L *is Boolean if and only if prime and maximal filters of* L *coincide.*

Proof. The necessity of the condition is well known. To show its sufficiency we first observe that by theorem 2 L is 0-distributive. Consequently the relation $\sim$ defined by $a \sim b$ iff $a \wedge x = 0$ is equivalent to $b \wedge x = 0$, is a congruence with kernel $\{0\}$ (see theorem 3 of [9]). Since L is also very weakly complemented, $\sim$ has to be the equality relation and condition (M_3) is fulfilled. Condition (P_3) is also satisfied and we can conclude that L is distributive. To complete the proof, it remains to put forward Nachbin's statement.

4. FINAL REMARK

Theorem 4 includes four conditions :

(1) L is bounded;

(2) L is very weakly complemented;

(3) every maximal filter of L is prime (0-distributivity);

(4) every prime filter of L is maximal.

To prove their independence we consider the following examples :

(A) Generalized Boolean lattice, i.e. a distributive and relatively complemented lattice bounded below but without greatest element.

(B) Five-element non-modular lattice.

(C) Five-element modular but non-distributive lattice.

(D) Lattice (A) to which we adjoin a maximum element 1.

It is easy to make sure that each of these lattices satisfies all conditions but one. (A) is not bounded. (B) is not very weakly complemented. Maximal and prime filters coincide as in (A). (C) is not 0-distributive and has no prime filter. Finally condition (4) is not satisfied by (D) since the prime filter [1) is not maximal.

We note that a lattice which satisfies (1), (2) and (3) but not (4) is necessarily infinite. Indeed we observed in [9] that a 0-distributive lattice which is compactly generated is pseudo-complemented. Since a pseudo-complemented and very weakly complemented lattice is Boolean, a finite lattice satisfying (1), (2) and (3) also fulfils (4).

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